FINITE ELEMENT METHOD

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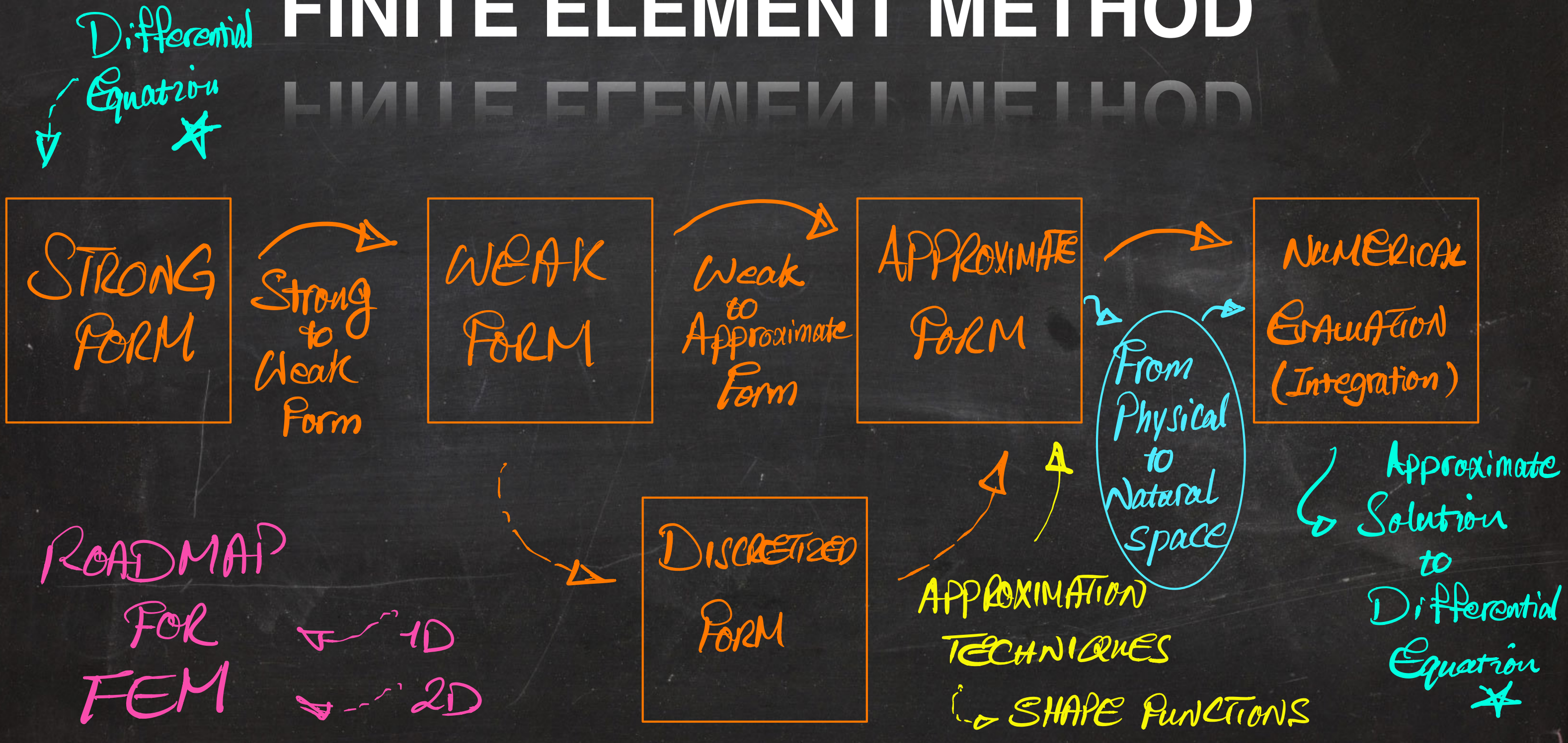
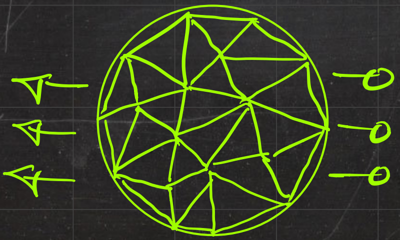


