

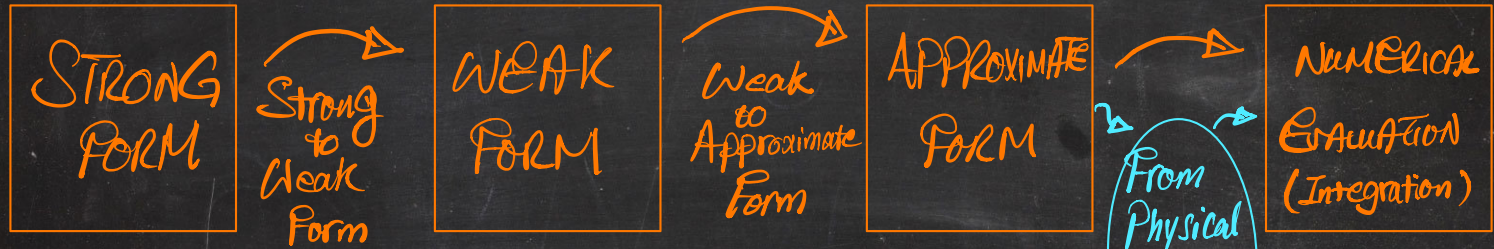
FINITE ELEMENT METHOD

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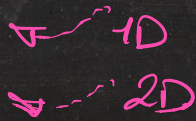
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FINITE ELEMENT METHOD

Differential Equation *



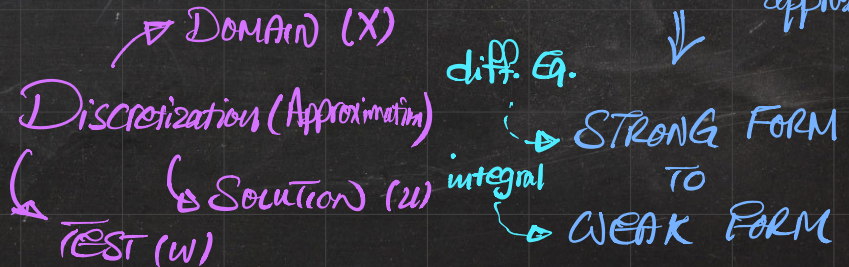
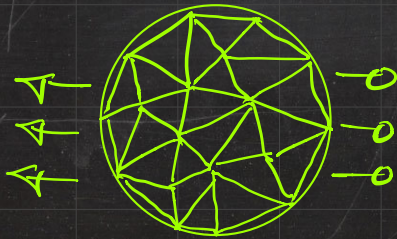
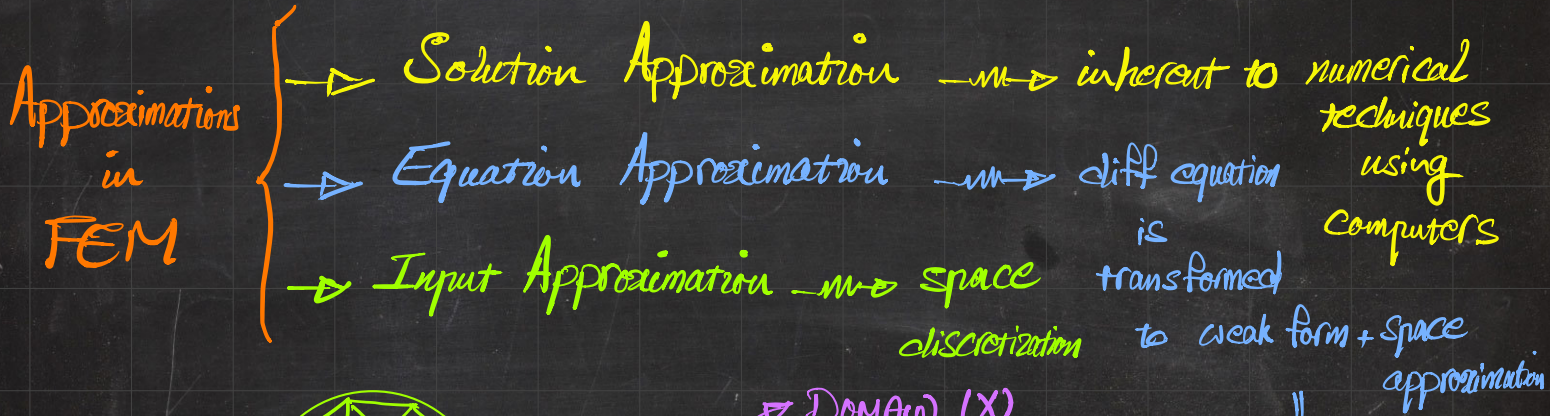
ROADMAP FOR FEM



DISCRETIZED FORM



UNDERSTANDING FEM VIA AN ANALOGY (A BRUTAL SIMPLIFICATION)



$$\frac{d}{dx} \left(EA \frac{du}{dx} \right) + bA = 0 \quad \text{subject to BCs}$$

$$\hookrightarrow E, A: \text{const.} \quad \rightarrow EA u'' + bA = 0 \quad \leftarrow f := \frac{b}{E}$$

STRONG
FORM

$$\rightarrow u'' + f = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = u_0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = t \quad \leftarrow$$

FROM STRONG TO WEAK FORM

STRONG FORM \leftrightarrow Differential Eq.

(I) MULTIPLY BY TEST FUNCTION w

(II) INTEGRATE OVER THE DOMAIN

Integral Form \leftrightarrow WEAK FORM

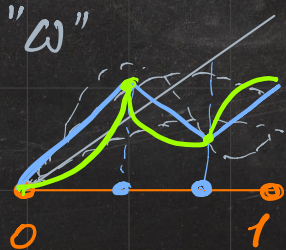
\rightarrow BECAUSE LOWER ORDER DIFFERENTIATION OF DISPLACEMENT u

STRONG : u''

WEAK : u'

$$u'' + f = 0 \quad 0 \leq x \leq 1$$
$$D: u(0) = u_0 \quad \leftarrow \text{prescribed}$$
$$N: u'(1) = t \quad \checkmark$$

$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \quad \leftarrow \text{ZERO @ DIRICHLET BOUNDARY CONDITIONS} \end{cases}$



FROM STRONG TO WEAK FORM

STRONG FORM

(I) MULTIPLY BY TEST FUNCTION w

(II) INTEGRATE OVER THE DOMAIN

$$u'' + f = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = u_0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = t \quad \checkmark$$

$$w: \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$$

$$I) [u'' + f = 0] \times w \Rightarrow wu'' + wf = 0$$

$$II) \int_0^1 [wu'' + wf] dx = 0 \quad wu'' = (wu')' - w'u'$$
$$\int_0^1 (wu')' dx - \int_0^1 w'u' dx + \int_0^1 wf dx = 0$$

FROM STRONG TO WEAK FORM

$$I) [u'' + f = 0] \times w \Rightarrow wu'' + wf = 0$$

$$II) \int_0^1 [wu'' + wf] dx = 0 \quad wu'' = (wu')' - w'u'$$
$$\int_0^1 (wu')' dx - \int_0^1 w'u' dx + \int_0^1 wf dx = 0$$

$$\int_0^1 w'u' dx = \int_0^1 wf dx + wu' \Big|_0^1$$

$$\int_0^1 w'u' dx = \int_0^1 wf dx + w(1)u'(1) - w(0)u'(0)$$

$$u'' + f = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = u_0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = t \quad \checkmark$$

$$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$$

FROM STRONG TO WEAK FORM

$$u'' + f = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = u_0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = t \quad \checkmark$$

$$I) [u'' + f = 0] \times \omega \Rightarrow \omega u'' + \omega f = 0$$

$$II) \int_0^1 [\omega u'' + \omega f] dx = 0 \quad \omega u'' = (\omega u')' - \omega' u'$$

$$\int_0^1 \omega' u' dx = \int_0^1 \omega f dx + \underbrace{\omega(1)u'(1) - \omega(0)u'(0)}_{\substack{\text{TEST FUNCTION @ 1} \\ \text{TEST FUNCTION @ 0}}}$$

$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$

BC:

DIRICHLET	$u \checkmark$	$u' ?$
NEUMANN	$u ?$	$u' \checkmark$

FROM STRONG TO WEAK FORM

$$I) [u'' + f = 0] \times \omega \Rightarrow \omega u'' + \omega f = 0$$

$$II) \int_0^1 [\omega u'' + \omega f] dx = 0 \quad \omega u'' = (\omega u')' - \omega' u'$$

WEAK FORM

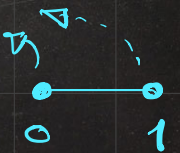
$$\int_0^1 \omega' u' dx = \int_0^1 \omega f dx + \omega(1)u'(1) - \omega(0)u'(0)$$

$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$

INTERNAL CONTRIBUTIONS OVER THE DOMAIN

EXTERNAL CONTRIBUTIONS OVER THE DOMAIN

EXTERNAL CONTRIBUTIONS OVER THE BOUNDARY OF THE DOMAIN



$$u'' + f = 0 \quad 0 \leq x \leq 1$$

D: $u(0) = u_0$ ← prescribed

N: $u'(1) = t$ ✓

FROM STRONG TO WEAK FORM

$$u'' = -1 \Rightarrow u' = -x + C_1$$

$$\Rightarrow u = -\frac{1}{2}x^2 + C_1x + C_2$$

$$\hookrightarrow u(0) = 0 \Rightarrow C_2 = 0$$

$$\hookrightarrow u'(1) = 0 \Rightarrow C_1 = 1$$

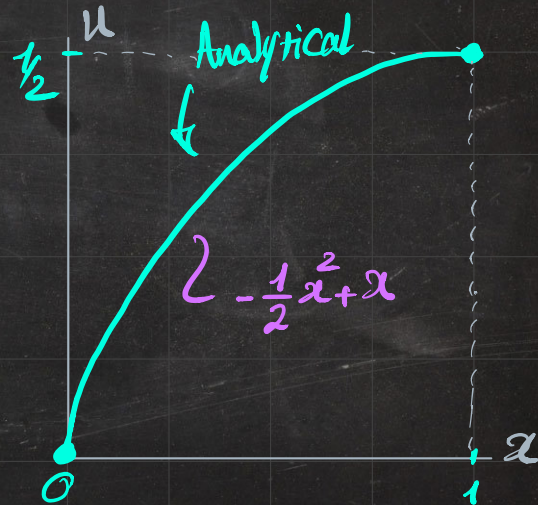
Analytical
Solution

$$\Rightarrow u = -\frac{1}{2}x^2 + x$$

$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$



FROM STRONG TO WEAK FORM

$$\int_0^1 \omega' u' dx = \int_0^1 \omega f dx + \omega(1) u'(1) - \omega(0) u'(0)$$

$$\Rightarrow \int_0^1 \omega' u' dx = \int_0^1 \omega dx \quad \checkmark \text{ WEAK FORM}$$

$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = 0 \quad \checkmark \text{ prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$$

Compute approximate solution \rightarrow from different spaces

\Downarrow
EXERCISE $n \rightarrow \dots$

UNDERSTANDING THE SIGNIFICANCE OF

WEAK FORM

$$\int_0^1 \omega' u' dx = \int_0^1 \omega dx$$

BY EXAMPLE

$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

→ 1-PIECE LINEAR APPROXIMATION

→ 2-PIECE LINEAR (UNIFORM) APPROXIMATION

→ 2-PIECE LINEAR (NON-UNIFORM) APPROXIMATION (I)

→ 2-PIECE LINEAR (NON-UNIFORM) APPROXIMATION (II)

→ 2-PIECE LINEAR (GENERAL) APPROXIMATION

$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$



UNDERSTANDING THE SIGNIFICANCE OF

WEAK FORM

$$\int_0^1 \omega' u' dx = \int_0^1 \omega dx$$

BY EXAMPLE

→ 3-PIECE LINEAR (UNIFORM) APPROXIMATION

→ 3-PIECE LINEAR (GENERAL) APPROXIMATION

→ 4-PIECE LINEAR (UNIFORM) APPROXIMATION

→ 4-PIECE LINEAR (GENERAL) APPROXIMATION

→ 1-PIECE QUADRATIC

→ 1-PIECE CUBIC

$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$$

$$\int_0^1 \omega' u' dx = \int_0^1 \omega dx \quad x \in [0,1]$$

$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

→ 1-PECE LINEAR APPROXIMATION

$$\omega = C_1 x + C_2 \quad C_2 = 0 \quad u = D_1 x + D_2 \quad D_2 = 0$$

$$\omega(0) = 0 \quad \omega|_D = 0 \quad \leftarrow u(0) \text{ is GIVEN}$$

$$u(0) = 0$$

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$$

$$\int_0^1 C_1 D_1 dx = \int_0^1 C_1 x dx \Rightarrow C_1 D_1 x \Big|_0^1 = \frac{1}{2} C_1 x^2 \Big|_0^1$$

$$\Rightarrow D_1 = \frac{1}{2} \quad C_1: \text{cancels out}$$

$$\int_0^1 \omega' u' dx = \int_0^1 \omega dx \quad x \in [0,1]$$

$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

→ 1-PIECE LINEAR APPROXIMATION

$$\omega = C_1 x + C_2 \quad C_2 = 0 \quad u = D_1 x + D_2 \quad D_2 = 0$$

$$\omega(0) = 0 \quad \omega|_D = 0 \quad \leftarrow u(0) \text{ is GIVEN}$$

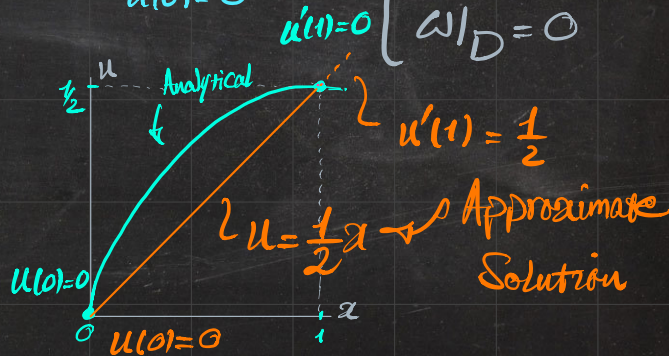
$$u(0) = 0$$

ω :
 { ARBITRARY
 CONTINUOUS
 $\omega|_D = 0$

$$C_1: \text{cancels out} \quad \Rightarrow \quad D_1 = \frac{1}{2}$$

APPROXIMATE
 SOLUTION FOR
 u

$$\Rightarrow u = \frac{1}{2}x$$



$$\int_0^1 w' u' dx = \int_0^1 w dx \quad x \in [0, 1]$$

$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

→ 1-PIECE LINEAR APPROXIMATION

$$w = C_1 x + C_2 \quad C_2 = 0 \quad u = D_1 x + D_2 \quad D_2 = 0$$

$$w(0) = 0 \quad \nearrow \quad w|_D = 0 \quad \Leftarrow \quad u(0) \text{ is GIVEN}$$

