FINITE ELEMENT METHOD

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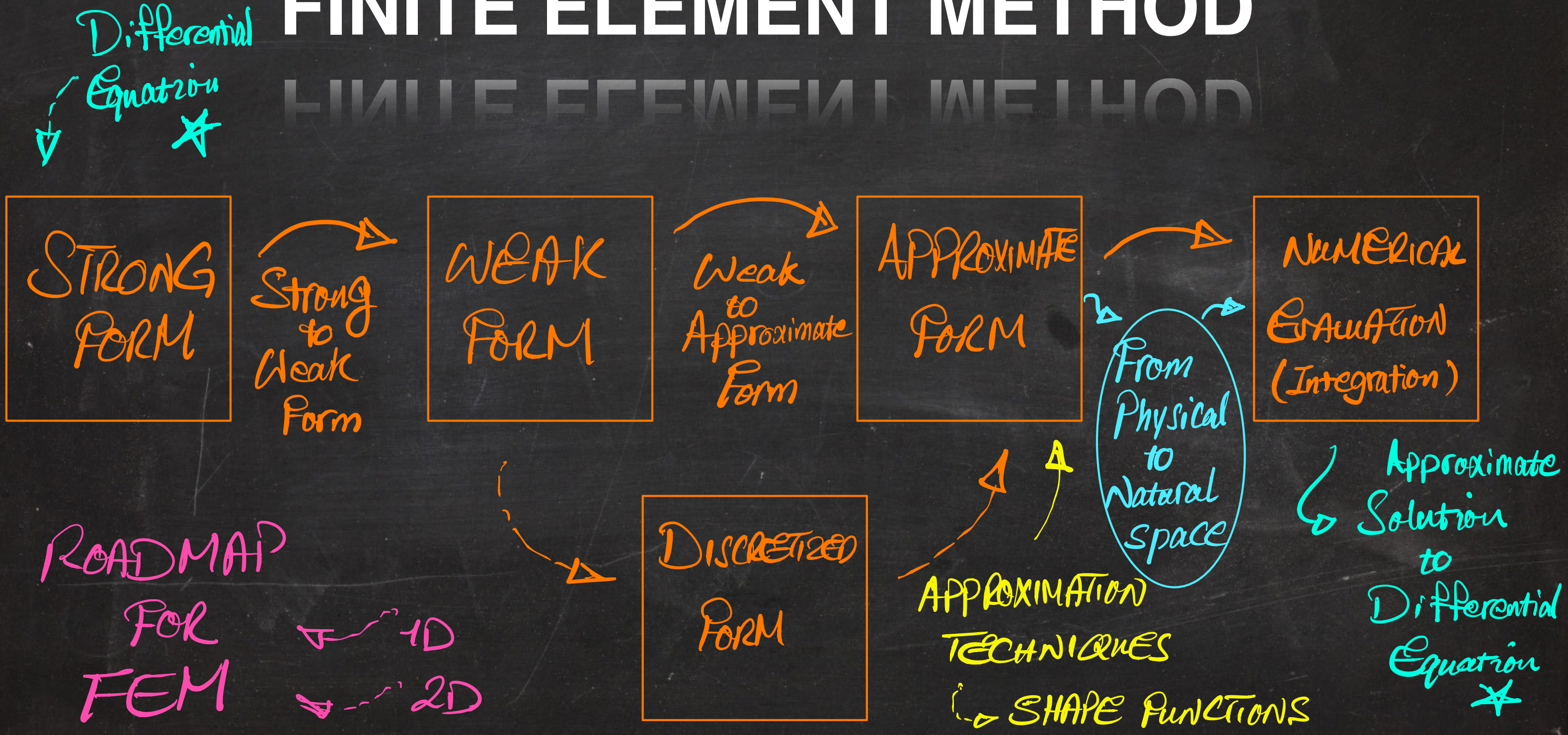
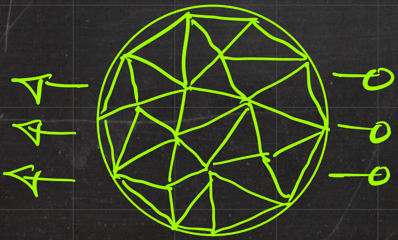


UNDERSTANDING FEM VIA AN ANALOGY (A BRUTAL SIMPLIFICATION)

Approximations in FEM

- → Solution Approximation → inherent to numerical techniques using computers
- → Equation Approximation → diff equation is transformed to weak form + space approximation
- → Input Approximation → space discretization

↓



Discretization (Approximation)
- → DOMAIN (X)
- → SOLUTION (u)
- → TEST (w)

diff. Eq.
→ STRONG FORM
TO
integral → WEAK FORM

$$\frac{d}{dx}\left(EA\,\frac{du}{dx}\right) + bA = 0 \quad \text{Subject to BCs}$$

$$\hookrightarrow E, A : \text{Const.} \quad \leadsto \quad EA\,u'' + bA = 0 \quad \swarrow \quad f := \frac{b}{E}$$

$$\text{STRONG FORM} \longrightarrow \leadsto \boxed{\begin{array}{ll} u'' + f = 0 & 0 \leqslant x \leqslant 1 \\ \\ D:\; u(0) = u_0 & \leftarrow \text{prescribed} \\ N:\; u'(1) = t & \swarrow \end{array}}$$

FROM STRONG TO WEAK FORM

$$u'' + f = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = u_0$ ← prescribed

N: $u'(1) = t$ ←

STRONG FORM ← Differential Eq.

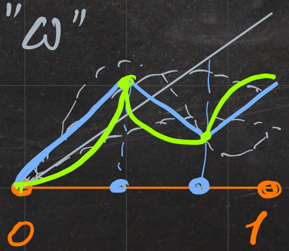
(I) MULTIPLY BY TEST FUNCTION w

w :
- ARBITRARY
- CONTINUOUS
- $w|_D = 0$ ← ZERO @ DIRICHLET BOUNDARY CONDITIONS

(II) INTEGRATE OVER THE DOMAIN

Integral form ← WEAK FORM

→ BECAUSE LOWER ORDER DIFFERENTIATION OF DISPLACEMENT u



STRONG : u''

WEAK : u'

FROM STRONG TO WEAK FORM

$$u'' + f = 0 \qquad 0 \leqslant x \leqslant 1$$

$$D: \; u(0) = u_0 \leftarrow \text{prescribed}$$

$$N: \; u'(1) = t \leftarrow$$

STRONG FORM

(I) MULTIPLY BY TEST FUNCTION w

(II) INTEGRATE OVER THE DOMAIN

$$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$$

I) $\left[u'' + f = 0 \right] \times w \Rightarrow w u'' + w f = 0$

II) $\displaystyle\int_0^1 [w u'' + w f]\, dx = 0$

$$w u'' = (w u')' - w' u'$$

$$\int_0^1 (w u')'\, dx - \int_0^1 w' u'\, dx + \int_0^1 w f\, dx = 0$$

FROM STRONG TO WEAK FORM

$$u'' + f = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = u_0 \leftarrow$ prescribed

N: $u'(1) = t \swarrow$

I) $[u'' + f = 0] \times \omega \Rightarrow \omega u'' + \omega f = 0$

II) $\int_0^1 [\omega u'' + \omega f]\, dx = 0$

$\omega u'' = (\omega u')' - \omega' u'$

$$\int_0^1 (\omega u')'\, dx - \int_0^1 \omega' u'\, dx + \int_0^1 \omega f\, dx = 0$$

$$\int_0^1 \omega' u'\, dx = \int_0^1 \omega f\, dx + \omega u' \Big]_0^1$$

$$\int_0^1 \omega' u'\, dx = \int_0^1 \omega f\, dx + \omega(1)\, u'(1) - \omega(0)\, u'(0)$$

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$$

FROM STRONG TO WEAK FORM

$$u'' + f = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = u_0$ ← prescribed

N: $u'(1) = t$ ↙

I) $[u'' + f = 0] \times w \Rightarrow w u'' + w f = 0$

II) $\int_0^1 [w u'' + w f]\, dx = 0$ $\qquad w u'' = (w u')' - w' u'$

$$\int_0^1 w' u' \, dx = \int_0^1 w f \, dx + \overbrace{\underbrace{w(1)}_{\text{TEST FUNCTION @ 1}} u'(1) - \underbrace{w(0)}_{\text{TEST FUNCTION @ 0}} u'(0)}$$

$$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$$

BC:

DIRICHLET u ✓ u' ?

NEUMANN u ? u' ✓

FROM STRONG TO WEAK FORM

$$u'' + f = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = u_0$ ← prescribed

N: $u'(1) = t$ ←

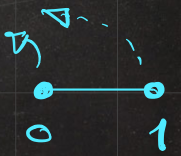
I) $[u'' + f = 0] \times \omega \Rightarrow \omega u'' + \omega f = 0$

II) $\int_0^1 [\omega u'' + \omega f]\, dx = 0$ $\qquad \omega u'' = (\omega u')' - \omega' u'$

↙ WEAK FORM

$$\underbrace{\int_0^1 \omega' u' \, dx}_{\text{INTERNAL CONTRIBUTIONS OVER THE DOMAIN}} = \underbrace{\int_0^1 \omega f \, dx}_{\text{EXTERNAL CONTRIBUTIONS OVER THE DOMAIN}} + \underbrace{\omega(1)\, u'(1) - \omega(0)\, u'(0)}_{\text{EXTERNAL CONTRIBUTIONS OVER THE BOUNDARY OF THE DOMAIN}}$$

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$$



FROM STRONG TO WEAK FORM

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = 0$

N: $u'(1) = 0$

← prescribed

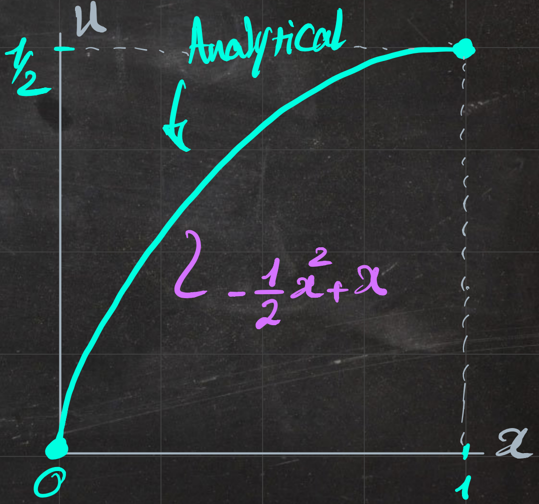
$$u'' = -1 \Rightarrow u' = -x + C_1$$

$$\Rightarrow u = -\frac{1}{2}x^2 + C_1 x + C_2$$

$\hookrightarrow u(0)=0 \Rightarrow C_2=0$

$\hookrightarrow u'(1)=0 \Rightarrow C_1 = 1$

Analytical Solution $\Rightarrow u = -\frac{1}{2}x^2 + x$



FROM STRONG TO WEAK FORM

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

$$\text{D:}\ u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$\text{N:}\ u'(1) = 0 \quad \swarrow$$

$$\int_0^1 w'u'\,dx = \int_0^1 wf\,dx + w(1)\,u'(1) - w(0)\,u'(0)$$

$$\Rightarrow \int_0^1 w'u'\,dx = \int_0^1 w\,dx \quad \leftarrow \text{WEAK FORM}$$

$$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$$

↳ Compute approximate solution ↳ from different spaces

⇓

EXERCISE ↦ ...