UNDERSTANDING FEM VIA AN ANALOGY (A BRUTAL SIMPLIFICATION)

Approximations in FEM

- → Solution Approximation → inherent to numerical techniques using computers
- → Equation Approximation → diff equation is transformed to weak form + space approximation
- → Input Approximation → space discretization



Discretization (Approximation)
- → DOMAIN (X)
- → SOLUTION (u)
- → TEST (w)

diff. Eq. → STRONG FORM

TO

integral → WEAK FORM

$$\frac{d}{dx}\left(EA\,\frac{du}{dx}\right)+bA=0 \quad \text{Subject to BCs}$$

$\hookrightarrow$ E, A: Const. $\rightsquigarrow$ $EA\,u'' + bA = 0$ $\leftarrow$ $f := \frac{b}{E}$

$\rightsquigarrow$

$$\boxed{\begin{array}{ll} u'' + f = 0 & 0 \leqslant x \leqslant 1 \\ D:\ u(0) = u_0 & \leftarrow \text{prescribed} \\ N:\ u'(1) = t & \leftarrow \end{array}}$$

STRONG FORM $\rightarrow$

FROM STRONG TO WEAK FORM

$$u'' + f = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = u_0$ ← prescribed

N: $u'(1) = t$

STRONG FORM ← Differential Eq.

(I) MULTIPLY BY TEST FUNCTION w

(II) INTEGRATE OVER THE DOMAIN

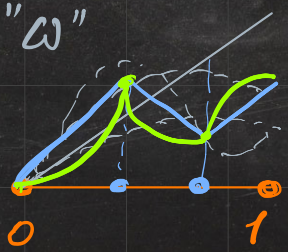
Integral form ← WEAK FORM

→ BECAUSE LOWER ORDER DIFFERENTIATION OF DISPLACEMENT u

w :
- ARBITRARY
- CONTINUOUS
- $w|_D = 0$ ← ZERO @ DIRICHLET BOUNDARY CONDITIONS

STRONG : u''

WEAK : u'



FROM STRONG TO WEAK FORM

$$u'' + f = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = u_0$ ← prescribed

N: $u'(1) = t$ ←

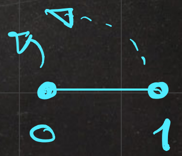
I) $[u'' + f = 0] \times w \Rightarrow wu'' + wf = 0$

II) $\int_0^1 [wu'' + wf]\,dx = 0$ $\qquad wu'' = (wu')' - w'u'$

↙ WEAK FORM

$$\underbrace{\int_0^1 w'u'\,dx}_{\text{INTERNAL CONTRIBUTIONS OVER THE DOMAIN}} = \underbrace{\int_0^1 wf\,dx}_{\text{EXTERNAL CONTRIBUTIONS OVER THE DOMAIN}} + \underbrace{w(1)u'(1) - w(0)u'(0)}_{\text{EXTERNAL CONTRIBUTIONS OVER THE BOUNDARY OF THE DOMAIN}}$$