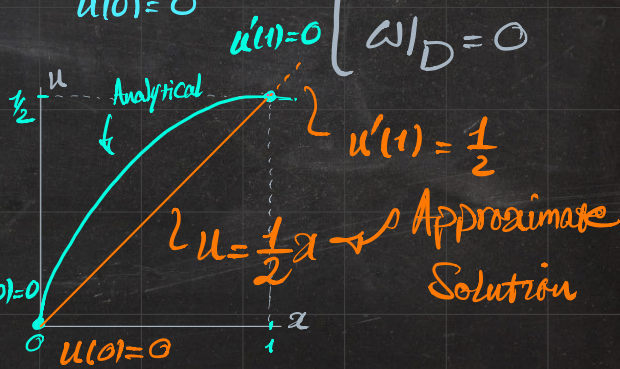
$$u(0) = 0$$

$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$

$$\Rightarrow u = \frac{1}{2} x$$

DIRICHLET BCs ARE STRONGLY SATISFIED

NEUMANN BCs ARE WEAKLY SATISFIED $u(0) = 0$



$$\int_0^1 w' u' dx = \int_0^1 w dx \quad x \in [0, 1]$$

$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

→ 2-PECE LINEAR (UNIFORM) APPROXIMATION

$$\begin{array}{ll} x \in [0, 0.5] & w = C_1 x + C_2 \quad u = E_1 x + E_2 \\ x \in [0.5, 1] & w = D_1 x + D_2 \quad u = F_1 x + F_2 \end{array}$$

$0 \leftarrow w|_D = 0$ $0 \leftarrow u(0) = 0$

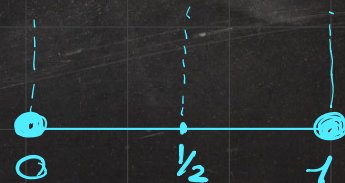
$$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$$

$\Rightarrow \frac{1}{2} C_1 + C_2 = \frac{1}{2} D_1 + D_2$ $\Rightarrow \frac{1}{2} E_1 + E_2 = \frac{1}{2} F_1 + F_2$

↳ Employ BCs and Continuity Conditions

↳ w continuous @ 0.5

↳ u continuous @ 0.5



$$\int_0^1 w' u' dx = \int_0^1 w dx \quad x \in [0, 1]$$

$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

→ 2-PECE LINEAR (UNIFORM) APPROXIMATION

$$x \in [0, 0.5]$$

$$w = C_1 x + C_2$$

$$u = E_1 x + E_2$$

$$x \in [0.5, 1]$$

$$w = D_1 x + D_2$$

$$u = F_1 x + F_2$$

$$\Rightarrow \frac{1}{2} C_1 + C_2 = \frac{1}{2} D_1 + D_2$$

⇓

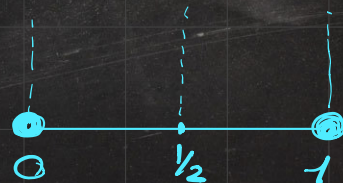
$$D_2 = \frac{1}{2} [C_1 - D_1]$$

$$\Rightarrow \frac{1}{2} E_1 + E_2 = \frac{1}{2} F_1 + F_2$$

⇓

$$F_2 = \frac{1}{2} [E_1 - F_1]$$

$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$



$$\int_0^1 w' u' dx = \int_0^1 w dx \quad x \in [0, 1]$$

$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

→ 2-PECE LINEAR (UNIFORM) APPROXIMATION

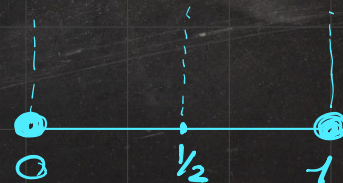
$$x \in [0, 0.5] \quad w = C_1 x + C_2 \quad u = E_1 x + E_2$$

$$x \in [0.5, 1] \quad w = D_1 x + D_2 \quad u = F_1 x + F_2$$

$$D_2 = \frac{1}{2} [C_1 - D_1] \quad F_2 = \frac{1}{2} [E_1 - F_1]$$

w : $\left\{ \begin{array}{l} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{array} \right.$

$$\int_0^{0.5} w' u' dx + \int_{0.5}^1 w' u' dx = \int_0^{0.5} w dx + \int_{0.5}^1 w dx$$



$$\int_0^1 w' u' dx = \int_0^1 w dx \quad x \in [0, 1]$$

$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

→ 2-PECE LINEAR (UNIFORM) APPROXIMATION

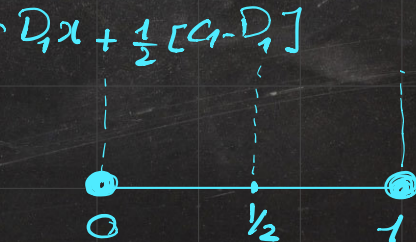
$$\begin{array}{ll} x \in [0, 0.5] & w = C_1 x + C_2 \quad u = E_1 x + E_2 \\ x \in [0.5, 1] & w = D_1 x + D_2 \quad u = F_1 x + F_2 \end{array} \quad \left\{ \begin{array}{l} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{array} \right.$$

$C_2 = 0 \leftarrow w|_D = 0$ $E_2 = 0 \leftarrow u(0) = 0$

$$D_2 = \frac{1}{2} [C_1 - D_1] \quad F_2 = \frac{1}{2} [E_1 - F_1]$$

$$\int_0^{0.5} w' u' dx + \int_{0.5}^1 w' u' dx = \int_0^{0.5} w dx + \int_{0.5}^1 w dx$$

$\uparrow E_1$ $\uparrow F_1$ $\uparrow C_1 x$



$$\int_0^1 \omega' u' d\alpha = \int_0^1 \omega d\alpha \quad \alpha \in [0,1]$$

$$u'' + 1 = 0 \quad 0 \leq \alpha \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

→ 2-PECE LINEAR (UNIFORM) APPROXIMATION

$$\int_0^{0.5} C_1 E_1 d\alpha + \int_{0.5}^1 D_1 F_1 d\alpha$$

$$= \int_0^{0.5} C_1 \alpha d\alpha + \int_{0.5}^1 [D_1 \alpha + \frac{1}{2} [C_1 - D_1]] d\alpha$$

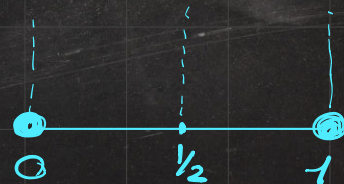
$$u = E_1 \alpha + E_2$$

$$u = F_1 \alpha + F_2$$

$$F_2 = \frac{1}{2} [E_1 - F_1]$$

ω :
 { ARBITRARY
 CONTINUOUS
 $\omega|_D = 0$

$$\frac{1}{2} C_1 E_1 + \frac{1}{2} D_1 F_1 = \frac{1}{8} C_1 + \frac{3}{8} D_1 + \frac{1}{4} [C_1 - D_1]$$



$$\int_0^1 \omega' u' dx = \int_0^1 \omega dx \quad \alpha \in [0,1]$$

$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

→ 2-PECE LINEAR (UNIFORM) APPROXIMATION

$$\frac{1}{2} C_1 E_1 + \frac{1}{2} D_1 F_1$$

$$= \frac{1}{8} C_1 + \frac{3}{8} D_1 + \frac{1}{4} [C_1 - D_1]$$

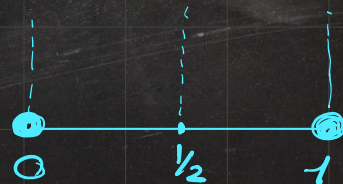
$$u = E_1 x + E_2 \quad \omega : \left\{ \begin{array}{l} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{array} \right.$$

$$u = F_1 x + F_2$$

$$F_2 = \frac{1}{2} [E_1 - F_1]$$

THIS SEEMS LIKE 1 EQUATION

BUT IT IS NOT!



$$\int_0^1 \omega' u' dx = \int_0^1 \omega dx \quad \alpha \in [0,1]$$

$$u'' + 1 = 0 \quad 0 \leq \alpha \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

→ 2-PECE LINEAR (UNIFORM) APPROXIMATION

$$\frac{1}{2} C_1 E_1 + \frac{1}{2} D_1 F_1$$

✓ C_1, D_1

$$= \frac{1}{8} C_1 + \frac{3}{8} D_1 + \frac{1}{4} [C_1 - D_1]$$

$$u = E_1 \alpha + E_2 \quad \omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$$

$$u = F_1 \alpha + F_2$$

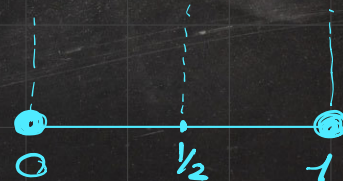
$$F_2 = \frac{1}{2} [E_1 - F_1]$$

$$C_1 = 1, D_1 = 0 \Rightarrow \begin{cases} \frac{1}{2} E_1 - \frac{3}{8} = 0 \end{cases} \Rightarrow E_1 = 3/4$$

$$C_1 = 0, D_1 = 1 \Rightarrow \begin{cases} \frac{1}{2} F_1 - \frac{1}{8} = 0 \end{cases} \Rightarrow F_1 = 1/4$$

$$E_1 = 3/4$$

$$F_1 = 1/4$$



$$\int_0^1 \omega' u' dx = \int_0^1 \omega dx \quad x \in [0,1]$$

$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

→ 2-PECE LINEAR (UNIFORM) APPROXIMATION

$$\frac{1}{2} C_1 E_1 + \frac{1}{2} D_1 F_1$$

$\checkmark C_1, D_1$

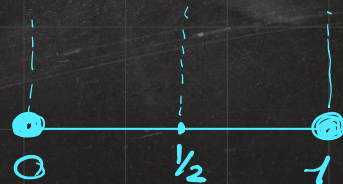
$$= \frac{1}{8} C_1 + \frac{3}{8} D_1 + \frac{1}{4} [C_1 - D_1]$$

$$u = E_1 x + E_2 x^2$$

$$u = F_1 x + F_2 \quad \omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$$

$$F_2 = \frac{1}{2} [E_1 - F_1]$$

ω : ARBITRARY $\Rightarrow C_1$ & D_1 : ARBITRARY



$$\int_0^1 \omega' u' dx = \int_0^1 \omega dx \quad x \in [0, 1]$$

$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

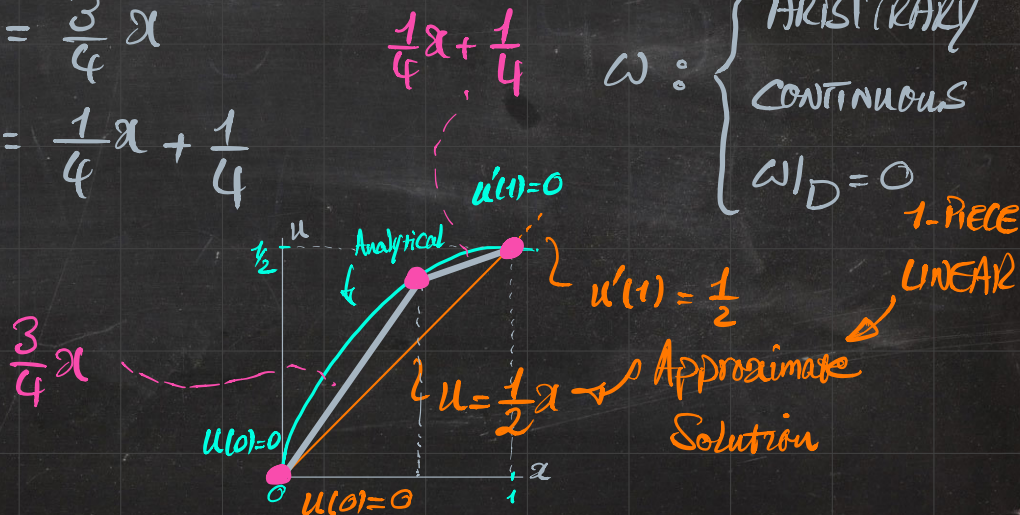
$$N: u'(1) = 0 \quad \checkmark$$

→ 2-PIECE LINEAR (UNIFORM) APPROXIMATION

$$x \in [0, 0.5] \quad u = \frac{3}{4}x$$

$$x \in [0.5, 1] \quad u = \frac{1}{4}x + \frac{1}{4}$$

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$$



$$\int_0^1 \omega' u' dx = \int_0^1 \omega dx \quad x \in [0, 1]$$

→ 2-PIECE LINEAR (NON-UNIFORM) APPROXIMATION

$$x \in [0, 0.6] \quad \omega = C_1 x + C_2 \quad u = E_1 x + E_2$$

$$x \in [0.6, 1] \quad \omega = D_1 x + D_2 \quad u = F_1 x + F_2$$

$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$$

$$\int_0^{0.6} \omega' u' dx + \int_{0.6}^1 \omega' u' dx = \int_0^{0.6} \omega dx + \int_{0.6}^1 \omega dx$$

$$\begin{matrix} C_1 & E_1 & D_1 & F_1 & C_1 x & D_1 x + 0.6 [C_1 - D_1] \end{matrix}$$

$$\int_0^1 \omega' u' dx = \int_0^1 \omega dx \quad x \in [0, 1]$$

→ 2-PIECE LINEAR (NON-UNIFORM) APPROXIMATION

$$x \in [0, 0.6] \quad \omega = C_1 x + C_2 \quad u = E_1 x + E_2$$

$$x \in [0.6, 1] \quad \omega = D_1 x + D_2 \quad u = F_1 x + F_2$$

$$0.6 [C_1 - D_1] \quad 0.6 [E_1 - F_1]$$

$$\int_0^{0.6} C_1 E_1 dx + \int_{0.6}^1 D_1 F_1 dx = \int_0^{0.6} C_1 x dx + \int_{0.6}^1 [D_1 x + 0.6 [C_1 - D_1]] dx$$

$$C_1 [0.6 E_1 - 0.42] + D_1 [0.4 F_1 - 0.08] = 0$$

$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$$

$$\int_0^1 \omega' u' dx = \int_0^1 \omega dx \quad x \in [0, 1]$$

→ 2-PIECE LINEAR (NON-UNIFORM) APPROXIMATION

$$x \in [0, 0.6] \quad u = E_1 x + E_2$$

$$x \in [0.6, 1] \quad u = F_1 x + F_2$$

$$C_1 [0.6 E_1 - 0.42] + D_1 [0.4 F_1 - 0.08] = 0$$

$$\Rightarrow E_1 = 0.7, F_1 = 0.2$$

$$\Rightarrow \begin{cases} u = 0.7x & 0 \leq x \leq 0.6 \\ u = 0.2x + 0.3 & 0.6 \leq x \leq 1 \end{cases}$$