UNDERSTANDING THE SIGNIFICANCE OF WEAK FORM BY EXAMPLE

$$\int_0^1 w'u'dx = \int_0^1 w\,dx$$

$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$

D: $u(0) = 0$

N: $u'(1) = 0$

prescribed

$$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$$

→ 1-PIECE LINEAR APPROXIMATION

→ 2-PIECE LINEAR (UNIFORM) APPROXIMATION

→ 2-PIECE LINEAR (NON-UNIFORM) APPROXIMATION (I)

→ 2-PIECE LINEAR (NON-UNIFORM) APPROXIMATION (II)

→ 2-PIECE LINEAR (GENERAL) APPROXIMATION

UNDERSTANDING THE SIGNIFICANCE OF WEAK FORM BY EXAMPLE

$$\int_0^1 w' u' dx = \int_0^1 w \, dx$$

$$u'' + 1 = 0 \qquad 0 \leq x \leq 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$

$$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$$

- → 3-PIECE LINEAR (UNIFORM) APPROXIMATION
- → 3-PIECE LINEAR (GENERIC) APPROXIMATION
- → 4-PIECE LINEAR (UNIFORM) APPROXIMATION
- → 4-PIECE LINEAR (GENERIC) APPROXIMATION
- → 1-PIECE QUADRATIC
- → 1-PIECE CUBIC

$$\int_0^1 \omega' u' \, dx = \int_0^1 \omega \, dx \qquad x \in [0,1]$$

$u'' + 1 = 0 \qquad 0 \leq x \leq 1$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

→ 1- PIECE LINEAR APPROXIMATION

$\omega = C_1 x + C_2 \to C_2 = 0 \qquad u = D_1 x + D_2 \to D_2 = 0$

$\omega(0) = 0 \Uparrow \qquad \omega|_D = 0 \Leftarrow u(0)$ is GIVEN $\qquad u(0) = 0$

ω : ARBITRARY, CONTINUOUS, $\omega|_D = 0$

$$\int_0^1 C_1 D_1 \, dx = \int_0^1 C_1 x \, dx \Rightarrow C_1 D_1 x \Big]_0^1 = \frac{1}{2} C_1 x^2 \Big]_0^1$$

$$\Rightarrow D_1 = \frac{1}{2}$$

C_1 : Cancels out

$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \qquad x\in[0,1]$$

$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

→ 1_PIECE LINEAR APPROXIMATION

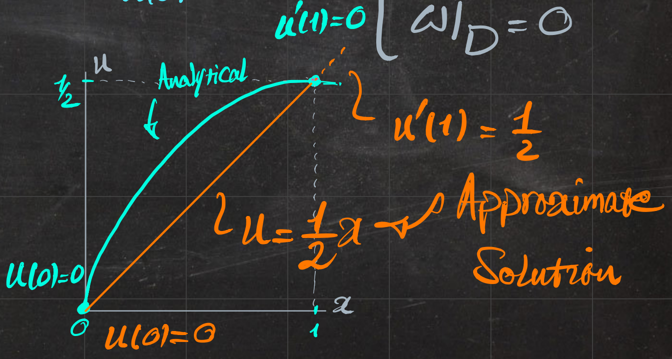
$w = C_1 x + C_2 \to C_2 = 0 \qquad u = D_1 x + D_2 \to D_2 = 0$

$w(0) = 0 \qquad w|_D = 0 \Leftarrow u(0)$ is GIVEN $\qquad u(0) = 0$

w : ARBITRARY, CONTINUOUS, $w|_D = 0$

C_1 : Cancels out $\Rightarrow D_1 = \frac{1}{2}$

APPROXIMATE SOLUTION FOR u $\Rightarrow u = \frac{1}{2}x$



$$\int_0^1 w'u'dx = \int_0^1 w\,dx \qquad x\in[0,1]$$

$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

→ 1-PIECE LINEAR APPROXIMATION

$w = C_1 x + C_2 \to C_2 = 0 \qquad u = D_1 x + D_2 \to D_2 = 0$

$w(0) = 0$ ⇑ $\qquad w|_D = 0 \Leftarrow u(0)$ is GIVEN $\qquad u(0) = 0$

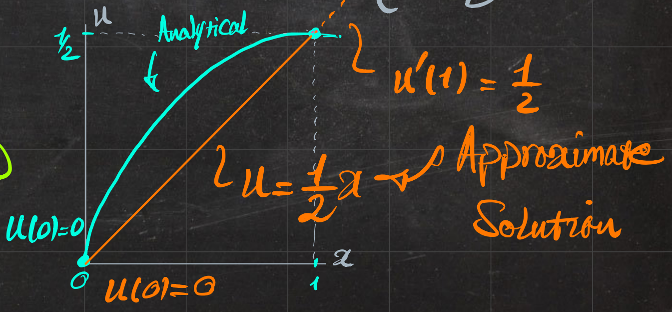
w : ARBITRARY, CONTINUOUS, $w|_D = 0$

$\Rightarrow u = \frac{1}{2}x$

DIRICHLET BCs ARE STRONGLY SATISFIED

NEUMANN BCs ARE WEAKLY SATISFIED

$u'(1)=0$



$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \qquad x\in[0,1]$$

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

$D:\ u(0) = 0$ ← prescribed

$N:\ u'(1) = 0$ ←

→ 2-PIECE LINEAR (UNIFORM) APPROXIMATION

$0 \leftarrow w|_D = 0$ $\qquad$ $0 \leftarrow u(0)=0$

$$x\in[0,0.5] \qquad w = C_1 x + \cancel{C_2} \qquad u = E_1 x + \cancel{E_2}$$

$$x\in[0.5,1] \qquad w = D_1 x + D_2 \qquad u = F_1 x + F_2$$

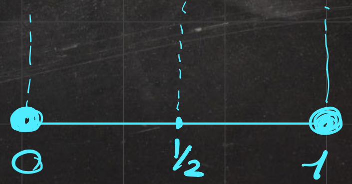
$w:$ ARBITRARY, CONTINUOUS, $w|_D = 0$

$$\Rightarrow \tfrac{1}{2}C_1 + C_2 = \tfrac{1}{2}D_1 + D_2 \qquad \Rightarrow \tfrac{1}{2}E_1 + E_2 = \tfrac{1}{2}F_1 + F_2$$

↳ Employ BCs and Continuity Conditions

↳ w Continuous @ 0.5

↳ u Continuous @ 0.5



$$\int_0^1 w'u'dx = \int_0^1 w\,dx \qquad x\in[0,1]$$

$u''+1=0 \qquad 0\leqslant x\leqslant 1$

D: $u(0)=0$ ← prescribed

N: $u'(1)=0$ ←

→ 2-PIECE LINEAR (UNIFORM) APPROXIMATION

$0 \leftarrow w|_D=0$ $\qquad$ $0 \leftarrow u(0)=0$

$$x\in[0,0.5] \qquad w=C_1x+\cancel{C_2} \qquad u=E_1x+\cancel{E_2}$$

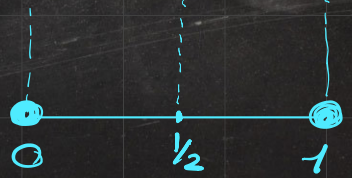
$$x\in[0.5,1] \qquad w=D_1x+D_2 \qquad u=F_1x+F_2$$

$$w:\begin{cases}\text{ARBITRARY}\\ \text{CONTINUOUS}\\ w|_D=0\end{cases}$$

$$\Rightarrow \tfrac{1}{2}C_1+C_2=\tfrac{1}{2}D_1+D_2 \qquad \Rightarrow \tfrac{1}{2}E_1+E_2=\tfrac{1}{2}F_1+F_2$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$D_2=\tfrac{1}{2}[C_1-D_1] \qquad F_2=\tfrac{1}{2}[E_1-F_1]$$



$$\int_0^1 w'u'dx = \int_0^1 w\,dx \qquad x\in[0,1]$$

$$u''+1=0 \qquad 0\leqslant x\leqslant 1$$

$D:\ u(0)=0$ ← prescribed

$N:\ u'(1)=0$ ←

→ 2-PIECE LINEAR (UNIFORM) APPROXIMATION

$0 \leftarrow w|_D=0$

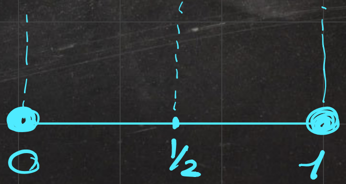
$x\in[0,0.5]$ $\quad w=C_1x+\cancel{C_2}$ $\quad u=E_1x+\cancel{E_2}$ $\quad 0 \leftarrow u(0)=0$

$x\in[0.5,1]$ $\quad w=D_1x+D_2$ $\quad u=F_1x+F_2$

$w:$ ARBITRARY, CONTINUOUS, $w|_D=0$

$D_2=\frac{1}{2}[C_1-D_1]$ $\qquad F_2=\frac{1}{2}[E_1-F_1]$

$$\int_0^{0.5} w'u'dx + \int_{0.5}^1 w'u'dx = \int_0^{0.5} w\,dx + \int_{0.5}^1 w\,dx$$



$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \qquad x \in [0,1]$$

$u'' + 1 = 0 \qquad 0 \leq x \leq 1$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

→ 2-PIECE LINEAR (UNIFORM) APPROXIMATION

$0 \leftarrow w|_D = 0$

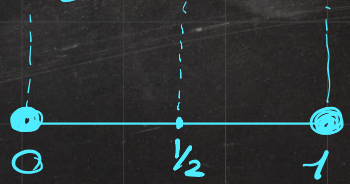
$x \in [0, 0.5] \qquad w = C_1 x + \cancel{C_2} \qquad u = E_1 x + \cancel{E_2}$ $0 \leftarrow u(0)=0$

$x \in [0.5, 1] \qquad w = D_1 x + D_2 \qquad u = F_1 x + F_2$

w : ARBITRARY
CONTINUOUS
$w|_D = 0$

$D_2 = \frac{1}{2}[C_1 - D_1] \qquad F_2 = \frac{1}{2}[E_1 - F_1]$

$$\int_0^{0.5} \overset{C_1}{w'}\underset{E_1}{u'}\,dx + \int_{0.5}^{1} \overset{D_1}{w'}\underset{F_1}{u'}\,dx = \int_0^{0.5} \underset{C_1 x}{w}\,dx + \int_{0.5}^{1} \overset{D_1 x + \frac{1}{2}[C_1 - D_1]}{w}\,dx$$



$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \qquad x\in[0,1]$$

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

→ 2-PIECE LINEAR (UNIFORM) APPROXIMATION

$$\int_0^{0.5} C_1 E_1\,dx + \int_{0.5}^{1} D_1 F_1\,dx$$

$$= \int_0^{0.5} C_1 x\,dx + \int_{0.5}^{1} \left[D_1 x + \tfrac{1}{2}[C_1 - D_1]\right] dx$$

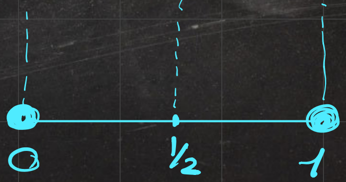
$$u = E_1 x + \cancel{E_2}^{\,0}$$

$$u = F_1 x + F_2$$

$$F_2 = \tfrac{1}{2}[E_1 - F_1]$$

w : { ARBITRARY, CONTINUOUS, $w|_D = 0$

$$\frac{1}{2} C_1 E_1 + \frac{1}{2} D_1 F_1 = \frac{1}{8} C_1 + \frac{3}{8} D_1 + \frac{1}{4}[C_1 - D_1]$$



$$\int_0^1 \omega' u' \, dx = \int_0^1 \omega \, dx \qquad x \in [0,1]$$

$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

→ 2-PIECE LINEAR (UNIFORM) APPROXIMATION

$$\frac{1}{2} C_1 E_1 + \frac{1}{2} D_1 F_1$$

$$= \frac{1}{8} C_1 + \frac{3}{8} D_1 + \frac{1}{4} [C_1 - D_1]$$

$u = E_1 x + \cancel{E_2}^{\,0}$

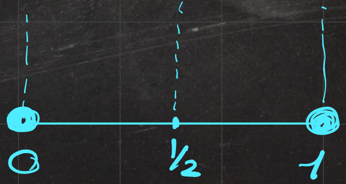
$u = F_1 x + F_2$

$F_2 = \frac{1}{2} [E_1 - F_1]$

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$$

THIS SEEMS LIKE 1 EQUATION

BUT IT IS NOT !