$$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$$



FROM STRONG TO WEAK FORM

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ↙

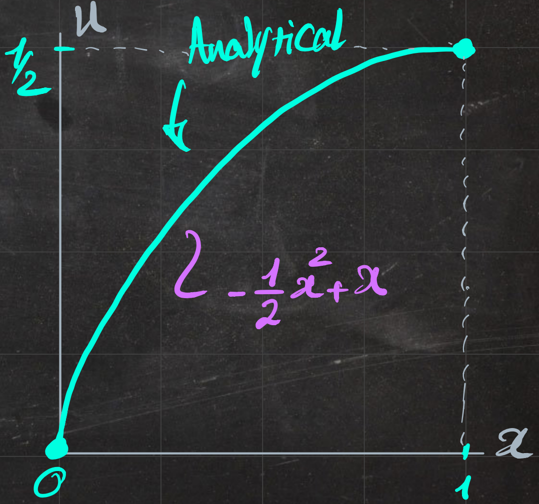
$$u'' = -1 \Rightarrow u' = -x + C_1$$

$$\Rightarrow u = -\frac{1}{2}x^2 + C_1 x + C_2$$

$\hookrightarrow u(0) = 0 \Rightarrow C_2 = 0$

$\hookrightarrow u'(1) = 0 \Rightarrow C_1 = 1$

Analytical Solution $\Rightarrow u = -\frac{1}{2}x^2 + x$



UNDERSTANDING THE SIGNIFICANCE OF WEAK FORM BY EXAMPLE

$$\int_0^1 \omega' u' dx = \int_0^1 \omega \, dx$$

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

- → 1-PIECE LINEAR APPROXIMATION
- → 2-PIECE LINEAR (UNIFORM) APPROXIMATION
- → 2-PIECE LINEAR (NON-UNIFORM) APPROXIMATION (I)
- → 2-PIECE LINEAR (NON-UNIFORM) APPROXIMATION (II)
- → 2-PIECE LINEAR (GENERAL) APPROXIMATION

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$$

UNDERSTANDING THE SIGNIFICANCE OF WEAK FORM BY EXAMPLE

$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx$$

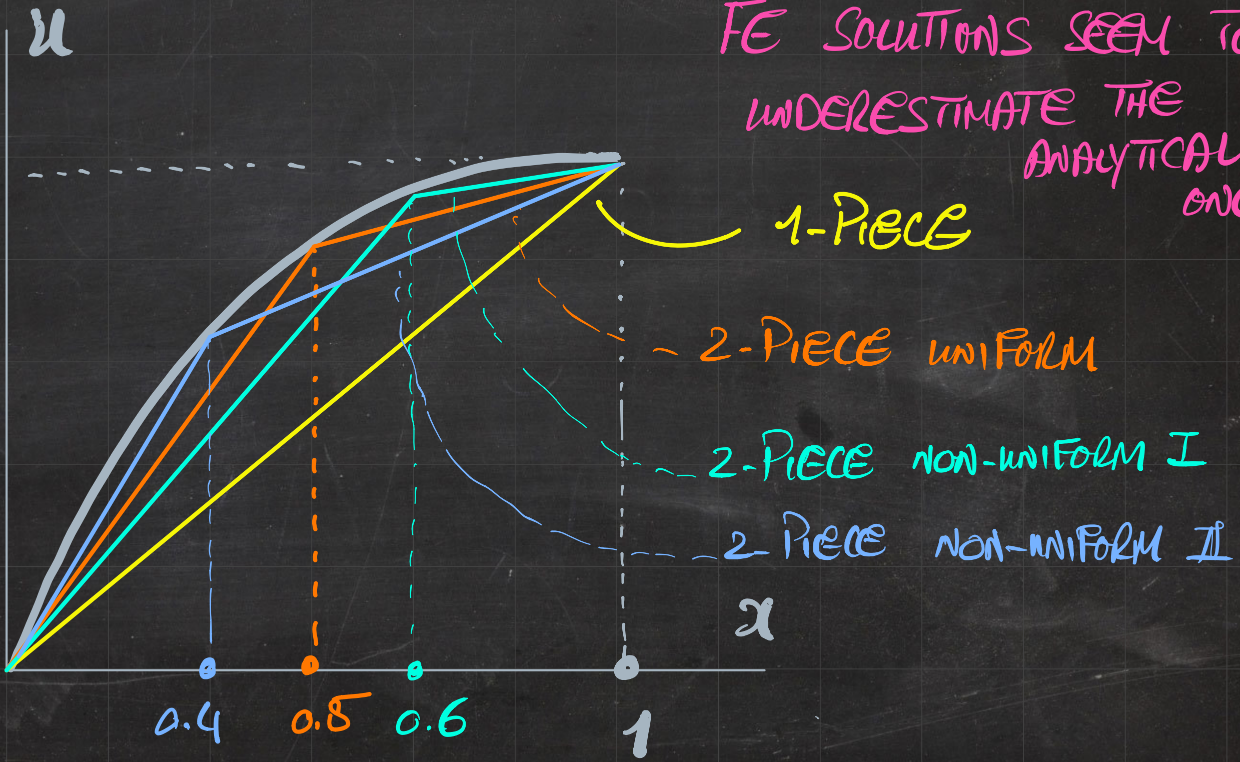
$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$

$$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$$

- → 3-PIECE LINEAR (UNIFORM) APPROXIMATION
- → 3-PIECE LINEAR (GENERIC) APPROXIMATION
- → 4-PIECE LINEAR (UNIFORM) APPROXIMATION
- → 4-PIECE LINEAR (GENERIC) APPROXIMATION
- → 1-PIECE QUADRATIC
- → 1-PIECE CUBIC



FE SOLUTIONS SEEM TO UNDERESTIMATE THE ANALYTICAL ONE!