$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$$

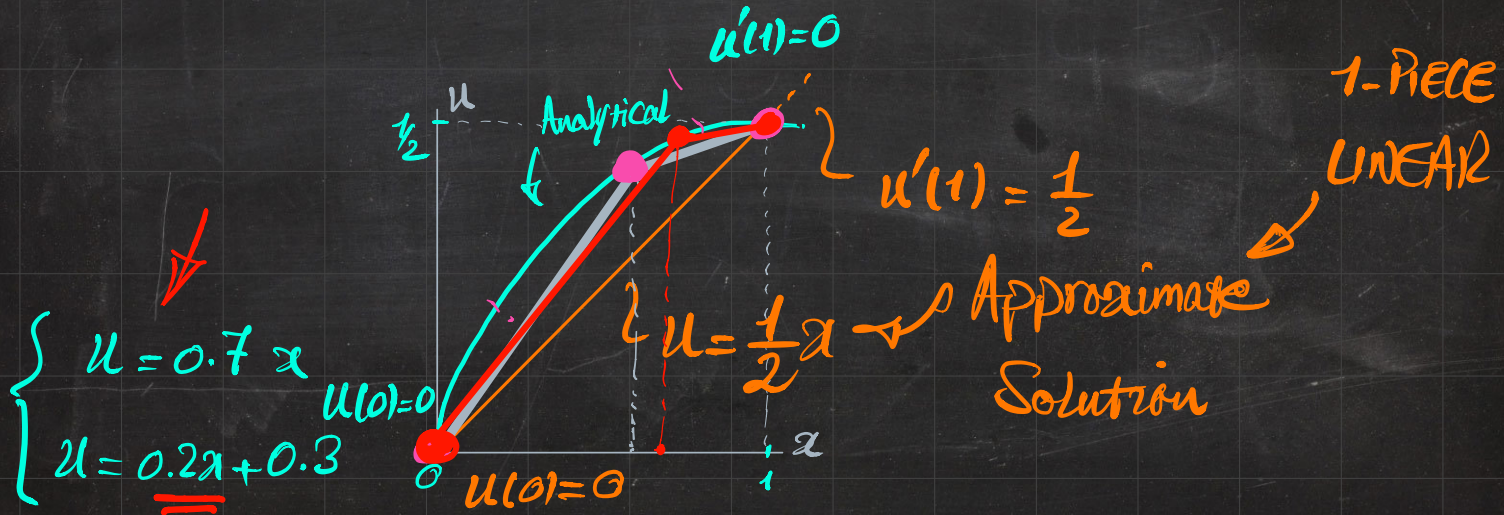
$$\int_0^1 \omega' u' dx = \int_0^1 \omega dx \quad x \in [0,1]$$

→ 2-PIECE LINEAR (NON-UNIFORM) APPROXIMATION

$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$



$$\int_0^1 \omega' u' dx = \int_0^1 \omega dx \quad x \in [0, 1]$$

→ 2-PIECE LINEAR (NON-UNIFORM) APPROXIMATION

$$x \in [0, 0.4] \quad \omega = C_1 x + C_2 \quad u = E_1 x + E_2$$

$$x \in [0.4, 1] \quad \omega = D_1 x + D_2 \quad u = F_1 x + F_2$$

$$0.4 [C_1 - D_1] \quad 0.4 [E_1 - F_1]$$

$$\int_0^{0.4} \omega' u' dx + \int_{0.4}^1 \omega' u' dx = \int_0^{0.4} \omega dx + \int_{0.4}^1 \omega dx$$

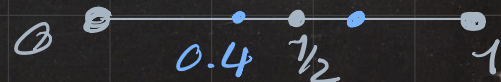
$\left(\begin{array}{cc} C_1 & E_1 \\ D_1 & F_1 \end{array} \right) \quad \left(\begin{array}{c} C_1 x \\ D_1 x + 0.4 [C_1 - D_1] \end{array} \right)$

$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$$



$$\int_0^1 \omega' u' dx = \int_0^1 \omega dx \quad x \in [0, 1]$$

→ 2-PIECE LINEAR (NON-UNIFORM) APPROXIMATION

$$x \in [0, 0.4] \quad \omega = C_1 x + C_2 \quad u = E_1 x + E_2$$

$$x \in [0.4, 1] \quad \omega = D_1 x + D_2 \quad u = F_1 x + F_2$$

$$0.4 [C_1 - D_1] \quad 0.4 [E_1 - F_1]$$

$$\int_0^{0.4} C_1 E_1 dx + \int_{0.4}^1 D_1 F_1 dx = \int_0^{0.4} C_1 x dx + \int_{0.4}^1 [D_1 x + 0.4 [C_1 - D_1]] dx$$

$$C_1 [0.4 E_1 - 0.32] + D_1 [0.6 F_1 - 0.18] = 0$$

$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$$

$$\int_0^1 \omega' u' dx = \int_0^1 \omega dx \quad x \in [0, 1]$$

→ 2-PIECE LINEAR (NON-UNIFORM) APPROXIMATION

$$x \in [0, 0.4] \quad u = E_1 x + E_2$$

$$x \in [0.4, 1] \quad u = F_1 x + F_2$$

$$C_1 [0.4 E_1 - 0.32] + D_1 [0.6 F_1 - 0.18] = 0$$

$$\Rightarrow E_1 = 0.8, F_1 = 0.3$$

$$\Rightarrow \begin{cases} u = 0.8x & 0 \leq x \leq 0.4 \\ u = 0.3x + 0.2 & 0.4 \leq x \leq 1 \end{cases}$$

$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$$

$$\int_0^1 \omega' u' dx = \int_0^1 \omega dx \quad x \in [0,1]$$

→ 2-PIECE LINEAR (NON-UNIFORM) APPROXIMATION

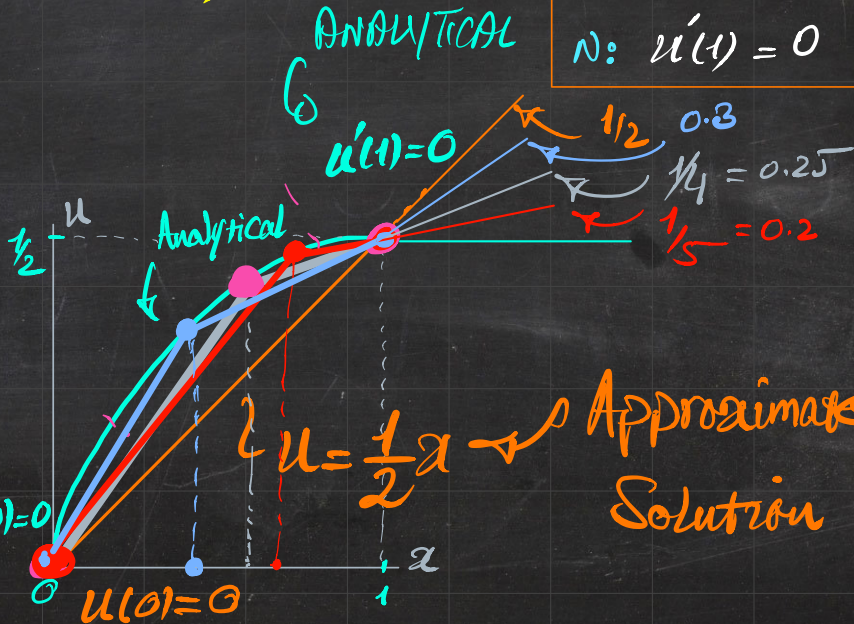
$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

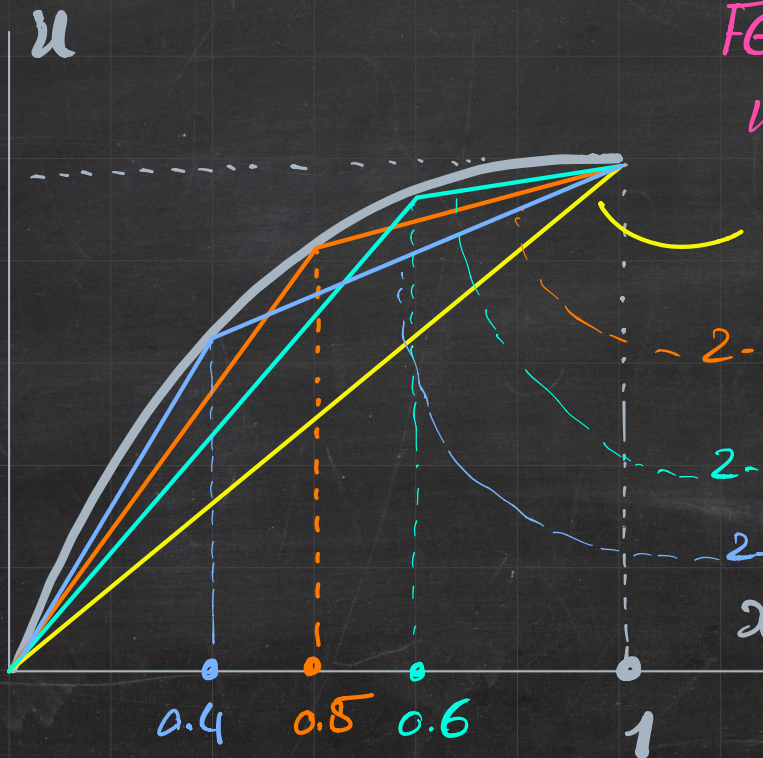
$$\begin{cases} u = 0.8x \\ u = 0.3x + 0.2 \end{cases}$$

$$\begin{cases} u = 0.7x \\ u = 0.2x + 0.3 \end{cases}$$



1-PIECE LINEAR

Approximate Solution



FE SOLUTIONS SEEM TO UNDERESTIMATE THE ANALYTICAL ONE!

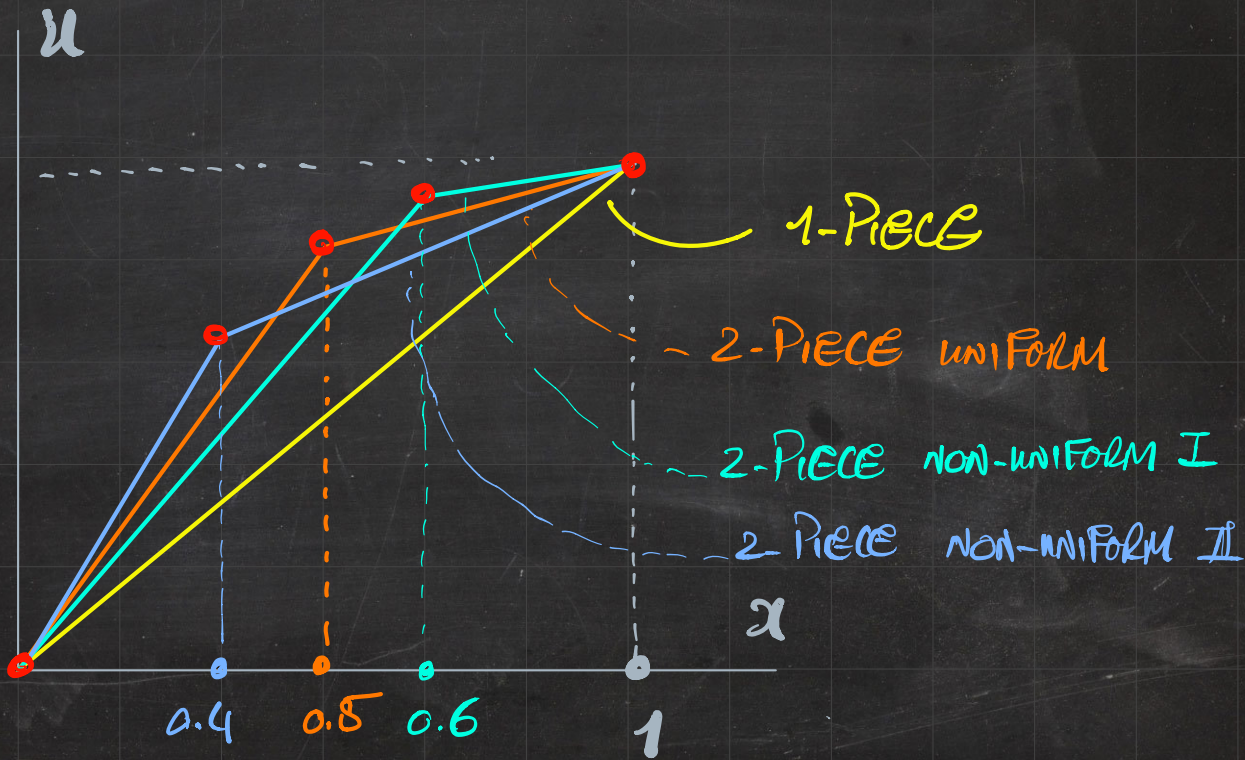
1-PIECE

2-PIECE UNIFORM

2-PIECE NON-UNIFORM I

2-PIECE NON-UNIFORM II

FE Solution approaches analytical one from below!