$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \qquad x\in[0,1]$$

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

→ 2-PIECE LINEAR (UNIFORM) APPROXIMATION

$$\frac{1}{2}C_1E_1 + \frac{1}{2}D_1F_1 \qquad \forall C_1, D_1$$

$$= \frac{1}{8}C_1 + \frac{3}{8}D_1 + \frac{1}{4}[C_1 - D_1]$$

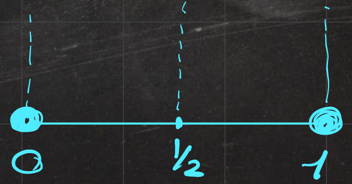
$$u = E_1x + \cancel{E_2}^{\,0}$$

$$u = F_1x + F_2$$

$$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$$

$$F_2 = \frac{1}{2}[E_1 - F_1]$$

$$\begin{matrix} C_1 = 1,\ D_1 = 0 \Rightarrow \\ C_1 = 0,\ D_1 = 1 \Rightarrow \end{matrix} \begin{cases} \frac{1}{2}E_1 - \frac{3}{8} = 0 \\ \frac{1}{2}F_1 - \frac{1}{8} = 0 \end{cases} \Rightarrow \begin{matrix} E_1 = 3/4 \\ F_1 = 1/4 \end{matrix}$$



$$\int_0^1 \omega' u' \, dx = \int_0^1 \omega \, dx \qquad x \in [0,1]$$

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

→ 2-PIECE LINEAR (UNIFORM) APPROXIMATION

$$\frac{1}{2} C_1 E_1 + \frac{1}{2} D_1 F_1 \qquad \forall C_1, D_1$$

$$= \frac{1}{8} C_1 + \frac{3}{8} D_1 + \frac{1}{4}[C_1 - D_1]$$

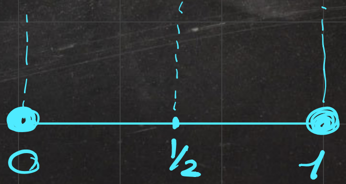
$$u = E_1 x + \cancel{E_2}^{\,0}$$

$$u = F_1 x + F_2$$

$$F_2 = \frac{1}{2}[E_1 - F_1]$$

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$$

ω : ARBITRARY $\Rightarrow$ C_1 & D_1 : ARBITRARY



$$\int_0^1 \omega' u' \, dx = \int_0^1 \omega \, dx \qquad x \in [0,1]$$

$u'' + 1 = 0 \qquad 0 \leq x \leq 1$

D: $u(0) = 0$ ← prescribed

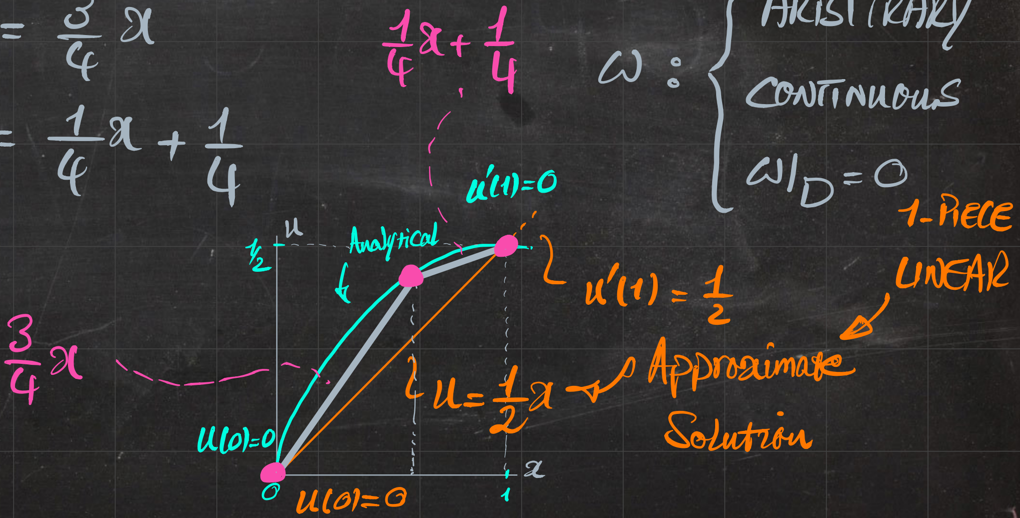
N: $u'(1) = 0$

→ 2-PIECE LINEAR (UNIFORM) APPROXIMATION

$x \in [0, 0.5] \qquad u = \frac{3}{4}x$

$x \in [0.5, 1] \qquad u = \frac{1}{4}x + \frac{1}{4}$

ω : ARBITRARY, CONTINUOUS, $\omega|_D = 0$



$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \qquad x\in[0,1]$$

$$u''+1=0 \qquad 0\leqslant x\leqslant 1$$

$D:\ u(0)=0$ ← prescribed

$N:\ u'(1)=0$ ←

→ 2-PIECE LINEAR (NON-UNIFORM) APPROXIMATION

$$x\in[0,0.6] \qquad w=C_1x+\overset{0}{\cancel{C_2}} \qquad u=E_1x+\overset{0}{\cancel{E_2}}$$

$$x\in[0.6,1] \qquad w=D_1x+\underset{0.6[C_1-D_1]}{\cancel{D_2}} \qquad u=F_1x+\underset{0.6[E_1-F_1]}{\cancel{F_2}}$$

$$w:\begin{cases}\text{ARBITRARY}\\ \text{CONTINUOUS}\\ w|_D=0\end{cases}$$

$$\int_0^{0.6}\underbrace{w'}_{C_1}\underbrace{u'}_{E_1}dx+\int_{0.6}^{1}\underbrace{w'}_{D_1}\underbrace{u'}_{F_1}dx=\int_0^{0.6}\underbrace{w}_{C_1x}dx+\int_{0.6}^{1}\underbrace{w}_{D_1x+0.6[C_1-D_1]}dx$$

$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \qquad x \in [0,1]$$

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

→ 2-PIECE LINEAR (NON-UNIFORM) APPROXIMATION

$$x \in [0, 0.6] \qquad w = C_1 x + \overset{0}{\cancel{C_2}} \qquad u = E_1 x + \overset{0}{\cancel{E_2}}$$

$$x \in [0.6, 1] \qquad w = D_1 x + D_2 \qquad u = F_1 x + F_2$$

$$D_2 = 0.6[C_1 - D_1] \qquad F_2 = 0.6[E_1 - F_1]$$

$$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$$

$$\int_0^{0.6} C_1 E_1\,dx + \int_{0.6}^{1} D_1 F_1\,dx = \int_0^{0.6} C_1 x\,dx + \int_{0.6}^{1} [D_1 x + 0.6[C_1 - D_1]]\,dx$$

$\dots$

$$C_1\,[0.6 E_1 - 0.42] + D_1\,[0.4 F_1 - 0.08] = 0$$

$$\int_0^1 w'u'dx = \int_0^1 w\,dx \qquad x\in[0,1]$$

$\rightarrow$ 2-PIECE LINEAR (NON-UNIFORM) APPROXIMATION

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ↙

$$x\in[0,0.6] \qquad u = E_1 x + \overset{0}{\cancel{E_2}}$$

$$x\in[0.6,1] \qquad u = F_1 x + F_2$$

$$0.6[E_1 - F_1]$$

$$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$$

$$C_1[0.6E_1 - 0.42] \qquad \checkmark\, C_1, D_1$$

$$+ D_1[0.4F_1 - 0.08] = 0$$

$$\Rightarrow E_1 = 0.7,\; F_1 = 0.2 \qquad \Rightarrow \begin{cases} u = 0.7x & 0 \leqslant x \leqslant 0.6 \\ u = 0.2x + 0.3 & 0.6 \leqslant x \leqslant 1 \end{cases}$$

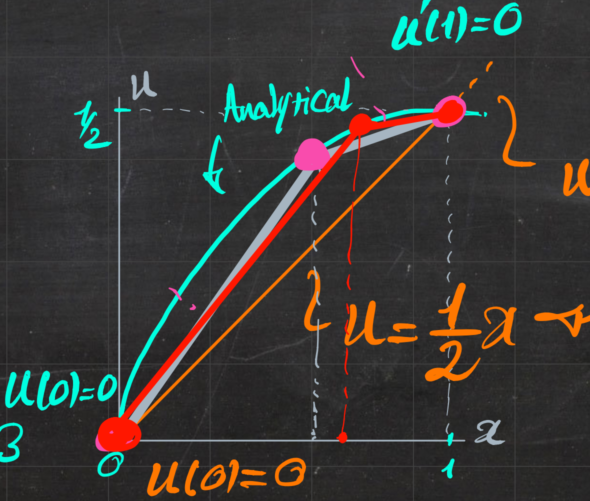
$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \qquad x\in[0,1]$$

→ 2-PIECE LINEAR (NON-UNIFORM) APPROXIMATION

$u''+1=0 \qquad 0\leqslant x\leqslant 1$

D: $u(0)=0$ ← prescribed

N: $u'(1)=0$ ←



1-PIECE LINEAR

Approximate Solution

$$\begin{cases} u = 0.7\,x \\ u = \underline{0.2x} + 0.3 \end{cases}$$

$$\int_0^1 w'u'dx = \int_0^1 w\,dx \qquad x \in [0,1]$$

$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$

D: $u(0) = 0$ $\leftarrow$ prescribed

N: $u'(1) = 0$ $\leftarrow$

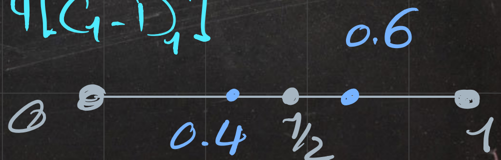
→ 2-PIECE LINEAR (NON-UNIFORM) APPROXIMATION