FE Solution approaches analytical one from below!

$$\int_0^1 w'u'dx = \int_0^1 w\,dx \qquad x \in [0,1]$$

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$

→ 1-PIECE QUADRATIC APPROXIMATION

$w|_D = 0$

$$x \in [0,1] \qquad w = C_1 x^2 + C_2 x + \overset{0}{\cancel{C_3}} \qquad u = D_1 x^2 + D_2 x + \overset{0}{\cancel{D_3}}$$

$u(0) = 0$

w : ANALYTICAL { ARBITRARY, CONTINUOUS, $w|_D = 0$ }

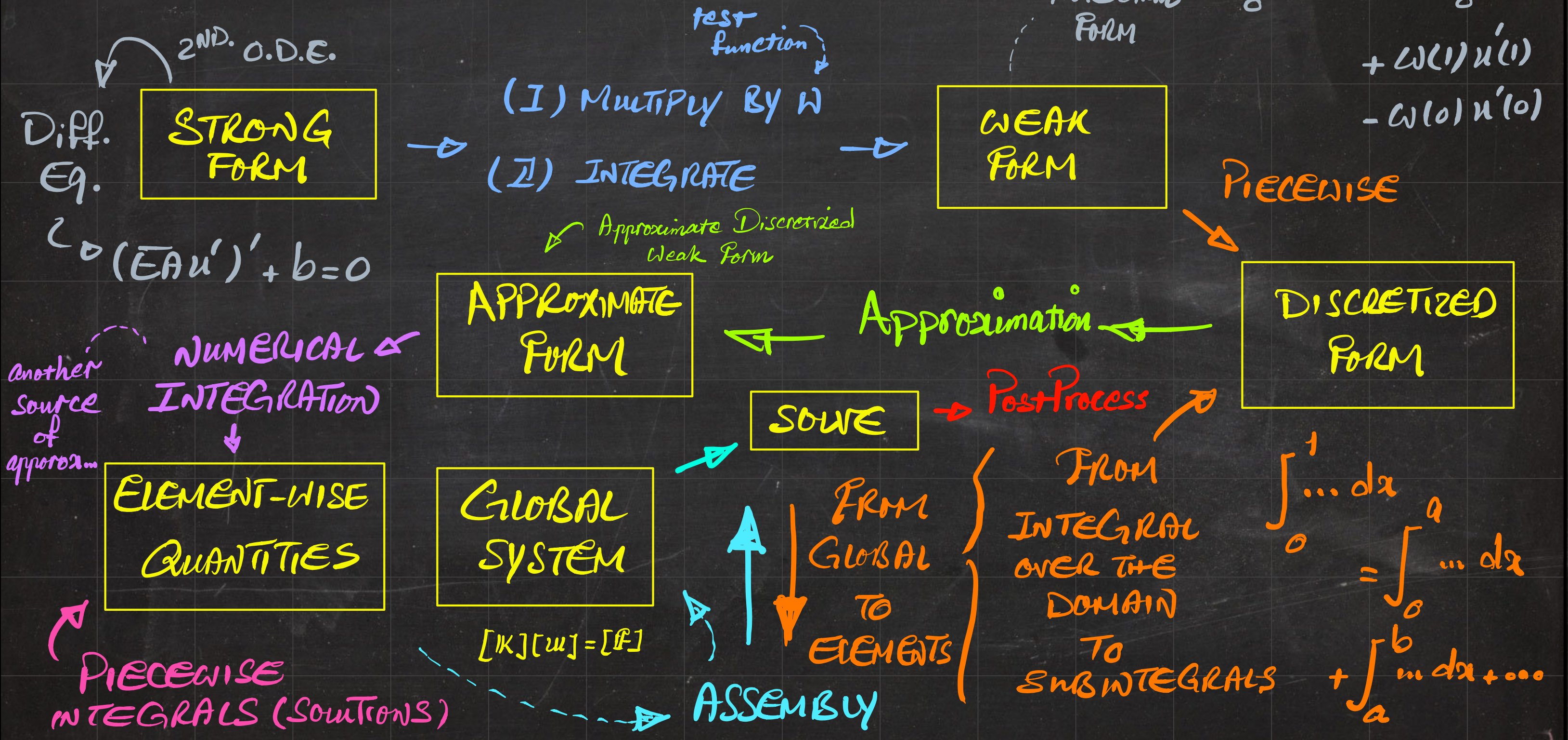
IF THE APPROXIMATION SPACE IS LARGE ENOUGH, IT CAN INCLUDE THE EXACT SOLUTION !