$$\int_0^1 \omega' u' dx = \int_0^1 \omega dx \quad x \in [0,1]$$

→ 2-PIECE LINEAR (GENERIC) APPROXIMATION

$$x \in [0, a] \quad \omega = C_1 x + C_2 \quad u = E_1 x + E_2$$

$$x \in [a, 1] \quad \omega = D_1 x + D_2 \quad u = F_1 x + F_2$$

$$a [C_1 - D_1]$$

$$a [E_1 - F_1]$$

$$\int_0^a \omega' u' dx + \int_a^1 \omega' u' dx = \int_0^a \omega dx + \int_a^1 \omega dx$$

$\begin{matrix} C_1 & E_1 & D_1 & F_1 \\ \left. \int_0^a \right) & \left. \int_a^1 \right) & \left. \int_0^a \right) & \left. \int_a^1 \right) \end{matrix}$

$$D_1 x + a [C_1 - D_1]$$

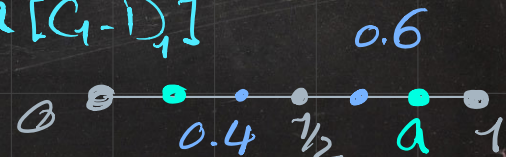
$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$

$$0 \leq a \leq 1$$



$$\int_0^1 w' u' dx = \int_0^1 w dx \quad x \in [0, 1]$$

→ 2-PIECE LINEAR (GENERIC) APPROXIMATION

$$x \in [0, a] \quad w = C_1 x + C_2 \quad u = E_1 x + E_2$$

$$x \in [a, 1] \quad w = D_1 x + D_2 \quad u = F_1 x + F_2$$

$a [C_1 - D_1]$ $a [E_1 - F_1]$

$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

$$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$$

$$a_1 C_1 E_1 + [1-a] D_1 F_1 = \dots + \dots$$

$$= a C_1 [1 - \frac{1}{2}a] + D_1 [1-a] [\frac{1}{2} - \frac{1}{2}a]$$

$$\int_0^1 \omega' u' dx = \int_0^1 \omega dx \quad x \in [0, 1]$$

→ 2-PIECE LINEAR (GENERIC) APPROXIMATION

$$x \in [0, a] \quad \omega = C_1 x + C_2 \quad u = E_1 x + E_2$$

$$x \in [a, 1] \quad \omega = D_1 x + D_2 \quad u = F_1 x + F_2$$

$a [C_1 - D_1]$ $a [E_1 - F_1]$

$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$$

$$a_1 C_1 E_1 + [1-a] D_1 F_1$$

$$= a C_1 [1 - \frac{1}{2}a] + D_1 [1-a] [\frac{1}{2} - \frac{1}{2}a] \quad \checkmark C_1, D_1$$

$$\Rightarrow E_1 = 1 - \frac{1}{2}a, \quad F_1 = \frac{1}{2} - \frac{1}{2}a$$

$$\int_0^1 \omega' u' dx = \int_0^1 \omega dx \quad x \in [0,1]$$

→ 2-PIECE LINEAR (GENERIC) APPROXIMATION

$$x \in [0, a] \Rightarrow u = [1 - \frac{1}{2}a] x$$

$$x \in [a, 1] \Rightarrow u = [\frac{1}{2} - \frac{1}{2}a] x + \frac{1}{2} a$$

$$a = 0.5$$

$$a = 0.6$$

$$a = 0.4$$

$$\left\{ \begin{array}{l} u = 0.75x \\ u = 0.25x + 0.25 \end{array} \right.$$

$$\left\{ \begin{array}{l} u = 0.7x \\ u = 0.2x + 0.3 \end{array} \right.$$

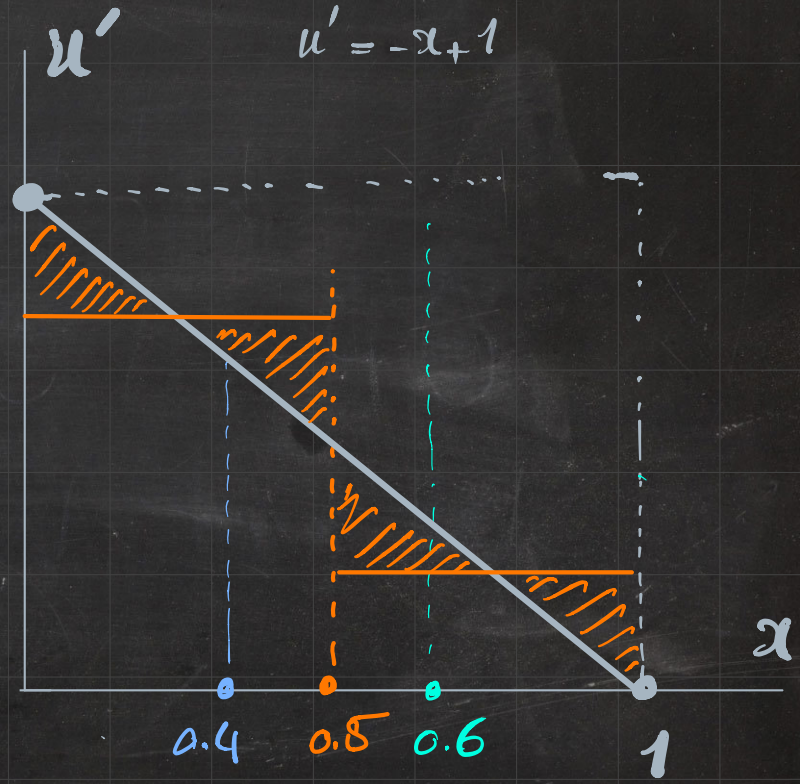
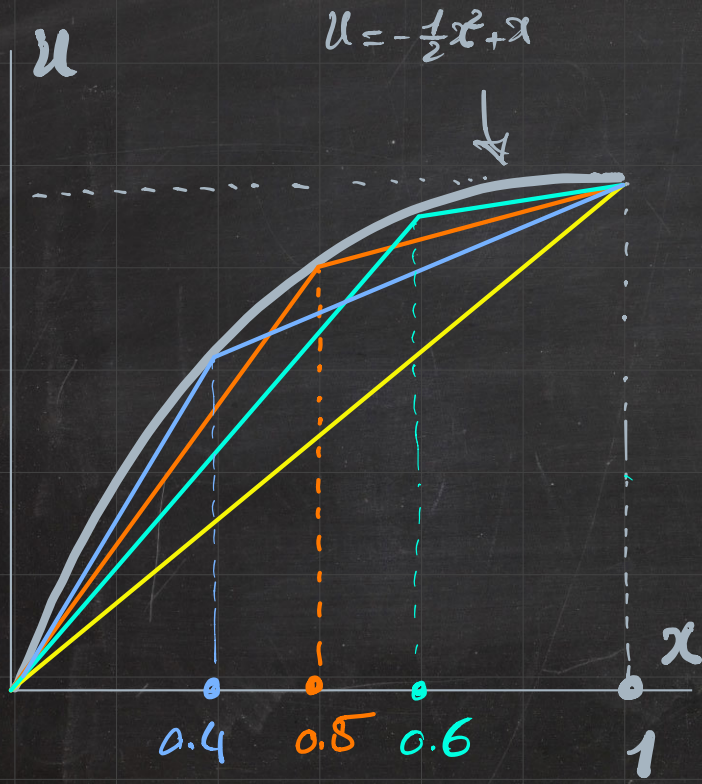
$$\left\{ \begin{array}{l} u = 0.8x \\ u = 0.3x + 2 \end{array} \right.$$

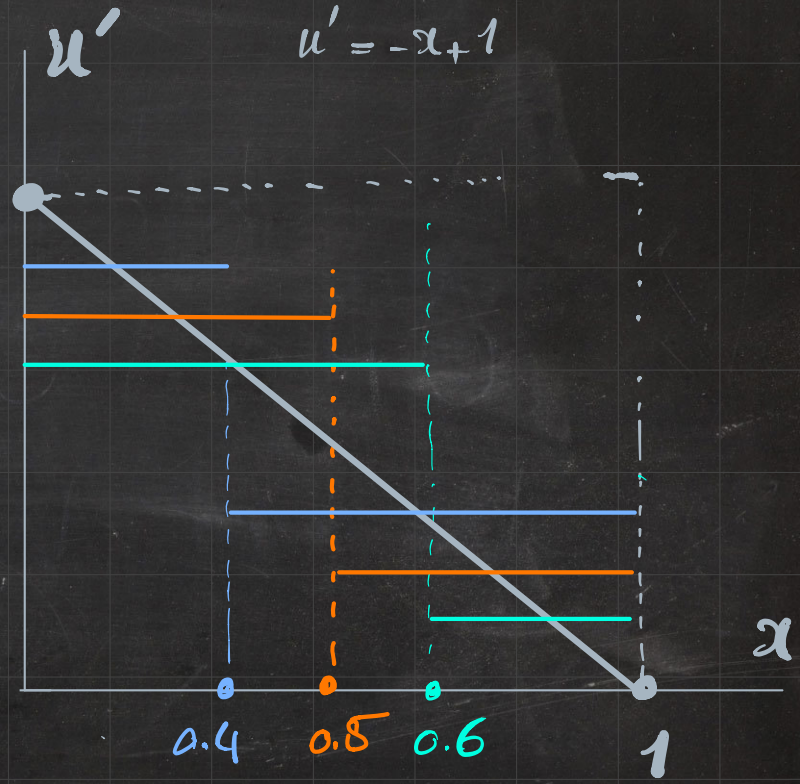
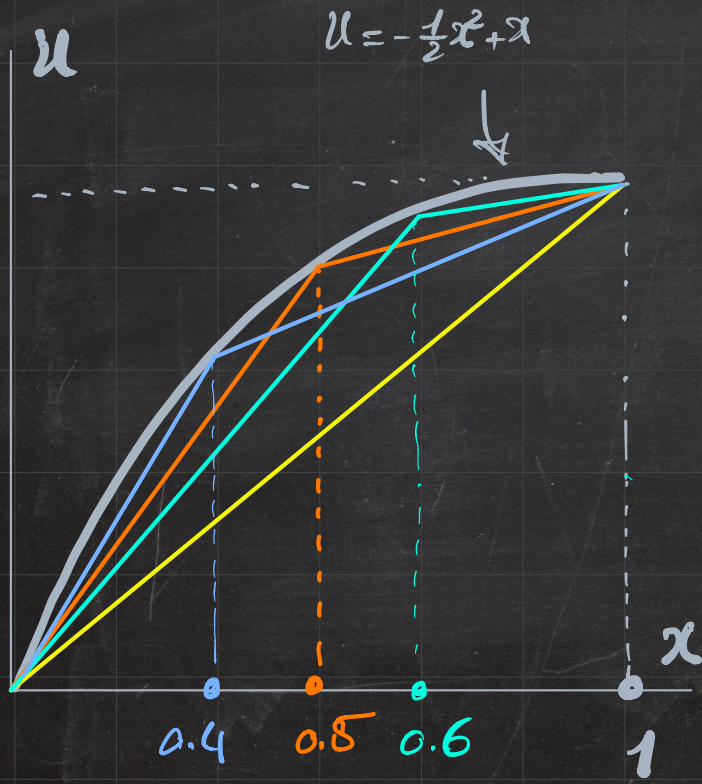
$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

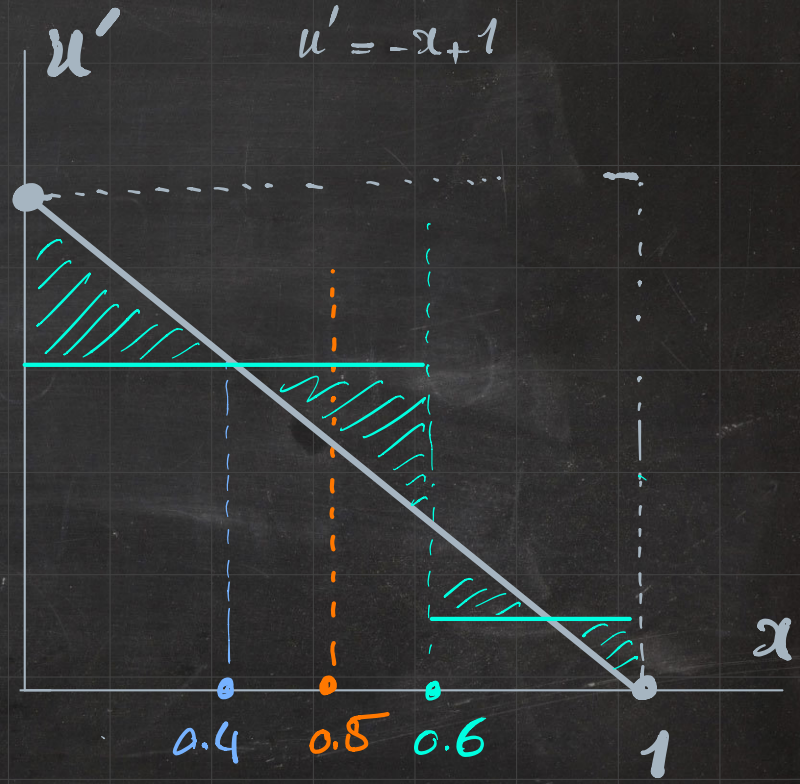
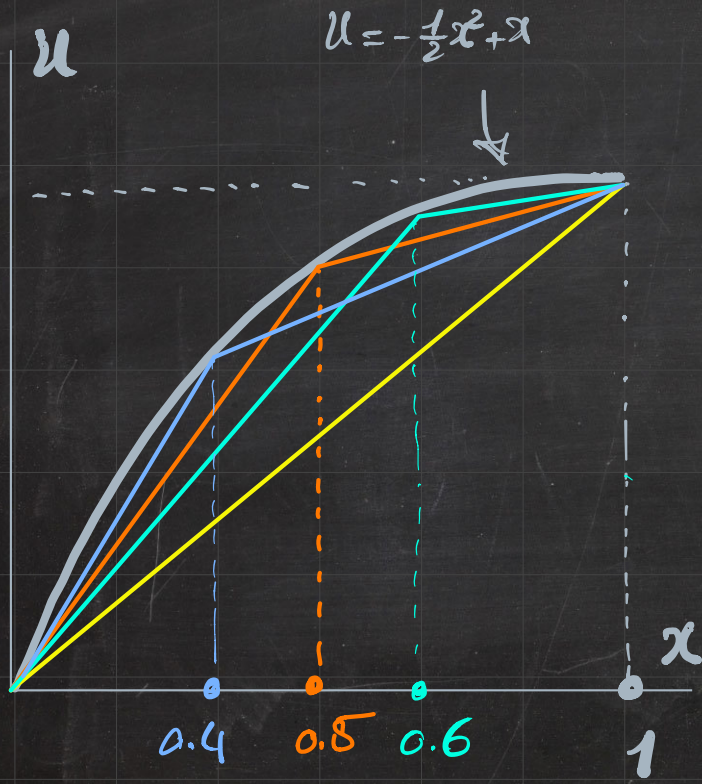
$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$$