$x \in [0, 0.4] \qquad w = C_1 x + \overset{\overset{0}{\uparrow}}{C_2} \qquad u = E_1 x + \overset{\overset{0}{\uparrow}}{E_2}$

$x \in [0.4, 1] \qquad w = D_1 x + \underset{\underset{0.4[C_1 - D_1]}{\downarrow}}{D_2} \qquad u = F_1 x + \underset{\underset{0.4[E_1 - F_1]}{\downarrow}}{F_2}$

$$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$$

$$\int_0^{0.4} \underset{C_1}{w'}\,\underset{E_1}{u'}\,dx + \int_{0.4}^1 \underset{D_1}{w'}\,\underset{F_1}{u'}\,dx = \int_0^{0.4} \underset{C_1 x}{w}\,dx + \int_{0.4}^1 \underset{D_1 x + 0.4[C_1 - D_1]}{w}\,dx$$



$$\int_0^1 \omega' u' \, dx = \int_0^1 \omega \, dx \qquad x \in [0,1]$$

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

→ 2-PIECE LINEAR (NON-UNIFORM) APPROXIMATION

$$x \in [0, 0.4] \qquad \omega = C_1 x + \overset{0}{\cancel{C_2}} \qquad u = E_1 x + \overset{0}{\cancel{E_2}}$$

$$x \in [0.4, 1] \qquad \omega = D_1 x + \underset{0.4[C_1 - D_1]}{\cancel{D_2}} \qquad u = F_1 x + \underset{0.4[E_1 - F_1]}{\cancel{F_2}}$$

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$$

$$\int_0^{0.4} C_1 E_1 \, dx + \int_{0.4}^1 D_1 F_1 \, dx = \int_0^{0.4} C_1 x \, dx + \int_{0.4}^1 [D_1 x + 0.4[C_1 - D_1]] \, dx$$

$\circ\circ\circ$

$$C_1 [0.4 E_1 - 0.32] + D_1 [0.6 F_1 - 0.18] = 0$$

$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \qquad x \in [0,1]$$

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ↙

→ 2-PIECE LINEAR (NON-UNIFORM) APPROXIMATION

$$x \in [0, 0.4] \qquad u = E_1 x + \overset{0}{\cancel{E_2}}$$

$$x \in [0.4, 1] \qquad u = F_1 x + F_2$$

$$0.4\,[E_1 - F_1]$$

$$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$$

$$C_1\,[0.4E_1 - 0.32] \qquad \checkmark\, C_1, D_1$$

$$+\, D_1\,[0.6F_1 - 0.18] = 0$$

$$\Rightarrow \; E_1 = 0.8, \quad F_1 = 0.3 \qquad \Rightarrow \begin{cases} u = 0.8x & 0 \leqslant x \leqslant 0.4 \\ u = 0.3x + 0.2 & 0.4 \leqslant x \leqslant 1 \end{cases}$$

$$\int_0^1 w'u'dx = \int_0^1 w\,dx \qquad x\in[0,1]$$

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

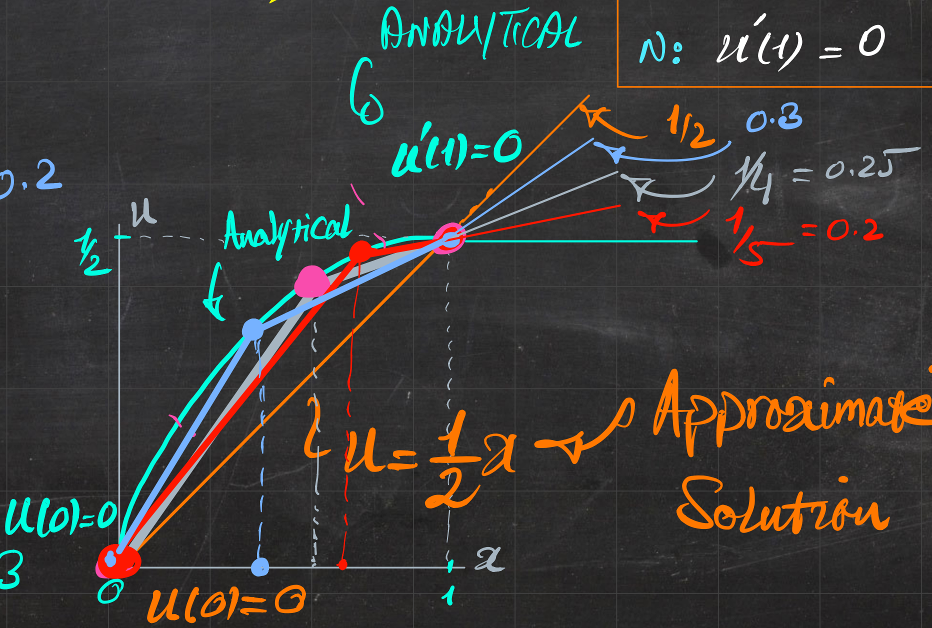
D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

→ 2-PIECE LINEAR (NON-UNIFORM) APPROXIMATION

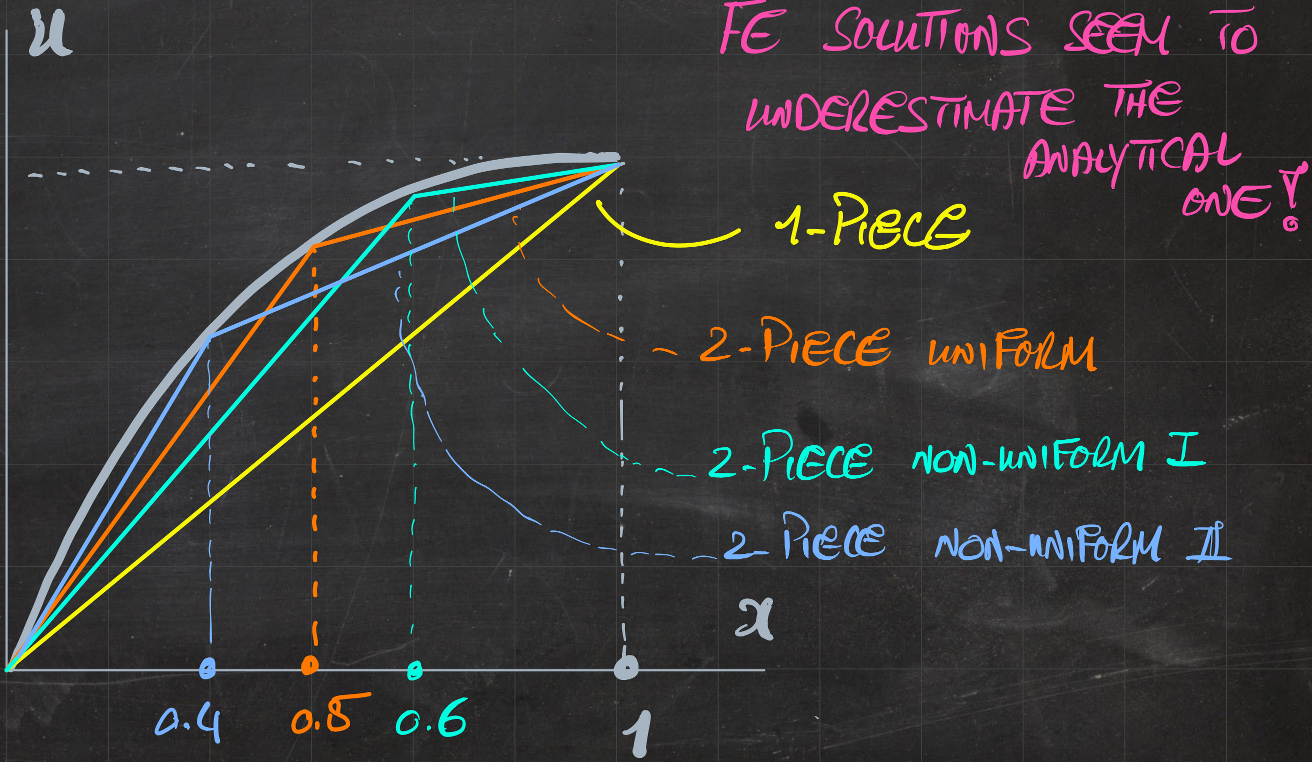
$$\begin{cases} u = 0.8x \\ u = 0.3x + 0.2 \end{cases}$$

$$\begin{cases} u = 0.7x \\ u = \underline{0.2x} + 0.3 \end{cases}$$



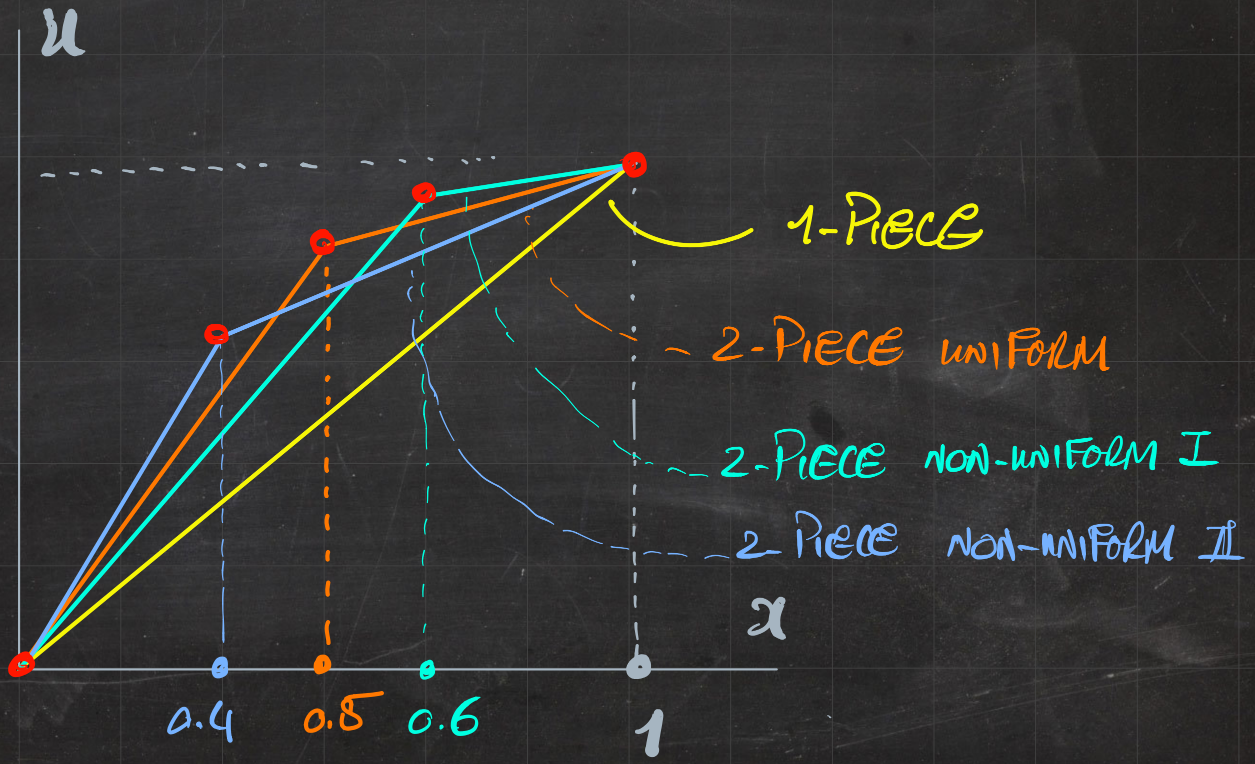
1-PIECE LINEAR

Approximate Solution



FE SOLUTIONS SEEM TO UNDERESTIMATE THE ANALYTICAL ONE!