$$\int_0^1 \omega' u' dx = \int_0^1 \omega dx \quad x \in [0, 1]$$

→ 2-PIECE LINEAR (GENERIC) APPROXIMATION

$$x \in [0, a] \Rightarrow u = [1 - \frac{1}{2}a] x$$

$$x \in [a, 1] \Rightarrow u = [\frac{1}{2} - \frac{1}{2}a] x + \frac{1}{2} a$$

$$x \in [0, a] \Rightarrow u' = [1 - \frac{1}{2}a]$$

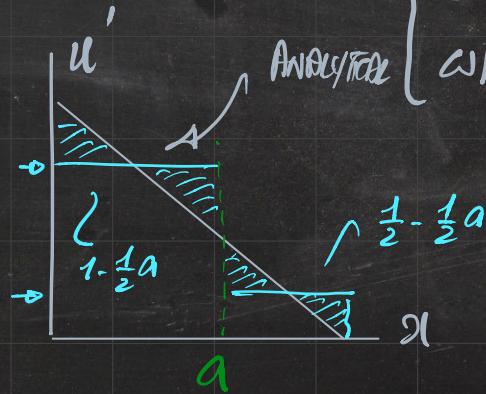
$$x \in [a, 1] \Rightarrow u' = [\frac{1}{2} - \frac{1}{2}a]$$

$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

ω : $\left\{ \begin{array}{l} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{array} \right.$



$$\int_0^1 \omega' u' dx = \int_0^1 \omega dx \quad \alpha \in [0,1]$$

→ 3-PIECE LINEAR (GENERIC) APPROXIMATION

$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$



$$0 \leq a < b \quad a < b \leq 1$$

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \text{ANALYTICAL} \end{cases} \quad \omega|_D = 0$$

$$\int_0^1 \omega' u' dx = \int_0^1 \omega dx \quad x \in [0,1]$$

→ 3-PIECE LINEAR (GENERIC) APPROXIMATION

$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

$$x \in [0, a] \quad \omega = C_1 x + C_2 \quad a [C_1 - D_1]$$

$$u = F_1 x + F_2 \quad a [F_1 - G_1]$$

$$x \in [a, b] \quad \omega = D_1 x + D_2$$

$$u = G_1 x + G_2$$

$$x \in [b, 1] \quad \omega = E_1 x + E_2$$

$$u = H_1 x + H_2$$

$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \text{ANALYTICAL} \end{cases} \quad \omega|_D = 0$

$$b [D_1 - E_1] + a [C_1 - D_1]$$

$$b [G_1 - H_1] + a [F_1 - G_1]$$

$$\checkmark C_1, D_1, E_1$$

$$F_1, G_1, H_1 = ?$$



$$\int_0^1 \omega' u' dx = \int_0^1 \omega dx \quad x \in [0,1]$$

→ 3-PIECE LINEAR (GENERIC) APPROXIMATION

$$x \in [0, a] \quad \omega = C_1 x + C_2 \quad u = F_1 x + F_2$$

(C₂ and F₂ are crossed out with orange arrows)

$$x \in [a, b] \quad \omega = D_1 x + D_2 \quad u = G_1 x + G_2$$

$$x \in [b, 1] \quad \omega = E_1 x + E_2 \quad u = H_1 x + H_2$$

(E₂ and H₂ are crossed out with orange arrows)

$$b[D_1 - E_1] + a[G_1 - D_1] \quad b[G_1 - H_1] + a[F_1 - G_1]$$

$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \text{ANALYTICAL} \end{cases} \quad \omega|_D = 0$$

$$\int_0^a \omega' u' dx + \int_a^b \omega' u' dx + \int_b^1 \omega' u' dx = \int_0^a \omega dx + \int_a^b \omega dx + \int_b^1 \omega dx$$

(E₁x + b[D₁ - E₁] + a[G₁ - D₁] is written above the last integral)

(C₁, F₁, D₁, G₁, E₁, H₁, C₁x, D₁x + a[G₁ - D₁] are written below the integrals)

$$\int_0^1 \omega' u' dx = \int_0^1 \omega dx \quad x \in [0,1]$$

→ 3-PIECE LINEAR (GENERIC) APPROXIMATION

$$x \in [0, a] \quad \omega = G_1 x + G_2 \quad u = F_1 x + F_2$$

(G2, F2 are crossed out with orange arrows)

$$x \in [a, b] \quad \omega = D_1 x + D_2 \quad u = G_1 x + G_2$$

$$x \in [b, 1] \quad \omega = E_1 x + E_2 \quad u = H_1 x + H_2$$

(E2, H2 are crossed out with orange arrows)

$$b[D_1 - E_1] + a[G_1 - D_1] \quad b[G_1 - H_1] + a[F_1 - G_1]$$

$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

ω : $\left\{ \begin{array}{l} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \text{ANALYTICAL} \end{array} \right. \omega|_D = 0$

$$\int_0^a \omega' u' dx + \int_a^b \omega' u' dx + \int_b^1 \omega' u' dx = \int_0^a \omega dx + \int_a^b \omega dx + \int_b^1 \omega dx$$

(Integrals are labeled with nodes: 0, a, b, 1 and functions: G1, D1, E1, F1, G2, H2, G1, D1, E1, F1, G2)

$$E_1 x + b[D_1 - E_1] + a[G_1 - D_1]$$

$$G_1 [000] + D_1 [000] + E_1 [000] = 0 \quad \sqrt{G_1, D_1, E_1}$$



$$\int_0^1 \omega' u' dx = \int_0^1 \omega dx \quad \alpha \in [0,1]$$

→ 3-PIECE LINEAR (GENERIC) APPROXIMATION

$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

$$C_1 [000] + D_1 [000] + E_1 [000] = 0$$

$\sqrt{C_1, D_1, E_1}$

$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \text{ANALYTICAL} \end{cases} \omega|_D = 0$

$D_1 = 0$
 $E_1 = 0$
 $C_1 = 1$

$$f(F_1) = 0$$

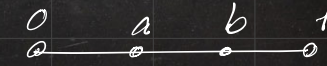
$D_1 = 1$
 $E_1 = 0$
 $C_1 = 0$

$$g(G_1) = 0$$

$$h(H_1) = 0$$

$\Rightarrow F_1, G_1, H_1 \checkmark$

$$F_1 = 000, G_1 = 000, H_1 = 000$$



$$\int_0^1 \omega' u' dx = \int_0^1 \omega dx \quad x \in [0,1]$$

→ 3-PIECE LINEAR (GENERIC) APPROXIMATION

$$x \in [0, a] \quad \omega = G_1 x + G_2 \quad u = F_1 x + F_2$$

(Annotations: G_2 and F_2 are crossed out with orange arrows pointing to a in $[G-D]$ and $[F-G]$ respectively)

$$x \in [a, b] \quad \omega = D_1 x + D_2 \quad u = G_1 x + G_2$$

$$x \in [b, 1] \quad \omega = E_1 x + E_2 \quad u = H_1 x + H_2$$

(Annotations: E_2 and H_2 are crossed out with orange arrows pointing to b in $[D_1-E_1] + a$ and $[G_1-H_1] + a$ respectively)

$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ✓

ω : { ARBITRARY
CONTINUOUS
ANALYTICAL } $\omega|_D = 0$

$$G_1 [000] + D_1 [000] + E_1 [000] = 0 \quad \sqrt{G_1, D_1, E_1}$$

$$\Rightarrow F_1 = 1 - \frac{1}{2}a \quad G_1 = 1 - \frac{1}{2}[a+b] \quad H_1 = 1 - \frac{1}{2}[b+1]$$

(Diagram: A number line from 0 to 1 with points 0, a, b, 1 marked)

$$\int_0^1 \omega' u' dx = \int_0^1 \omega dx \quad x \in [0, 1]$$

→ 3-PIECE LINEAR (GENERIC) APPROXIMATION

$$x \in [0, a]$$

$$x \in [a, b]$$

$$x \in [b, 1]$$

$$u = F_1 x + F_2 \overset{a}{\uparrow} [F_1 - F_2]$$

$$u = G_1 x + G_2$$

$$u = H_1 x + H_2$$

$$b[G_1 - H_1] + a[F_1 - G_1]$$

$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \text{ANALYTICAL} \end{cases} \quad \omega|_D = 0$$

$$\Rightarrow F_1 = 1 - \frac{1}{2}a \quad G_1 = 1 - \frac{1}{2}[a+b] \quad H_1 = 1 - \frac{1}{2}[b+1]$$



$$\int_0^1 \omega' u' dx = \int_0^1 \omega dx \quad \alpha \in [0,1]$$

→ 3-PIECE LINEAR (GENERIC) APPROXIMATION

$$\begin{cases} u = [1 - \frac{1}{2}\alpha]x & \alpha \in [0, a] \\ u = [1 - \frac{1}{2}(a+b)]x + \frac{1}{2}ab & \alpha \in [a, b] \\ u = [1 - \frac{1}{2}(b+1)]x + \frac{1}{2}b & \alpha \in [b, 1] \end{cases}$$

$$u'' + 1 = 0 \quad 0 \leq \alpha \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \text{ANALYTICAL} \end{cases} \quad \omega|_D = 0$$

$$\hookrightarrow u = [1 - \frac{1}{2}[\alpha, \beta]]x + \frac{1}{2}\alpha\beta \quad \leftarrow 0 \leq \alpha \leq \beta \leq 1$$

$$\{\alpha, \beta\} \rightarrow \{0, a\} \quad \{\alpha, \beta\} \rightarrow \{a, b\} \quad \{\alpha, \beta\} \rightarrow \{b, 1\}$$

$$\int_0^1 \omega' u' dx = \int_0^1 \omega dx \quad \alpha \in [0,1]$$

→ 3-PIECE LINEAR (GENERIC) APPROXIMATION

$$\begin{cases} u = [1 - \frac{1}{2}a]x & \alpha \in [0, a] \\ u = [1 - \frac{1}{2}(a+b)]x + \frac{1}{2}ab & \alpha \in [a, b] \\ u = [1 - \frac{1}{2}(b+1)]x + \frac{1}{2}b & \alpha \in [b, 1] \end{cases}$$

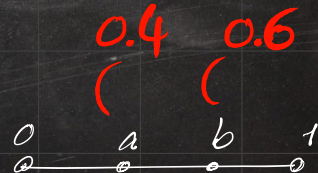
$$\begin{cases} u = 0.8x & \alpha \in [0, 0.4] \\ u = 0.5x + 0.12 & \alpha \in [0.4, 0.6] \\ u = 0.2x + 0.3 & \alpha \in [0.6, 1] \end{cases}$$

$$u'' + 1 = 0 \quad 0 \leq \alpha \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

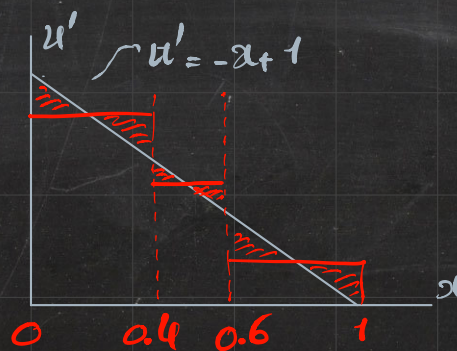
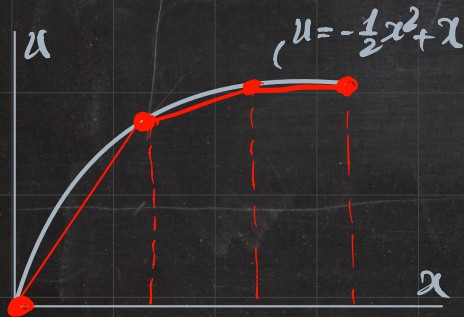
$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \text{ANALYTICAL} \end{cases} \quad \omega|_D = 0$$



$$\int_0^1 \omega' u' dx = \int_0^1 \omega dx \quad x \in [0,1]$$

→ 3-PIECE LINEAR (GENERIC) APPROXIMATION

$$\begin{cases} u = 0.8x & x \in [0, 0.4] \\ u = 0.5x + 0.12 & x \in [0.4, 0.6] \\ u = 0.2x + 0.3 & x \in [0.6, 1] \end{cases}$$