FE Solution approaches analytical one from below!


u
1-PIECE
2-PIECE UNIFORM
2-PIECE NON-UNIFORM I
2-PIECE NON-UNIFORM II
x
0.4
0.5
0.6
1

$$\int_0^1 \omega' u' dx = \int_0^1 \omega \, dx \qquad x \in [0,1]$$

$u'' + 1 = 0 \qquad 0 \leq x \leq 1$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

→ 2-PIECE LINEAR (GENERIC) APPROXIMATION

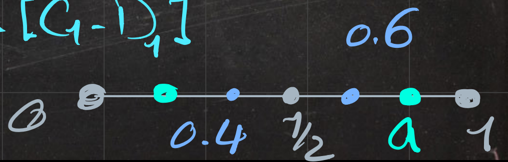
$x \in [0, a] \qquad \omega = C_1 x + \overset{0}{C_2} \qquad u = E_1 x + \overset{0}{E_2}$

$x \in [a, 1] \qquad \omega = D_1 x + \underset{a[C_1 - D_1]}{D_2} \qquad u = F_1 x + \underset{a[E_1 - F_1]}{F_2}$

$\omega :$ ARBITRARY, CONTINUOUS, $\omega|_D = 0$

$$\int_0^a \underset{C_1}{\omega'}\,\underset{E_1}{u'}\,dx + \int_a^1 \underset{D_1}{\omega'}\,\underset{F_1}{u'}\,dx = \int_0^a \underset{C_1 x}{\omega}\,dx + \int_a^1 \underset{D_1 x + a[C_1 - D_1]}{\omega}\,dx$$

$0 \leq a \leq 1$



$$\int_0^1 \omega' u' \, dx = \int_0^1 \omega \, dx \qquad x \in [0,1]$$

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

→ 2-PIECE LINEAR (GENERIC) APPROXIMATION

$$x \in [0, a] \qquad \omega = C_1 x + \overset{0}{\cancel{C_2}} \qquad u = E_1 x + \overset{0}{\cancel{E_2}}$$

$$x \in [a, 1] \qquad \omega = D_1 x + \underset{a[C_1 - D_1]}{\cancel{D_2}} \qquad u = F_1 x + \underset{a[E_1 - F_1]}{\cancel{F_2}}$$

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$$

$$a_1 C_1 E_1 + [1-a] D_1 F_1 = \circ\circ\circ + \circ\circ\circ$$

$$= a C_1 \left[1 - \tfrac{1}{2} a\right] + D_1 [1-a] \left[\tfrac{1}{2} - \tfrac{1}{2} a\right]$$

$$\int_0^1 w' u' \, dx = \int_0^1 w \, dx \qquad x \in [0,1]$$

$u'' + 1 = 0 \qquad 0 \leq x \leq 1$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

→ 2-PIECE LINEAR (GENERIC) APPROXIMATION

$x \in [0, a] \qquad w = C_1 x + \overset{0}{C_2} \qquad u = E_1 x + \overset{0}{E_2}$

$x \in [a, 1] \qquad w = D_1 x + D_2 \qquad u = F_1 x + F_2$

$a[C_1 - D_1] \qquad a[E_1 - F_1]$

$$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$$

$$a_1 C_1 E_1 + [1-a] D_1 F_1$$

$$= a C_1 \left[1 - \tfrac{1}{2} a\right] + D_1 [1-a]\left[\tfrac{1}{2} - \tfrac{1}{2} a\right] \qquad \checkmark \; C_1, D_1$$

$$\Rightarrow E_1 = 1 - \tfrac{1}{2} a \;, \quad F_1 = \tfrac{1}{2} - \tfrac{1}{2} a$$

$$\int_0^1 \omega' u' \, dx = \int_0^1 \omega \, dx \qquad x \in [0,1]$$

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

$D: \; u(0) = 0$ ← prescribed

$N: \; u'(1) = 0$ ←

→ 2-PIECE LINEAR (GENERIC) APPROXIMATION

$$x \in [0, a] \Rightarrow u = [1 - \tfrac{1}{2} a] \, x$$

$$x \in [a, 1] \Rightarrow u = [\tfrac{1}{2} - \tfrac{1}{2} a] \, x + \tfrac{1}{2} a$$

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$$

$a = 0.5$

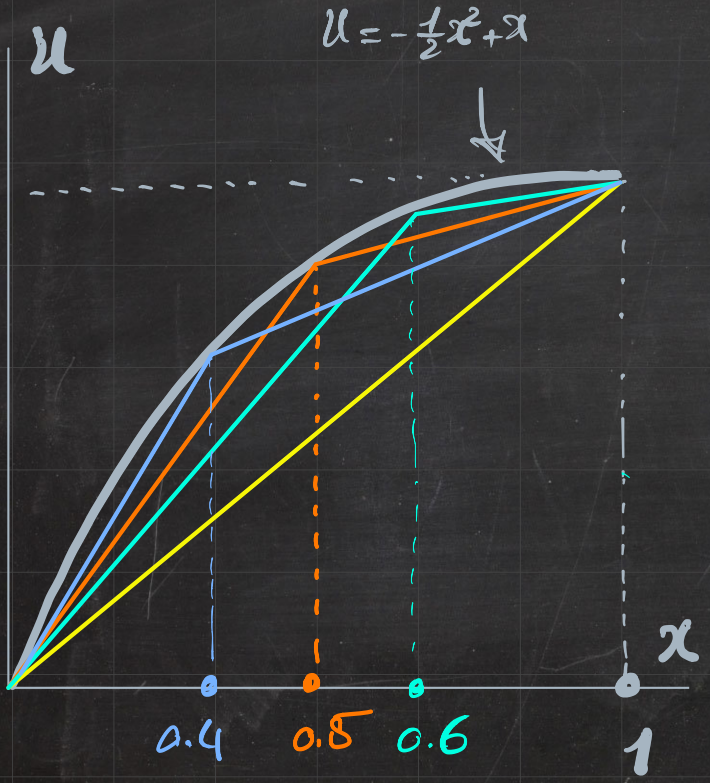
$$\begin{cases} u = 0.75x \\ u = 0.25x + 0.25 \end{cases}$$

$a = 0.6$

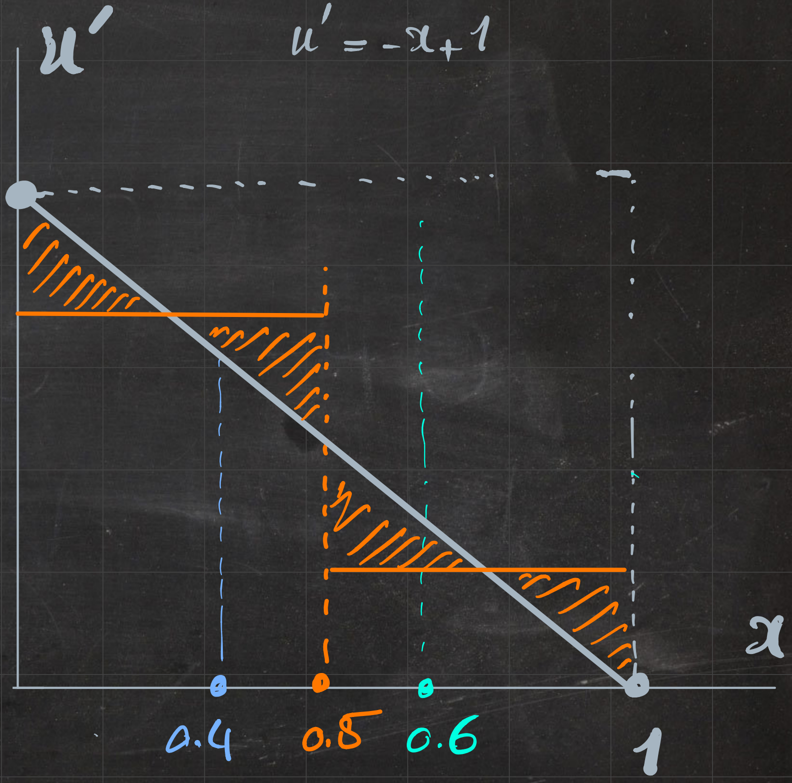
$$\begin{cases} u = 0.7x \\ u = 0.2x + 0.3 \end{cases}$$

$a = 0.4$

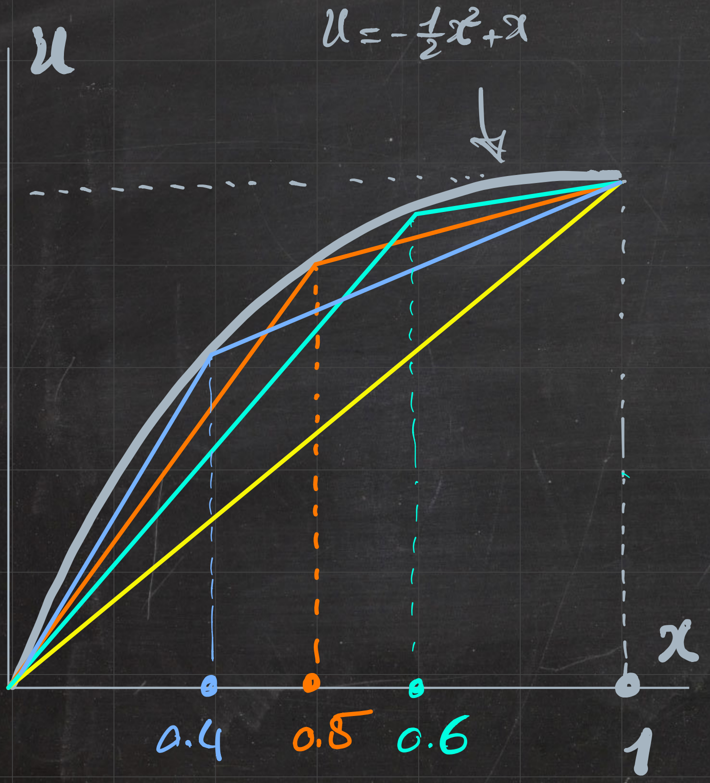
$$\begin{cases} u = 0.8x \\ u = 0.3x + 2 \end{cases}$$