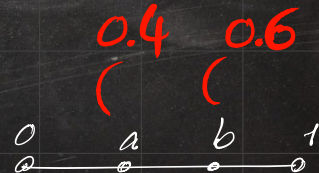


$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \text{ANALYTICAL} \end{cases} \quad \omega|_D = 0$$



$$\int_0^1 \omega' u' dx = \int_0^1 \omega dx \quad \alpha \in [0,1]$$

→ 4-PIECE LINEAR (GENERIC) APPROXIMATION

$$u'' + 1 = 0 \quad 0 \leq \alpha \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

$$\alpha \in [0, a] \quad \omega = C_1 \alpha + C_2$$

$$u = G_1 \alpha + G_2$$

$$\alpha \in [a, b] \quad \omega = D_1 \alpha + D_2$$

$$u = H_1 \alpha + H_2$$

$$\alpha \in [b, c] \quad \omega = E_1 \alpha + E_2$$

$$u = I_1 \alpha + I_2$$

$$\alpha \in [c, 1] \quad \omega = F_1 \alpha + F_2$$

$$u = J_1 \alpha + J_2$$

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \text{ANALYTICAL} \end{cases} \quad \omega|_D = 0$$



$$\int_0^1 \omega' u' dx = \int_0^1 \omega dx \quad x \in [0,1]$$

→ 4-PIECE LINEAR (GENERIC) APPROXIMATION

$$x \in [0,a] \quad \omega = G_1 x + G_2 \quad u = G_1 x + G_2$$

$$x \in [a,b] \quad \omega = D_1 x + D_2 \quad u = H_1 x + H_2$$

$$x \in [b,c] \quad \omega = E_1 x + E_2 \quad u = I_1 x + I_2$$

$$x \in [c,1] \quad \omega = F_1 x + F_2 \quad u = J_1 x + J_2$$

$$H_2 = a [G_1 - H_1]$$

$$I_2 = b [H_1 - I_1] + a [G_1 - H_1]$$

$$J_2 = c [I_1 - J_1]$$

$$+ b [H_1 - I_1] + a [G_1 - H_1]$$

$$D_2 = a [G_1 - D_1]$$

$$E_2 = b [D_1 - E_1] + a [G_1 - D_1]$$

$$F_2 = c [E_1 - F_1] + b [D_1 - E_1] + a [G_1 - D_1]$$

$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \text{ANALYTICAL} \end{cases} \quad \omega|_D = 0$$

$$0 < a < b < c < 1$$

$$\int_0^1 \omega' u' dx = \int_0^1 \omega dx \quad x \in [0,1]$$

→ 4-PIECE LINEAR (GENERIC) APPROXIMATION

$$x \in [0,a] \quad \omega = C_1 x + C_2 \quad u = G_1 x + G_2$$

$$x \in [a,b] \quad \omega = D_1 x + D_2 \quad u = H_1 x + H_2$$

$$x \in [b,c] \quad \omega = E_1 x + E_2 \quad u = I_1 x + I_2$$

$$x \in [c,1] \quad \omega = F_1 x + F_2 \quad u = J_1 x + J_2$$

$$\int_0^a \omega' u' dx + \int_a^b \omega' u' dx + \int_b^c \omega' u' dx + \int_c^1 \omega' u' dx$$

$$= \int_0^a \omega dx + \int_a^b \omega dx + \int_b^c \omega dx + \int_c^1 \omega dx$$

$$0 < a < b < c < 1$$


$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \text{ANALYTICAL} \end{cases} \quad \omega|_D = 0$$

$$\int_0^1 \omega' u' dx = \int_0^1 \omega dx \quad \alpha \in [0,1]$$

→ 4-PIECE LINEAR (GENERIC) APPROXIMATION

$$\alpha \in [0,a] \quad \omega = C_1 \alpha + C_2 \quad u = G_1 \alpha + G_2$$

$$\alpha \in [a,b] \quad \omega = D_1 \alpha + D_2 \quad u = H_1 \alpha + H_2$$

$$\alpha \in [b,c] \quad \omega = E_1 \alpha + E_2 \quad u = I_1 \alpha + I_2$$

$$\alpha \in [c,1] \quad \omega = F_1 \alpha + F_2 \quad u = J_1 \alpha + J_2$$

$$\begin{aligned} \int_0^a \omega' u' dx + \int_a^b \omega' u' dx + \int_b^c \omega' u' dx + \int_c^1 \omega' u' dx \\ = \int_0^a \omega dx + \int_a^b \omega dx + \int_b^c \omega dx + \int_c^1 \omega dx \end{aligned} \quad \dots$$

$$P(G_1) G_1 + P(H_1) D_1 \quad \checkmark C_1, D_1, E_1, F_1$$

$$+ P(I_1) E_1$$

$$+ P(J_1) F_1 = 0$$



$$u'' + 1 = 0 \quad 0 \leq \alpha \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \text{ANALYTICAL} \end{cases} \quad \omega|_D = 0$$

$$\int_0^1 \omega' u' dx = \int_0^1 \omega dx \quad \alpha \in [0,1]$$

→ 4-PIECE LINEAR (GENERIC) APPROXIMATION

$$\alpha \in [0,a] \quad \omega = G_1 \alpha + G_2 \quad u = G_1 \alpha + G_2$$

$$\alpha \in [a,b] \quad \omega = D_1 \alpha + D_2 \quad u = H_1 \alpha + H_2$$

$$\alpha \in [b,c] \quad \omega = E_1 \alpha + E_2 \quad u = I_1 \alpha + I_2$$

$$\alpha \in [c,1] \quad \omega = F_1 \alpha + F_2 \quad u = J_1 \alpha + J_2$$

$$\int_0^a \omega' u' dx + \int_a^b \omega' u' dx + \int_b^c \omega' u' dx + \int_c^1 \omega' u' dx$$

$$= \int_0^a \omega dx + \int_a^b \omega dx + \int_b^c \omega dx + \int_c^1 \omega dx \Rightarrow$$

✓ G_1, D_1, E_1, F_1

$$G_1 = 1 - \frac{1}{2}a \quad \text{ARBITRARY}$$

$$H_1 = 1 - \frac{1}{2}[a+b]$$

$$I_1 = 1 - \frac{1}{2}[b+c]$$

$$J_1 = 1 - \frac{1}{2}[c+1]$$

$\omega : \left\{ \begin{array}{l} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{array} \right.$



$$u'' + 1 = 0 \quad 0 \leq \alpha \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

$$\int_0^1 \omega' u' dx = \int_0^1 \omega dx \quad x \in [0,1]$$

→ 4-PIECE LINEAR (GENERIC) APPROXIMATION

$$\begin{cases} u = [1 - \frac{1}{2}a]x & x \in [0, a] \\ u = [1 - \frac{1}{2}(a+b)]x + \frac{1}{2}ab & x \in [a, b] \\ u = [1 - \frac{1}{2}(b+c)]x + \frac{1}{2}bc & x \in [b, c] \\ u = [1 - \frac{1}{2}(c+1)]x + \frac{1}{2}c & x \in [c, 1] \end{cases}$$

↳ $u = [1 - \frac{1}{2}(\alpha + \beta)]x + \frac{1}{2}\alpha\beta \quad x \in [\alpha, \beta]$

$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \text{ANALYTICAL} \end{cases} \quad \omega|_D = 0$$

... ⇒ n-piece LINEAR (GENERIC)

$$0 < a < b < c < 1$$

$$\int_0^1 \omega' u' dx = \int_0^1 \omega dx \quad x \in [0,1]$$

→ 4-PIECE LINEAR (GENERIC) APPROXIMATION

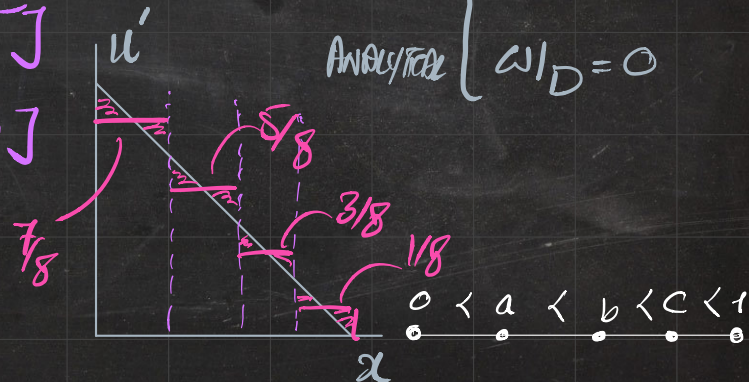
$$\left\{ \begin{array}{l} u = 7/8 x \quad x \in [0, 0.25] \\ u = 5/8 x + 1/6 \quad x \in [0.25, 0.50] \\ u = 3/8 x + 3/16 \quad x \in [0.50, 0.75] \\ u = 1/8 x + 6/16 \quad x \in [0.75, 1.00] \end{array} \right.$$

$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

$$\omega : \left\{ \begin{array}{l} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{array} \right.$$



$$\int_0^1 \omega' u' dx = \int_0^1 \omega dx \quad x \in [0,1]$$

→ 1-PECE QUADRATIC APPROXIMATION

$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

$$x \in [0,1] \quad \omega = C_1 x^2 + C_2 x + C_3 \quad \omega|_D = 0$$

$$u = D_1 x^2 + D_2 x + D_3 \quad u|_D = 0$$

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \text{ANALYTICAL} \end{cases} \quad \omega|_D = 0$$

$$\int_0^1 \omega' u' dx = \int_0^1 \omega dx \quad \dots$$

$\left. \begin{matrix} 2D_1 x + D_2 \\ 2C_1 x + C_2 \end{matrix} \right\} C_1 x^2 + C_2 x$

$$\Rightarrow C_1 \left[\frac{4D_1}{3} - \frac{1}{3} + \frac{D_2}{2} \right] + C_2 \left[D_1 - \frac{1}{2} + \frac{D_2}{2} \right] = 0$$

$$\sqrt{C_1, C_2}$$



$$\int_0^1 \omega' u' dx = \int_0^1 \omega dx \quad x \in [0,1]$$

→ 1-PECE QUADRATIC APPROXIMATION

$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ✓

$x \in [0,1]$ $\omega = C_1 x^2 + C_2 x + C_3$ $u = D_1 x^2 + D_2 x + D_3$

↖ $\omega|_D = 0$ ↖ $u|_D = 0$

ω : { ARBITRARY
CONTINUOUS
ANALYTICAL } $\omega|_D = 0$

$$\begin{cases} \frac{4}{3} D_1 + D_2 - \frac{1}{3} = 0 \\ D_1 + D_2 - \frac{1}{2} = 0 \end{cases} \Rightarrow \begin{cases} D_1 = -\frac{1}{2} \\ D_2 = 1 \end{cases}$$

approximation that has zero error

$$u = -\frac{1}{2} x^2 + x$$

IDENTICAL TO ANALYTICAL SOLUTION

0 ————— 1

$$\int_0^1 \omega' u' dx = \int_0^1 \omega dx \quad x \in [0,1]$$

→ 1-PECE QUADRATIC APPROXIMATION

$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

$$x \in [0,1] \quad \omega = C_1 x^2 + C_2 x + C_3 \quad \omega|_D = 0$$

$$u = D_1 x^2 + D_2 x + D_3 \quad u|_D = 0$$

ω :
 { ARBITRARY
 CONTINUOUS
 ANALYTICAL
 $\omega|_D = 0$

IF THE APPROXIMATION SPACE IS

LARGE ENOUGH, IT CAN INCLUDE

THE EXACT SOLUTION!

$$u = -\frac{1}{2}x^2 + x$$

IDENTICAL TO ANALYTICAL SOLUTION

approximation that has zero error



$$\int_0^1 \omega' u' d\alpha = \int_0^1 \omega d\alpha \quad \alpha \in [0,1]$$

→ 1-PECE QUADRATIC APPROXIMATION

$$u'' + 1 = 0 \quad 0 \leq \alpha \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

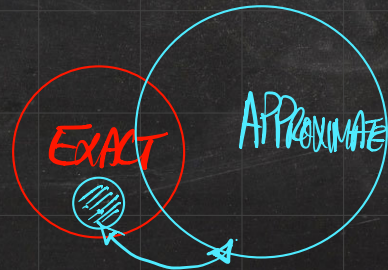
$$N: u'(1) = 0 \quad \checkmark$$

$$\alpha \in [0,1] \quad \omega = C_1 \alpha^2 + C_2 \alpha + C_3 \quad \omega|_D = 0$$

$$u = D_1 \alpha^2 + D_2 \alpha + D_3 \quad u|_D = 0$$

$\omega : \left\{ \begin{array}{l} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \text{ANALYTICAL} \end{array} \right. \omega|_D = 0$

IF THE APPROXIMATION SPACE IS
LARGE ENOUGH, IT CAN INCLUDE
THE EXACT SOLUTION!



$$\int_0^1 \omega' u' dx = \int_0^1 \omega dx \quad x \in [0,1]$$

→ 1-PECE QUADRATIC APPROXIMATION

$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

$$x \in [0,1] \quad \omega = C_1 x^2 + C_2 x + C_3 \quad \omega|_D = 0$$

$$u = D_1 x^2 + D_2 x + D_3 \quad u|_D = 0$$

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \text{ANALYTICAL} \end{cases} \quad \omega|_D = 0$$

IF THE APPROXIMATION SPACE IS
LARGE ENOUGH, IT CAN INCLUDE
THE EXACT SOLUTION!