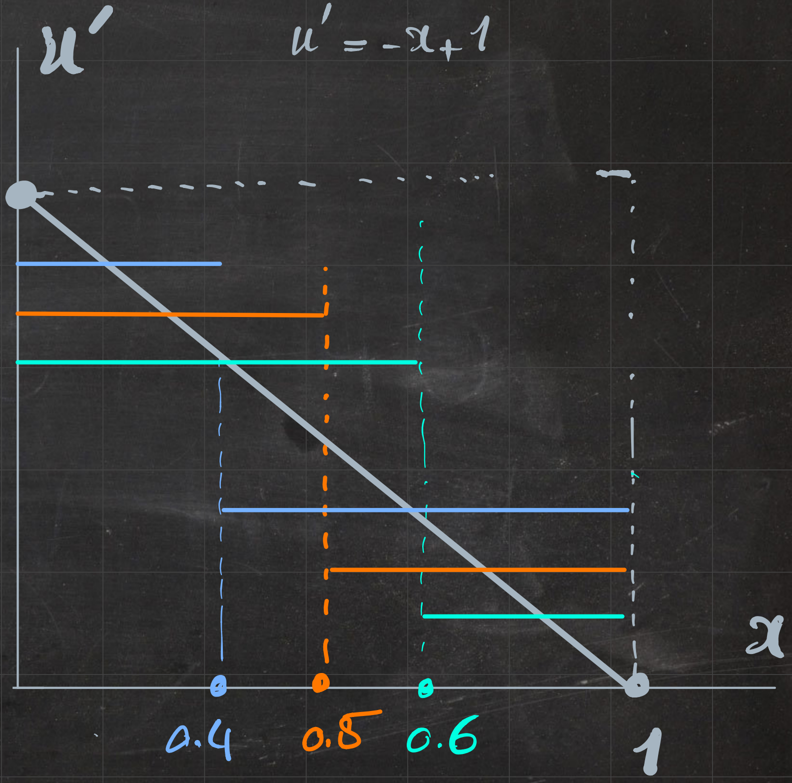


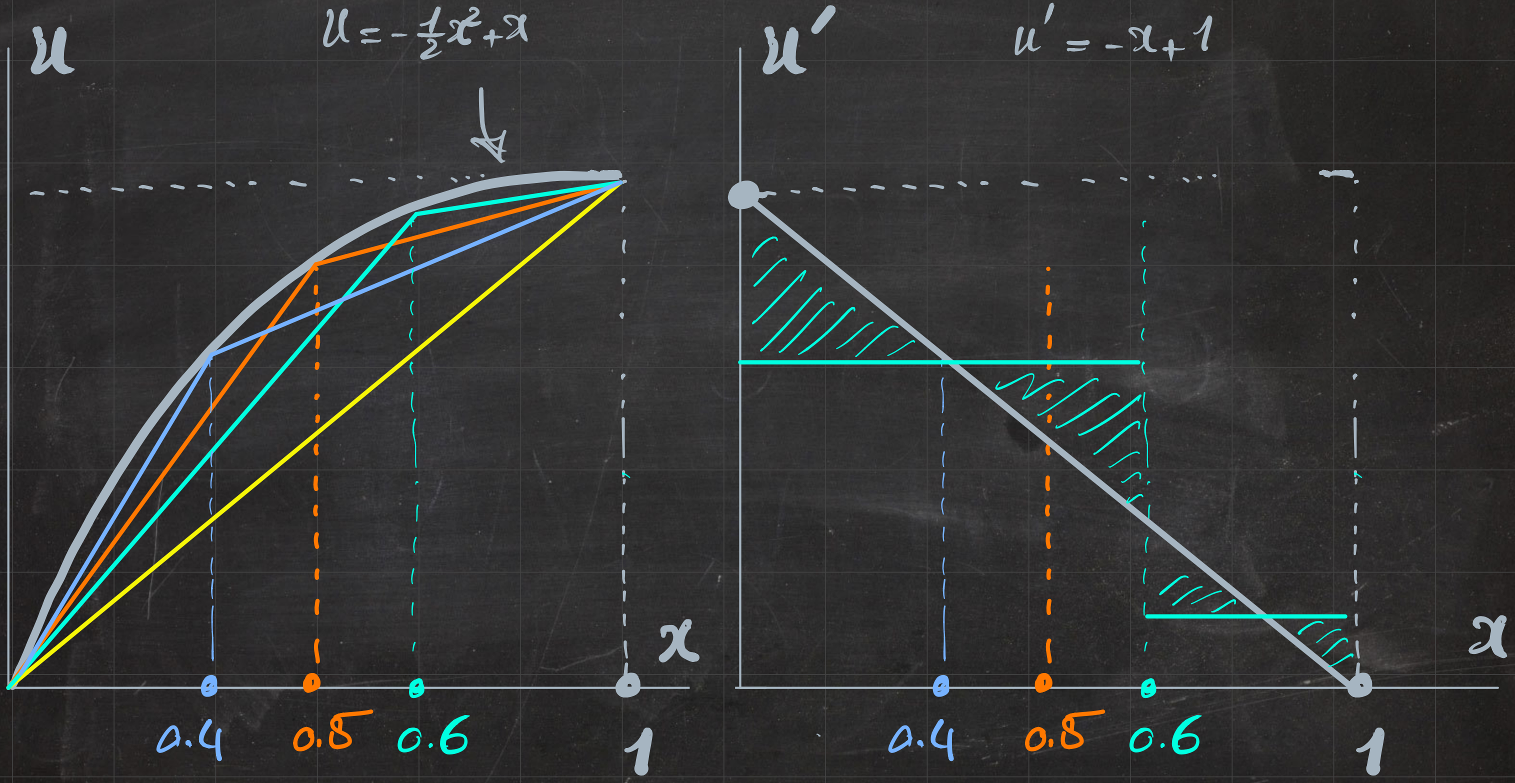
u
$u = -\frac{1}{2}x^2 + x$
x
0.4
0.5
0.6
1



u'
$u' = -x + 1$
x
0.4
0.5
0.6
1


u
$u = -\frac{1}{2}x^2 + x$
x
0.4
0.5
0.6
1


u'
$u' = -x + 1$
x
0.4
0.5
0.6
1


u
$u = -\frac{1}{2}x^2 + x$
x
0.4
0.5
0.6
1
u'
$u' = -x + 1$
x
0.4
0.5
0.6
1

$$\int_0^1 w'u'dx = \int_0^1 w\,dx \qquad x\in[0,1]$$

→ 2-PIECE LINEAR (GENERIC) APPROXIMATION

$$u''+1=0 \qquad 0\leqslant x\leqslant 1$$

D: $u(0)=0$ ← prescribed

N: $u'(1)=0$ ↙

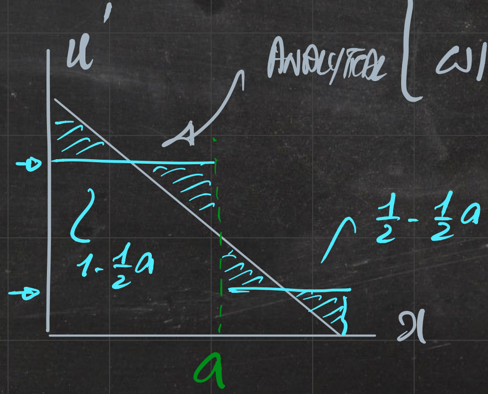
$$x\in[0,a] \Rightarrow u=\left[1-\tfrac{1}{2}a\right]x$$

$$x\in[a,1] \Rightarrow u=\left[\tfrac{1}{2}-\tfrac{1}{2}a\right]x+\tfrac{1}{2}a$$

$$w:\begin{cases}\text{ARBITRARY}\\ \text{CONTINUOUS}\\ w|_D=0\end{cases}$$

$$x\in[0,a] \Rightarrow u'=\left[1-\tfrac{1}{2}a\right]$$

$$x\in[a,1] \Rightarrow u'=\left[\tfrac{1}{2}-\tfrac{1}{2}a\right]$$



$$\int_0^1 w'u'dx = \int_0^1 w\,dx \qquad x\in[0,1]$$

$\rightarrow$ 3_PIECE LINEAR (GENERIC) APPROXIMATION

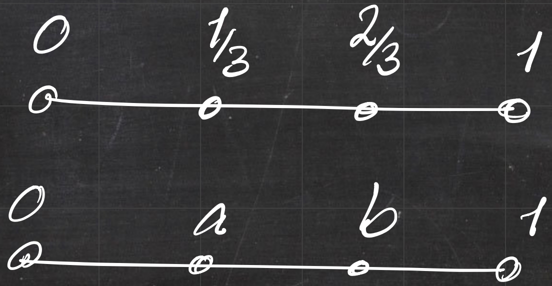
$u'' + 1 = 0 \qquad 0 \leq x \leq 1$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

w : ARBITRARY, CONTINUOUS, $w|_D = 0$

ANALYTICAL



$0 \leq a < b \qquad a < b \leq 1$

$$\int_0^1 \omega' u' \, dx = \int_0^1 \omega \, dx \qquad x \in [0,1]$$

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

→ 3_PIECE LINEAR (GENERIC) APPROXIMATION

$x \in [0,a] \qquad \omega = C_1 x + \overset{0}{\cancel{C_2}} \qquad u = F_1 x + \overset{0}{\cancel{F_2}}$

$a[C_1 - D_1] \qquad a[F_1 - G_1]$

$x \in [a,b] \qquad \omega = D_1 x + \cancel{D_2} \qquad u = G_1 x + \cancel{G_2}$

$x \in [b,1] \qquad \omega = E_1 x + \cancel{E_2} \qquad u = H_1 x + \cancel{H_2}$

$\omega :$ ANALYTICAL $\begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$

$b[D_1 - E_1] + a[C_1 - D_1] \qquad b[G_1 - H_1] + a[F_1 - G_1]$

$\rightarrow \forall \, C_1, D_1, E_1 \qquad \rightarrow F_1, G_1, H_1 = ?$

$$\int_0^1 \omega' u' \, dx = \int_0^1 \omega \, dx \qquad x \in [0,1]$$

$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

→ 3_PIECE LINEAR (GENERIC) APPROXIMATION

$x \in [0,a] \qquad \omega = C_1 x + \overset{0}{\cancel{C_2}} \qquad u = F_1 x + \overset{0}{\cancel{F_2}}$

$x \in [a,b] \qquad \omega = D_1 x + \overset{a[C_1 - D_1]}{\cancel{D_2}} \qquad u = G_1 x + \overset{a[F_1 - G_1]}{\cancel{G_2}}$

$x \in [b,1] \qquad \omega = E_1 x + \underset{b[D_1 - E_1] + a[C_1 - D_1]}{\cancel{E_2}} \qquad u = H_1 x + \underset{b[G_1 - H_1] + a[F_1 - G_1]}{\cancel{H_2}}$

$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$ ANALYTICAL

$$\int_0^a \underbrace{\omega'}_{C_1}\underbrace{u'}_{F_1} dx + \int_a^b \underbrace{\omega'}_{D_1}\underbrace{u'}_{G_1} dx + \int_b^1 \underbrace{\omega'}_{E_1}\underbrace{u'}_{H_1} dx = \int_0^a \underbrace{\omega}_{C_1 x} dx + \int_a^b \underbrace{\omega}_{D_1 x + a[C_1 - D_1]} dx + \int_b^1 \overbrace{\omega}^{E_1 x + b[D_1 - E_1] + a[C_1 - D_1]} dx$$