APPROXIMATION
SPACE COINCIDES
WITH THE
REQUIRED SPACE OF
EXACT SOLUTION

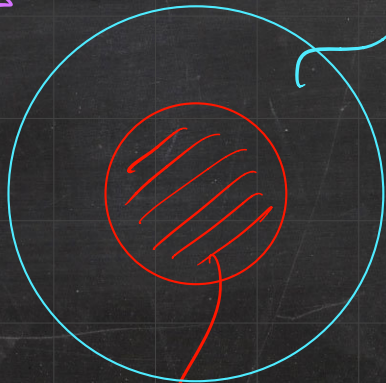
$$\int_0^1 w' u' dx = \int_0^1 w dx \quad x \in [0,1]$$

→ 1-PECE CUBIC APPROXIMATION

$$2 \rightarrow a x^3 + b x^2 + c x + d$$

APPROXIMATION SPACE (CUBIC)

$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \end{cases}$
 ANALYTICAL $w|_D = 0$



QUADRATIC SPACE



APPROXIMATION SPACE COINCIDES WITH THE REQUIRED SPACE OF EXACT SOLUTION

$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ✓

$$\int_0^1 \omega' u' dx = \int_0^1 \omega dx \quad x \in [0,1]$$

→ 1-PECE CUBIC APPROXIMATION

$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

$$x \in [0,1] \quad \omega = C_1 x^3 + C_2 x^2 + C_3 x + C_4$$

$$u = D_1 x^3 + D_2 x^2 + D_3 x + D_4$$

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \text{ANALYTICAL} \end{cases} \quad \omega|_D = 0$$

$$\int_0^1 [3C_1 x^2 + 2C_2 x + C_3] [3D_1 x^2 + 2D_2 x + D_3] dx = \int_0^1 [C_1 x^3 + C_2 x^2 + C_3 x] dx$$

$$\int_0^1 \omega' u' dx = \int_0^1 \omega dx \quad x \in [0,1]$$

→ 1-PECE CUBIC APPROXIMATION

$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

$$x \in [0,1] \quad \omega = C_1 x^3 + C_2 x^2 + C_3 x + C_4$$

$$u = D_1 x^3 + D_2 x^2 + D_3 x + D_4$$

$$\int_0^1 [3C_1 x^2 + 2C_2 x + C_3] [3D_1 x^2 + 2D_2 x + D_3] dx = \int_0^1 [C_1 x^3 + C_2 x^2 + C_3 x] dx$$

$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \text{ANALYTICAL} \end{cases} \quad \omega|_D = 0$

$$\dots \Rightarrow C_1 \left[\frac{9}{5} D_1 + \frac{6}{4} D_2 + D_3 - \frac{1}{4} \right]$$

✓ C_1, C_2, C_3

$$+ C_2 \left[\frac{6}{4} D_1 + \frac{4}{3} D_2 + D_3 - \frac{1}{3} \right] + C_3 \left[D_1 + \frac{1}{2} D_2 + D_3 - \frac{1}{2} \right] = 0$$

$$\int_0^1 \omega' u' dx = \int_0^1 \omega dx \quad \alpha \in [0,1]$$

→ 1-PECE CUBIC APPROXIMATION

$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

$$\alpha \in [0,1] \quad u = D_1 \alpha^3 + D_2 \alpha^2 + D_3 \alpha + \cancel{D_4}$$

$$\begin{aligned} \dots \Rightarrow C_1 \left[\frac{1}{5} D_1 + \frac{6}{4} D_2 + D_3 - \frac{1}{4} \right] & \quad \checkmark C_1, C_2, C_3 \\ + C_2 \left[\frac{6}{4} D_1 + \frac{4}{3} D_2 + D_3 - \frac{1}{3} \right] & + C_3 \left[D_1 + D_2 + D_3 - \frac{1}{2} \right] = 0 \end{aligned}$$

$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \text{ANALYTICAL} \end{cases} \quad \omega|_D = 0$

$$\begin{cases} \left[\frac{1}{5} D_1 + \frac{6}{4} D_2 + D_3 - \frac{1}{4} \right] = 0 \\ \left[\frac{6}{4} D_1 + \frac{4}{3} D_2 + D_3 - \frac{1}{3} \right] = 0 \\ \left[D_1 + D_2 + D_3 - \frac{1}{2} \right] = 0 \end{cases}$$

$$\Rightarrow \begin{cases} D_1 = 0 \\ D_2 = -\frac{1}{2} \\ D_3 = 1 \end{cases}$$

if approximation space is large enough, we recover the exact solution!

$$\Rightarrow u = -\frac{1}{2} \alpha^2 + \alpha$$

FINITE ELEMENT METHOD IN A NUTSHELL

Diff. Eq.
 2ND. O.D.E.
 $(EAu')' + b = 0$

STRONG FORM

(I) MULTIPLY BY w (test function)
 (II) INTEGRATE

WEAK FORM

INTEGRAL FORM

$$\int_0^1 w'u' dx = \int_0^1 w da + w(1)u'(1) - w(0)u'(0)$$

PIECEWISE

APPROXIMATE FORM

Approximate Discretized Weak Form

Approximation

DISCRETIZED FORM

NUMERICAL INTEGRATION
 another source of approx...

ELEMENT-WISE QUANTITIES

SOLVE

PostProcess

GLOBAL SYSTEM

FROM GLOBAL TO ELEMENTS

FROM INTEGRAL OVER THE DOMAIN TO SUBINTEGRALS

$$\int_0^1 \dots dx = \int_0^a \dots dx + \int_a^b \dots dx + \dots$$

$$[K][u] = [F]$$

ASSEMBLY

PIECEWISE INTEGRALS (SOLUTIONS)

APPROXIMATION:

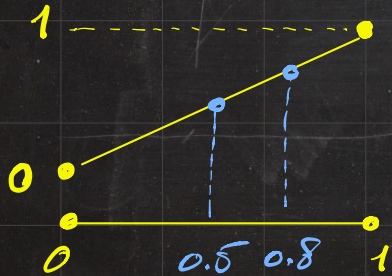
UNDERSTANDING VIA EXAMPLES

(I) $u(0) = 0$

$u(1) = 1$

$u(0.5) = ? \approx 0.5$

$u(0.8) = ? \approx 0.8$

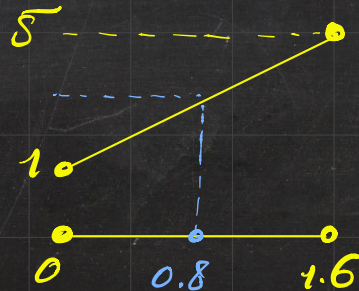


(II) $u(0) = 1$

$u(1.6) = 5$

$u(0.8) = ? \approx 3$

$u(1) = ? \approx 3.5$

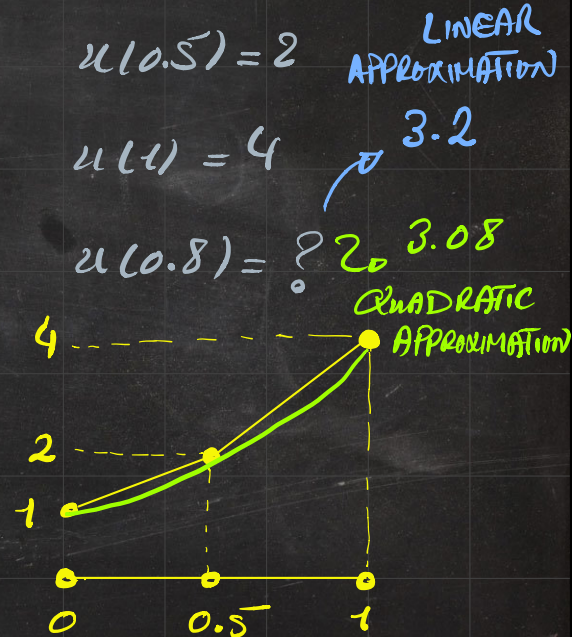


(III) $u(0) = 1$

$u(10.5) = 2$

$u(1) = 4$

$u(0.8) = ? \approx 3.08$



APPROXIMATION: UNDERSTANDING VIA EXAMPLES

QUADRATIC APPROXIMATION?

$$(III) u(0) = 1$$

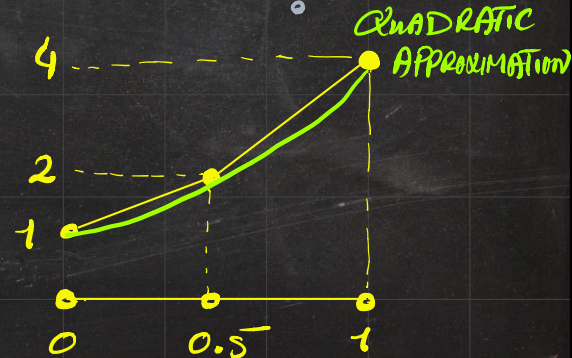
$$u(0.5) = 2 \quad \text{LINEAR APPROXIMATION}$$

$$u(1) = 4 \quad \rightarrow 3.2$$

$$u(0.8) = ? \rightarrow 3.08$$

$$f(x) = ax^2 + bx + c \Rightarrow f(x) = 2x^2 + x + 1$$

$$\left. \begin{array}{l} f(0) = 1 \\ f(0.5) = 2 \\ f(1) = 4 \end{array} \right\} \Rightarrow \begin{array}{l} 3 \text{ EQNS} \\ 3 \text{ UNKNOWN S} \end{array} \Rightarrow \begin{array}{l} a = 2 \\ b = 1 \\ c = 1 \end{array}$$



APPROXIMATION :

UNDERSTANDING VIA EXAMPLES

(IV)

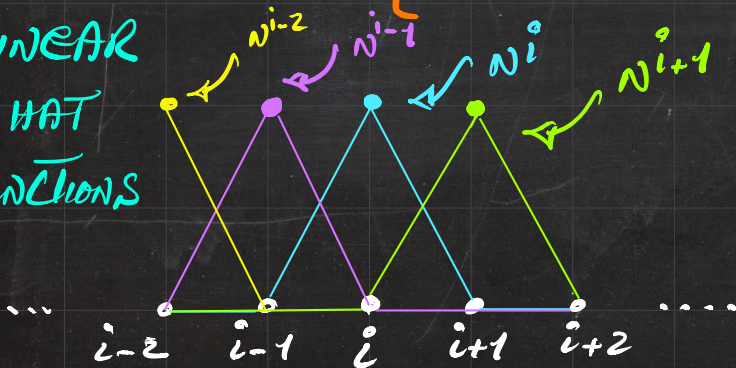
$$\left. \begin{array}{l} u(0) = 1 \\ u(0.2) = 2 \\ u(0.6) = 4 \\ u(1) = 8 \end{array} \right\} \Rightarrow u(0.8) = ?$$

SHAPE FUNCTIONS (HAT FUNCTIONS, TENT FUNCTIONS)

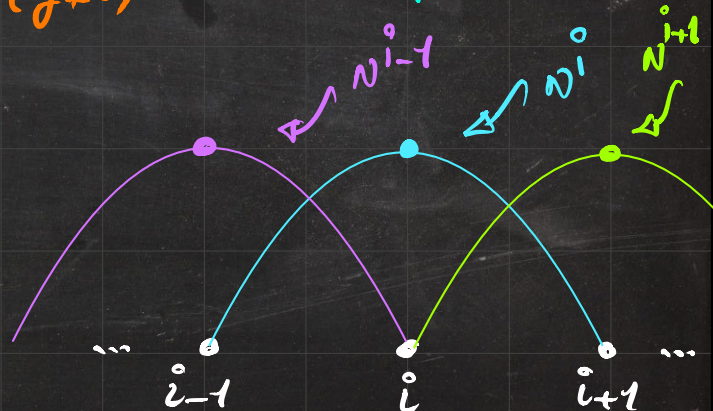
↳ A powerful tool for approximations \rightarrow SYSTEMATIC

$$N^i(x) \rightarrow \begin{cases} N^i = 1 @ x^j (j=i) \\ N^i = 0 @ x^j (j \neq i) \end{cases} \rightarrow \text{NEARLY IDENTICAL FOR 2D 3D}$$

LINEAR
HAT
FUNCTIONS



QUADRATIC HAT
FUNCTIONS



SHAPE FUNCTIONS (HAT FUNCTIONS, TENT FUNCTIONS)

↳ A powerful tool for approximations \rightarrow SYSTEMATIC

$$N^i(x) \rightarrow \begin{cases} N^i = 1 @ x^j (j=i) \\ N^i = 0 @ x^j (j \neq i) \end{cases} \rightarrow \text{NEARLY IDENTICAL FOR } \begin{matrix} 2D \\ 3D \end{matrix}$$

NODES
PER
ELEMENT \rightarrow
NPE

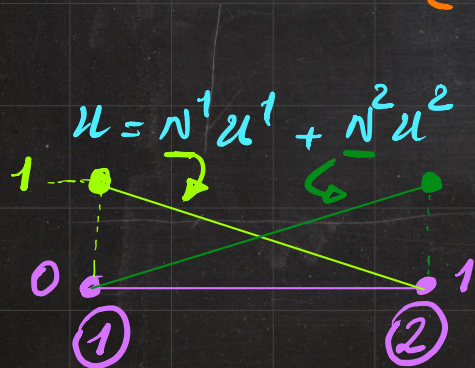
$$u = \sum_{i=1} N^i u^i$$

$$\left\{ \begin{array}{l} u = N^1 u^1 + N^2 u^2 \quad \swarrow \text{linear approximation} \\ u = N^1 u^1 + N^2 u^2 + N^3 u^3 \quad \swarrow \text{quadratic approximation} \end{array} \right.$$

SHAPE FUNCTIONS (HAT FUNCTIONS, TEST FUNCTIONS)

↳ A powerful tool for approximations \rightarrow SYSTEMATIC

$$N^i(x) \rightarrow \begin{cases} N^i = 1 @ x^j (j=i) \\ N^i = 0 @ x^j (j \neq i) \end{cases} \rightarrow \text{NEARLY IDENTICAL FOR } \begin{matrix} 2D \\ 3D \end{matrix}$$



$$(I) \Rightarrow \begin{cases} N^1 = 1-x \\ N^2 = x \end{cases}$$

$$u(0) = 0 \rightarrow u^1 = 0$$

$$u(1) = 1 \rightarrow u^2 = 1$$

$$u = [1-x] \times 0 + [x] \times 1 = x$$

$$u \approx x$$



SHAPE FUNCTIONS (HAT FUNCTIONS, TEST FUNCTIONS)

↳ A powerful tool for approximations \rightarrow SYSTEMATIC

$$N^i(x) \rightarrow \begin{cases} N^i = 1 @ x^j (j=i) \\ N^i = 0 @ x^j (j \neq i) \end{cases} \rightarrow \text{NEARLY IDENTICAL FOR } \begin{matrix} 2D \\ 3D \end{matrix}$$