$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \qquad x\in[0,1]$$

$u''+1=0 \qquad 0\leqslant x\leqslant 1$

D: $u(0)=0$ ← prescribed

N: $u'(1)=0$ ←

→ 3_PIECE LINEAR (GENERIC) APPROXIMATION

$x\in[0,a] \qquad w = C_1x+\cancelto{0}{C_2} \qquad u = F_1x+\cancelto{0}{F_2}$

$a[C_1-D_1] \qquad a[F_1-G_1]$

$x\in[a,b] \qquad w = D_1x+\cancel{D_2} \qquad u = G_1x+\cancel{G_2}$

$x\in[b,1] \qquad w = E_1x+\cancel{E_2} \qquad u = H_1x+\cancel{H_2}$

$b[D_1-E_1]+a[C_1-D_1] \qquad b[G_1-H_1]+a[F_1-G_1]$

$w:\begin{cases}\text{ARBITRARY}\\ \text{CONTINUOUS}\\ w|_D=0\end{cases}$

ANALYTICAL

$$\int_0^a \underset{C_1}{w'}\,\underset{F_1}{u'}\,dx+\int_a^b \underset{D_1}{w'}\,\underset{G_1}{u'}\,dx+\int_b^1 \underset{E_1}{w'}\,\underset{H_1}{u'}\,dx=\int_0^a \underset{C_1x}{w}\,dx+\int_a^b \underset{D_1x+a[C_1-D_1]}{w}\,dx+\int_b^1 \overset{E_1x+b[D_1-E_1]+a[C_1-D_1]}{w}\,dx$$

$$C_1[000]+D_1[000]+E_1[000]=0 \qquad \surd\, C_1, D_1, E_1$$

$$\int_0^1 \omega' u' dx = \int_0^1 \omega \, dx \qquad x \in [0,1]$$

→ 3_PIECE LINEAR (GENERIC) APPROXIMATION

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

$D: \; u(0) = 0$ ← prescribed

$N: \; u'(1) = 0$ ←

$$C_1 \underbrace{[\circ\circ\circ]}_{\omega} + D_1 \underbrace{[\circ\circ\circ]}_{\omega} + E_1 \underbrace{[\circ\circ\circ]}_{\omega} = 0 \qquad \checkmark C_1, D_1, E_1$$

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$$

ANALYTICAL

$D_1 = 0$, $E_1 = 0$, $C_1 = 1$ → $f(F_1) = 0$

$D_1 = 1$, $E_1 = 0$, $C_1 = 0$ → $g(G_1) = 0$

→ $h(H_1) = 0$

$$\Rightarrow F_1, G_1, H_1 \; \checkmark$$

$$\Downarrow$$

$$F_1 = \circ\circ\circ, \quad G_1 = \circ\circ\circ, \quad H_1 = \circ\circ\circ$$

$$\int_0^1 w'u'dx = \int_0^1 w\,dx \qquad x\in[0,1]$$

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

→ 3_PIECE LINEAR (GENERIC) APPROXIMATION

$x \in [0,a] \qquad w = C_1x + \overset{0}{\cancel{C_2}} \qquad u = F_1x + \overset{0}{\cancel{F_2}}$

$a[C_1 - D_1] \qquad a[F_1 - G_1]$

$x \in [a,b] \qquad w = D_1x + \cancel{D_2} \qquad u = G_1x + \cancel{G_2}$

$x \in [b,1] \qquad w = E_1x + \cancel{E_2} \qquad u = H_1x + \cancel{H_2}$

$b[D_1 - E_1] + a[C_1 - D_1] \qquad b[G_1 - H_1] + a[F_1 - G_1]$

w : ARBITRARY

CONTINUOUS

ANALYTICAL $w|_D = 0$

$$C_1[\circ\circ\circ] + D_1[\circ\circ\circ] + E_1[\circ\circ\circ] = 0 \qquad \forall C_1, D_1, E_1$$

$$\Rightarrow F_1 = 1 - \tfrac{1}{2}a \qquad G_1 = 1 - \tfrac{1}{2}[a+b] \qquad H_1 = 1 - \tfrac{1}{2}[b+1]$$

$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \qquad x\in[0,1]$$

$$u''+1=0 \qquad 0\leqslant x\leqslant 1$$

D: $u(0)=0$ ← prescribed

N: $u'(1)=0$ ←

→ 3_PIECE LINEAR (GENERIC) APPROXIMATION

$x\in[0,a]$ $\quad u = F_1x+\cancel{F_2} \rightarrow 0$

$x\in[a,b]$ $\quad u = G_1x+\cancel{G_2} \rightarrow a[F_1-G_1]$

$x\in[b,1]$ $\quad u = H_1x+\cancel{H_2} \rightarrow b[G_1-H_1]+a[F_1-G_1]$

$w:$ ANALYTICAL $\begin{cases}\text{ARBITRARY}\\ \text{CONTINUOUS}\\ w|_D=0\end{cases}$

$$\Rightarrow F_1 = 1-\tfrac{1}{2}a \qquad G_1 = 1-\tfrac{1}{2}[a+b] \qquad H_1 = 1-\tfrac{1}{2}[b+1]$$

$$\int_0^1 \omega' u' dx = \int_0^1 \omega \, dx \qquad x \in [0,1]$$

→ 3 _ PIECE LINEAR (GENERIC) APPROXIMATION

$$\begin{cases} u = [1 - \frac{1}{2} a] x & x \in [0,a] \\ u = [1 - \frac{1}{2}[a+b]] x + \frac{1}{2} ab & x \in [a,b] \\ u = [1 - \frac{1}{2}[b+1]] x + \frac{1}{2} b & x \in [b,1] \end{cases}$$

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$$

ANALYTICAL

$$\hookrightarrow \; u = \left[1 - \frac{1}{2} [\alpha, \beta] \right] x + \frac{1}{2} \alpha \beta \quad \leftarrow \quad 0 \leqslant \alpha < \beta \leqslant 1$$

$\{\alpha, \beta\} \to \{0, a\}$ $\{\alpha, \beta\} \to \{a, b\}$ $\{\alpha, \beta\} \to \{b, 1\}$

$$\int_0^1 \omega' u' dx = \int_0^1 \omega \, dx \qquad x \in [0,1]$$

$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

→ 3 _ PIECE LINEAR (GENERIC) APPROXIMATION

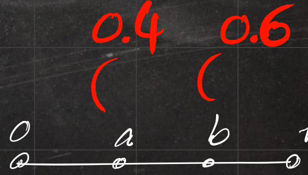
$$\begin{cases} u = [1 - \frac{1}{2}a]x & x \in [0,a] \\ u = [1 - \frac{1}{2}[a+b]]x + \frac{1}{2}ab & x \in [a,b] \\ u = [1 - \frac{1}{2}[b+1]]x + \frac{1}{2}b & x \in [b,1] \end{cases}$$

$\omega :$ ARBITRARY

CONTINUOUS

ANALYTICAL $\omega|_D = 0$

$$\begin{cases} u = 0.8x & x \in [0,0.4] \\ u = 0.5x + 0.12 & x \in [0.4,0.6] \\ u = 0.2x + 0.3 & x \in [0.6,1] \end{cases}$$



$$\int_0^1 w'u'dx = \int_0^1 w\,dx \qquad x\in[0,1]$$

$u''+1=0 \qquad 0\leqslant x\leqslant 1$

D: $u(0)=0$ ← prescribed

N: $u'(1)=0$ ←

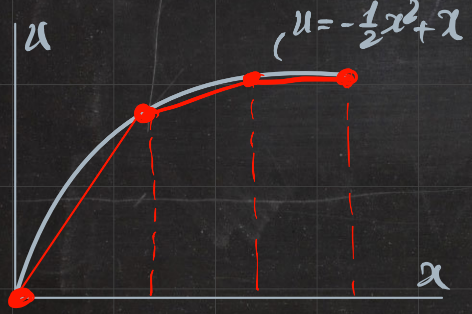
→ 3_PIECE LINEAR (GENERIC) APPROXIMATION

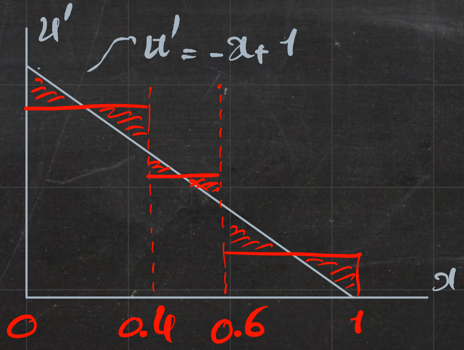
$$\begin{cases} u=0.8x & x\in[0,0.4] \\ u=0.5x+0.12 & x\in[0.4,0.6] \\ u=0.2x+0.3 & x\in[0.6,1] \end{cases}$$

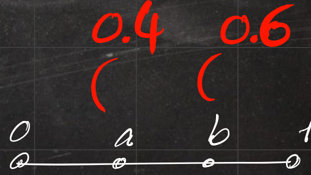
w : { ARBITRARY

CONTINUOUS

ANALYTICAL $w|_D=0$







$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \qquad x \in [0,1]$$

→ 4-PIECE LINEAR (GENERIC) APPROXIMATION

$$u'' + 1 = 0 \qquad 0 \leq x \leq 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$

$x \in [0,a]$ $\quad w = C_1 x + C_2 \qquad u = G_1 x + G_2$

$x \in [a,b]$ $\quad w = D_1 x + D_2 \qquad u = H_1 x + H_2$

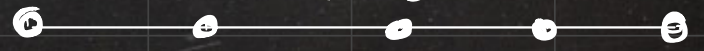
$x \in [b,c]$ $\quad w = E_1 x + E_2 \qquad u = I_1 x + I_2$

$x \in [c,1]$ $\quad w = F_1 x + F_2 \qquad u = J_1 x + J_2$

$$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$$

ANALYTICAL

$0 < a < b < c < 1$



$$\int_0^1 w'u'dx = \int_0^1 w\,dx \qquad x\in[0,1]$$

$$u''+1=0 \qquad 0\leqslant x\leqslant 1$$

D: $u(0)=0$ ← prescribed

N: $u'(1)=0$ ←

→ 4-PIECE LINEAR (GENERIC) APPROXIMATION

$x\in[0,a] \quad w = C_1x+\overset{0}{\cancel{C_2}} \qquad u = G_1x+\overset{0}{\cancel{G_2}}$

$x\in[a,b] \quad w = D_1x+D_2 \qquad u = H_1x+H_2$

$x\in[b,c] \quad w = E_1x+E_2 \qquad u = I_1x+I_2$

$x\in[c,1] \quad w = F_1x+F_2 \qquad u = J_1x+J_2$

$$H_2 = a\,[G_1-H_1]$$

$$I_2 = b[H_1-I_1]+a[G_1-H_1]$$

$$J_2 = c[I_1-J_1] + b[H_1-I_1]+a[G_1-H_1]$$

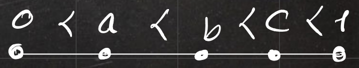
$$w : \begin{cases}\text{ARBITRARY}\\ \text{CONTINUOUS}\\ w|_D=0\end{cases}$$

ANALYTICAL

$$D_2 = a\,[C_1-D_1]$$

$$E_2 = b\,[D_1-E_1]+a[C_1-D_1]$$

$$F_2 = c\,[E_1-F_1]+b\,[D_1-E_1]+a\,[C_1-D_1]$$



$$\int_0^1 \omega' u' dx = \int_0^1 \omega \, dx \qquad x \in [0,1]$$

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

→ 4-PIECE LINEAR (GENERIC) APPROXIMATION

$x \in [0,a] \quad \omega = C_1 x + C_2 \qquad u = G_1 x + G_2$

$x \in [a,b] \quad \omega = D_1 x + D_2 \qquad u = H_1 x + H_2$

$x \in [b,c] \quad \omega = E_1 x + E_2 \qquad u = I_1 x + I_2$

$x \in [c,1] \quad \omega = F_1 x + F_2 \qquad u = J_1 x + J_2$

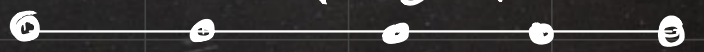
$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$$

ANALYTICAL

$$\int_0^a \omega' u' dx + \int_a^b \omega' u' dx + \int_b^c \omega' u' dx + \int_c^1 \omega' u' dx$$

$$= \int_0^a \omega \, dx + \int_a^b \omega dx + \int_b^c \omega dx + \int_c^1 \omega dx$$

$0 < a < b < c < 1$



$$\int_0^1 \omega' u' dx = \int_0^1 \omega \, dx \qquad x \in [0,1]$$

→ 4-PIECE LINEAR (GENERIC) APPROXIMATION

$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

$x \in [0,a] \quad \omega = C_1 x + C_2 \qquad u = G_1 x + G_2$

$x \in [a,b] \quad \omega = D_1 x + D_2 \qquad u = H_1 x + H_2$

$x \in [b,c] \quad \omega = E_1 x + E_2 \qquad u = I_1 x + I_2$

$x \in [c,1] \quad \omega = F_1 x + F_2 \qquad u = J_1 x + J_2$

$\omega :$ ARBITRARY

CONTINUOUS

ANALYTICAL

$\omega|_D = 0$

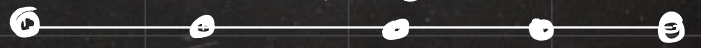
$$\int_0^a \omega' u' dx + \int_a^b \omega' u' dx + \int_b^c \omega' u' dx + \int_c^1 \omega' u' dx \quad \circ\circ\circ$$

$$= \int_0^a \omega \, dx + \int_a^b \omega \, dx + \int_b^c \omega \, dx + \int_c^1 \omega \, dx$$

$$F(G_1)\, C_1 + F^{*}(H_1)\, D_1 + F^{**}(I_1)\, E_1 + F^{***}(J_1)\, F_1 = 0$$

✓ C_1, D_1, E_1, F_1

$0 < a < b < c < 1$



$$\int_0^1 w'u'dx = \int_0^1 w\,dx \qquad x\in[0,1]$$

$u''+1=0 \qquad 0\leqslant x\leqslant 1$

D: $u(0)=0$ ← prescribed

N: $u'(1)=0$ ←

→ 4-PIECE LINEAR (GENERIC) APPROXIMATION

$x\in[0,a] \quad w=C_1x+C_2 \qquad u=G_1x+G_2$

$x\in[a,b] \quad w=D_1x+D_2 \qquad u=H_1x+H_2$

$x\in[b,c] \quad w=E_1x+E_2 \qquad u=I_1x+I_2$

$x\in[c,1] \quad w=F_1x+F_2 \qquad u=J_1x+J_2$

$w:$ { ARBITRARY, CONTINUOUS, $w|_D=0$ }

$$\int_0^a w'u'dx+\int_a^b w'u'dx+\int_b^c w'u'dx+\int_c^1 w'u'dx$$

$$=\int_0^a w\,dx+\int_a^b w\,dx+\int_b^c w\,dx+\int_c^1 w\,dx$$

✓ C_1, D_1, E_1, F_1

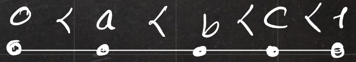
$\Rightarrow$

$G_1=1-\frac{1}{2}a$ ANALYTICAL

$H_1=1-\frac{1}{2}[a+b]$

$I_1=1-\frac{1}{2}[b+c]$

$J_1=1-\frac{1}{2}[c+1]$



$$\int_0^1 \omega' u' dx = \int_0^1 \omega \, dx \qquad x \in [0,1]$$

→ 4-PIECE LINEAR (GENERIC) APPROXIMATION

$$\begin{cases} u = [1 - \frac{1}{2}a]x & x \in [0,a] \\ u = [1 - \frac{1}{2}[a+b]]x + \frac{1}{2}ab & x \in [a,b] \\ u = [1 - \frac{1}{2}[b+c]]x + \frac{1}{2}bc & x \in [b,c] \\ u = [1 - \frac{1}{2}[c+1]]x + \frac{1}{2}c & x \in [c,1] \end{cases}$$

$$\hookrightarrow u = \left[1 - \frac{1}{2}[\alpha + \beta]\right]x + \frac{1}{2}\alpha\beta \qquad x \in [\alpha,\beta]$$

$$u'' + 1 = 0 \qquad 0 \leq x \leq 1$$

D: $u(0) = 0$ ← prescribed

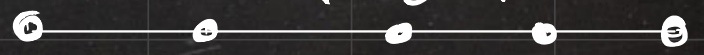
N: $u'(1) = 0$ ←

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$$

ANALYTICAL

... ⇒ n-piece LINEAR (GENERIC)

$0 < a < b < c < 1$



$$\int_0^1 \omega' u' dx = \int_0^1 \omega \, dx \qquad x \in [0,1]$$

→ 4-PIECE LINEAR (GENERIC) APPROXIMATION

$$\begin{cases} u = \frac{7}{8}x & x \in [0, 0.25] \\ u = \frac{5}{8}x + \frac{1}{16} & x \in [0.25, 0.50] \\ u = \frac{3}{8}x + \frac{3}{16} & x \in [0.50, 0.75] \\ u = \frac{1}{8}x + \frac{6}{16} & x \in [0.75, 1.00] \end{cases}$$

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

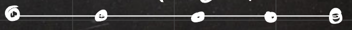
D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$$

ANALYTICAL

$0 < a < b < c < 1$



$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \qquad x \in [0,1]$$

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

→ 1-PIECE QUADRATIC APPROXIMATION

$$x \in [0,1] \qquad w = C_1 x^2 + C_2 x + \cancel{C_3}^{\,0} \quad \leftarrow w|_D = 0 \qquad u = D_1 x^2 + D_2 x + \cancel{D_3}^{\,0} \quad \leftarrow u(0) = 0$$

w : ARBITRARY, CONTINUOUS, ANALYTICAL, $w|_D = 0$

$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \quad \dots$$

$w' = 2C_1 x + C_2$, $u' = 2D_1 x + D_2$, $w = C_1 x^2 + C_2 x$

$$\Rightarrow C_1\Big[\overbrace{\tfrac{4}{3} D_1 - \tfrac{1}{3} + D_2}^{=0}\Big] + C_2\Big[\overbrace{D_1 - \tfrac{1}{2} + D_2}^{=0}\Big] = 0$$

$$\forall C_1, C_2$$

$$\int_0^1 w'u'dx = \int_0^1 w\,dx \qquad x \in [0,1]$$

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

→ 1-PIECE QUADRATIC APPROXIMATION

$$x \in [0,1] \qquad w = C_1 x^2 + C_2 x + \cancel{C_3}^{\,0} \leftarrow w|_D = 0 \qquad u = D_1 x^2 + D_2 x + \cancel{D_3}^{\,0} \leftarrow u(0) = 0$$

$$w : \left\{ \begin{array}{l} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{array} \right.$$

ANALYTICAL

$$\left\{ \begin{array}{l} \frac{4}{3} D_1 + D_2 - \frac{1}{3} = 0 \\ D_1 + D_2 - \frac{1}{2} = 0 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} D_1 = -1/2 \\ D_2 = 1 \end{array} \right.$$

$$u = -\frac{1}{2} x^2 + x$$

IDENTICAL TO ANALYTICAL SOLUTION

approximation that has zero error

$$\int_0^1 w'u'dx = \int_0^1 w\,dx \qquad x \in [0,1]$$

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

→ 1-PIECE QUADRATIC APPROXIMATION

← $w|_D = 0$

$$x \in [0,1] \qquad w = C_1 x^2 + C_2 x + \cancelto{0}{C_3} \qquad u = D_1 x^2 + D_2 x + \cancelto{0}{D_3}$$

← $u(0)=0$

w : ARBITRARY

CONTINUOUS

ANALYTICAL

$w|_D = 0$

IF THE APPROXIMATION SPACE IS LARGE ENOUGH, IT CAN INCLUDE THE EXACT SOLUTION !

$$u = -\frac{1}{2}x^2 + x$$

IDENTICAL TO ANALYTICAL SOLUTION

approximation that has zero error

$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \qquad x\in[0,1]$$

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = 0$ ← prescribed

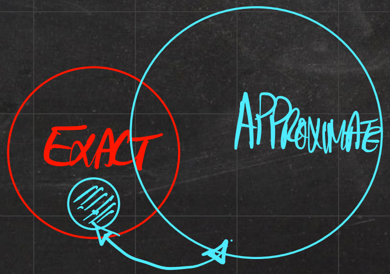
N: $u'(1) = 0$ ←

→ 1-PIECE QUADRATIC APPROXIMATION

$$x\in[0,1] \qquad w = C_1x^2 + C_2x + \overset{0}{\cancel{C_3}} \xleftarrow{} w|_D = 0 \qquad u = D_1x^2 + D_2x + \overset{0}{\cancel{D_3}} \xleftarrow{} u(0)=0$$

w : ANALYTICAL { ARBITRARY, CONTINUOUS, $w|_D = 0$ }

IF THE APPROXIMATION SPACE IS LARGE ENOUGH, IT CAN INCLUDE THE EXACT SOLUTION !



$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \qquad x \in [0,1]$$

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

→ 1-PIECE QUADRATIC APPROXIMATION

$$x \in [0,1] \qquad w = C_1 x^2 + C_2 x + \cancelto{0}{C_3} \leftarrow w|_D = 0 \qquad u = D_1 x^2 + D_2 x + \cancelto{0}{D_3} \leftarrow u(0) = 0$$

$$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$$

ANALYTICAL

IF THE APPROXIMATION SPACE IS LARGE ENOUGH, IT CAN INCLUDE THE EXACT SOLUTION !

APPROXIMATION SPACE COINCIDES WITH THE REQUIRED SPACE OF EXACT SOLUTION

$$\int_0^1 w'u'dx = \int_0^1 w\,dx \qquad x\in[0,1]$$

→ 1-PIECE CUBIC APPROXIMATION

$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

$$u_0 = \underset{0}{\cancel{a}}x^3 + \underset{-\frac{1}{2}}{\cancel{b}}x^2 + \overset{1}{\cancel{c}}x + \overset{0}{\cancel{d}}$$

✓

APPROXIMATION SPACE (CUBIC)

QUADRATIC SPACE

w : { ARBITRARY, CONTINUOUS, $w|_D = 0$ }

ANALYTICAL

APPROXIMATION SPACE COINCIDES WITH THE REQUIRED SPACE OF EXACT SOLUTION