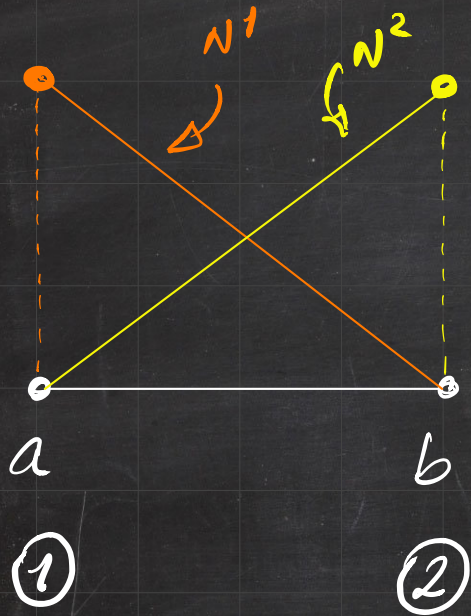
APPROXIMATION

OF u
ORDER M

$$u = \sum_{i=1}^{M+1} N^i u^i$$

Linear approximation $\rightarrow \sum_{i=1}^2$
 quadratic $\rightarrow \sum_{i=1}^3$
 cubic $\rightarrow \sum_{i=1}^4$

↳ NPE $\equiv M+1$



$$N^1 = \frac{x-b}{a-b}$$

$$N^2 = \frac{x-a}{b-a}$$

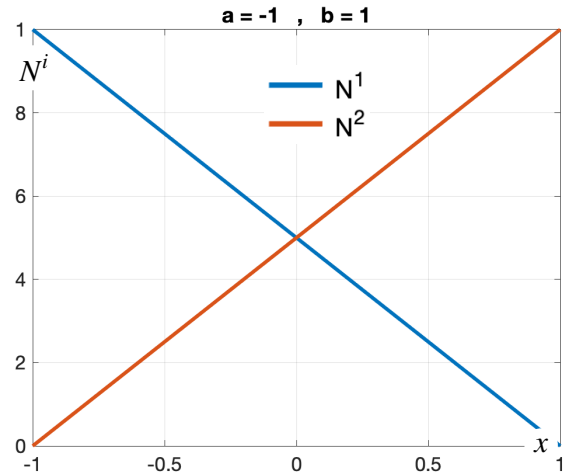
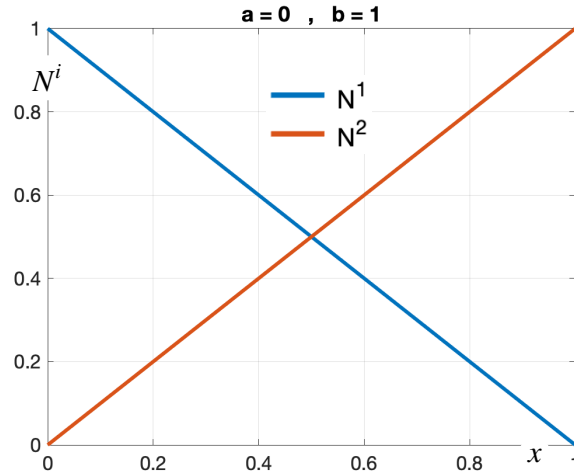
LINEAR
SHAPE
FUNCTIONS



1D Linear Shape Functions

$$N^1 = \frac{[x - b]}{[a - b]}$$

$$N^2 = \frac{[x - a]}{[b - a]}$$

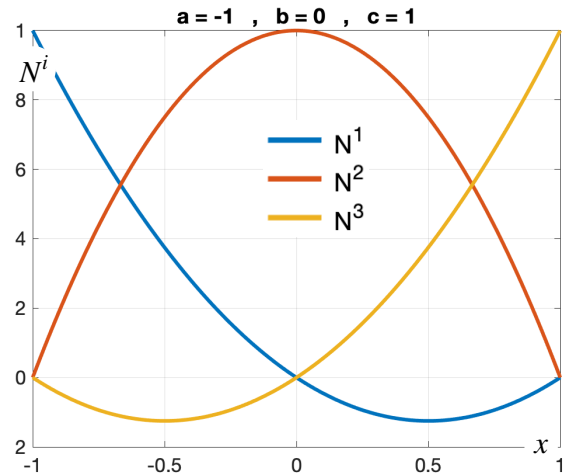
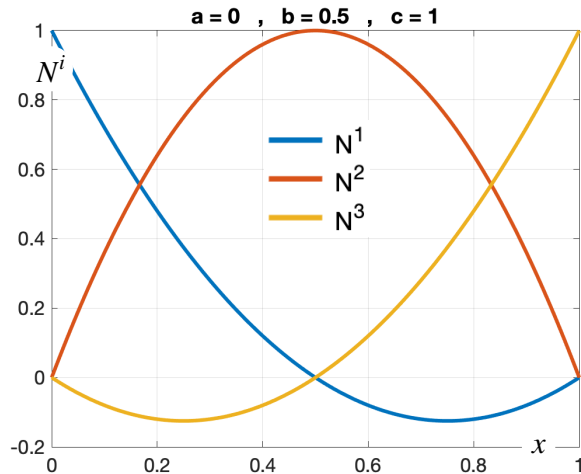


1D Quadratic Shape Functions

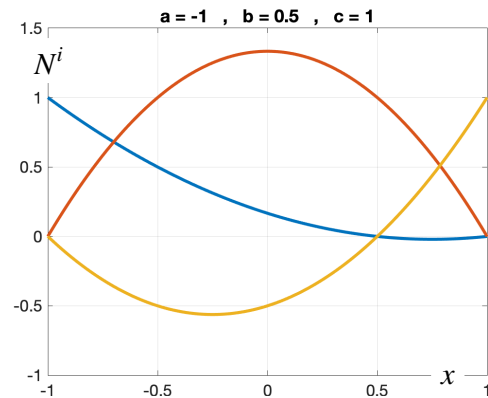
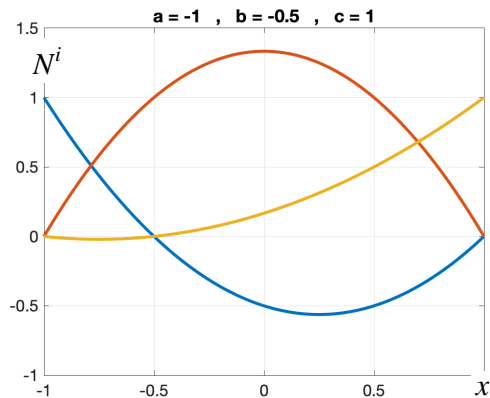
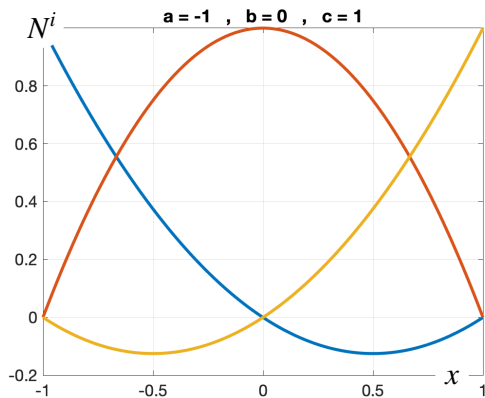
$$N^1 = \frac{[x - b][x - c]}{[a - b][a - c]}$$

$$N^2 = \frac{[x - a][x - c]}{[b - a][b - c]}$$

$$N^3 = \frac{[x - a][x - b]}{[c - a][c - b]}$$



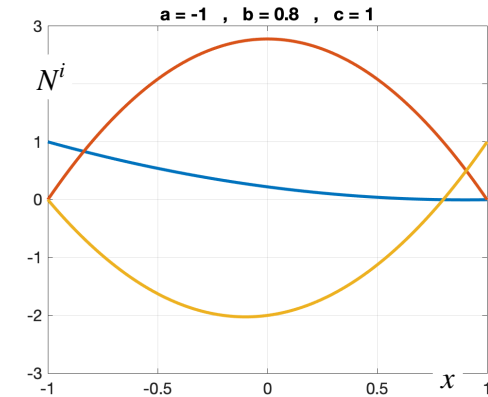
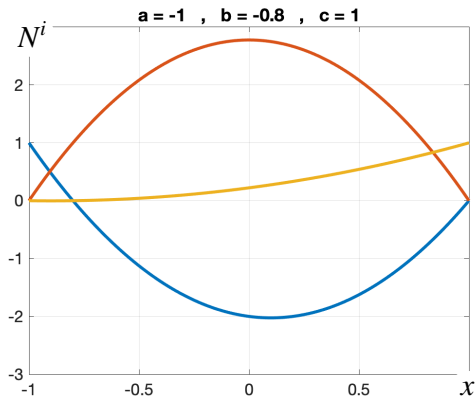
1D Quadratic Shape Functions



$$N^1 = \frac{[x - b][x - c]}{[a - b][a - c]}$$

$$N^2 = \frac{[x - a][x - c]}{[b - a][b - c]}$$

$$N^3 = \frac{[x - a][x - b]}{[c - a][c - b]}$$



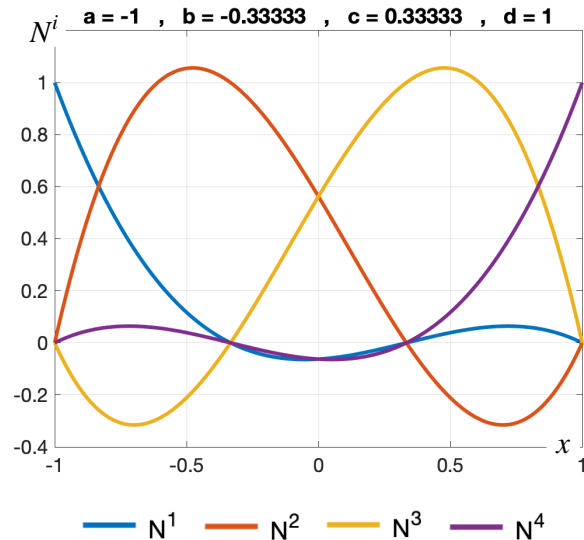
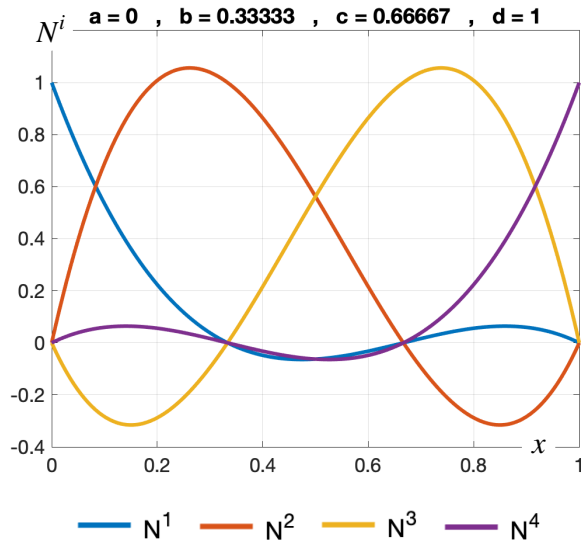
1D Cubic Shape Functions

$$N^1 = \frac{[x - b][x - c][x - d]}{[a - b][a - c][a - d]}$$

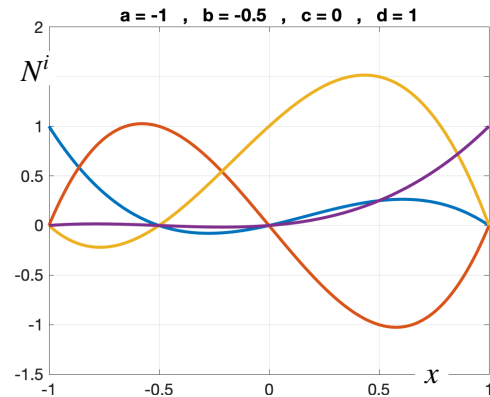
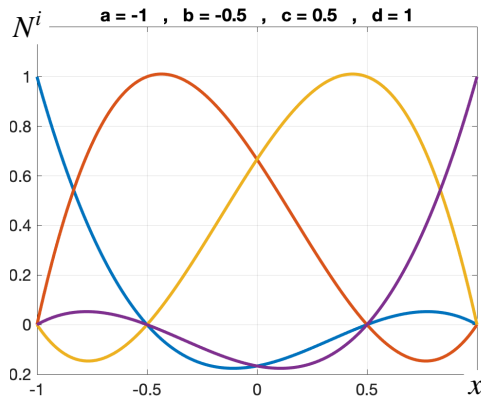
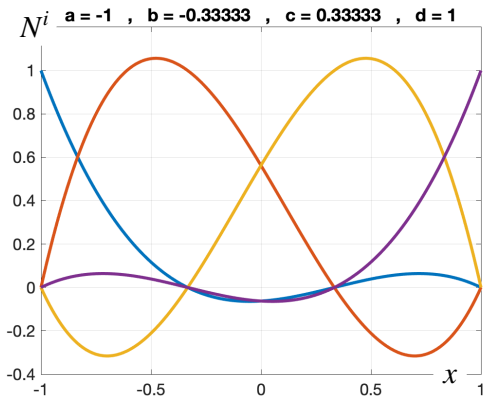
$$N^2 = \frac{[x - a][x - c][x - d]}{[b - a][b - c][b - d]}$$

$$N^3 = \frac{[x - a][x - b][x - d]}{[c - a][c - b][c - d]}$$

$$N^4 = \frac{[x - a][x - b][x - c]}{[d - a][d - b][d - c]}$$



1D Cubic Shape Functions



$$N^1 = \frac{[x - b][x - c][x - d]}{[a - b][a - c][a - d]}$$

$$N^2 = \frac{[x - a][x - c][x - d]}{[b - a][b - c][b - d]}$$

$$N^3 = \frac{[x - a][x - b][x - d]}{[c - a][c - b][c - d]}$$

$$N^4 = \frac{[x - a][x - b][x - c]}{[d - a][d - b][d - c]}$$