$$u = -\frac{1}{2}x^2 + x$$

approximation that has zero error

IDENTICAL TO ANALYTICAL SOLUTION

FINITE ELEMENT METHOD IN A NUTSHELL



$$\int_0^1 \omega' u' \, dx = \int_0^1 \omega \, dx + \omega(1)\, u'(1) - \omega(0)\, u'(0)$$

$$(EAu')' + b = 0$$

$$[K][u] = [F]$$

$$\int_0^1 \dots dx = \int_0^a \dots dx + \int_a^b \dots dx + \dots$$

APPROXIMATION : UNDERSTANDING VIA EXAMPLES

(I) $u(0) = 0$

$u(1) = 1$

$u(0.5) = ? \approx 0.5$

$u(0.8) = ? \approx 0.8$

(II) $u(0) = 1$

$u(1.6) = 5$

$u(0.8) = ? \approx 3$

$u(1) = ? \approx 3.5$

(III) $u(0) = 1$

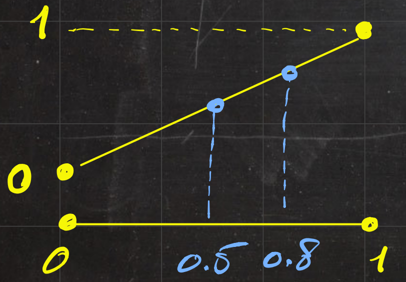
$u(0.5) = 2$

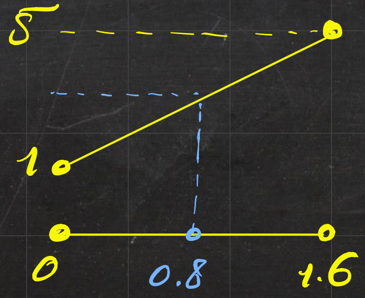
$u(1) = 4$

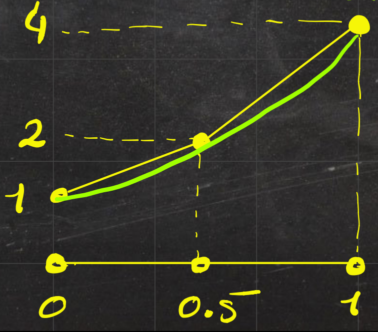
$u(0.8) = ? \approx 3.08$

LINEAR APPROXIMATION 3.2

QUADRATIC APPROXIMATION 3.08







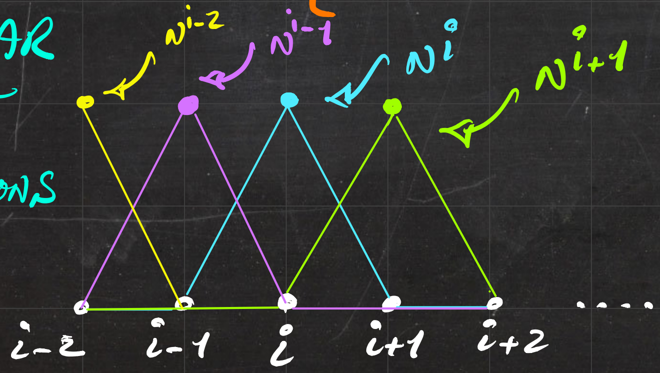
SHAPE FUNCTIONS (HAT FUNCTIONS, TENT FUNCTIONS)

↳ A powerful tool for approximations ⇝ SYSTEMATIC

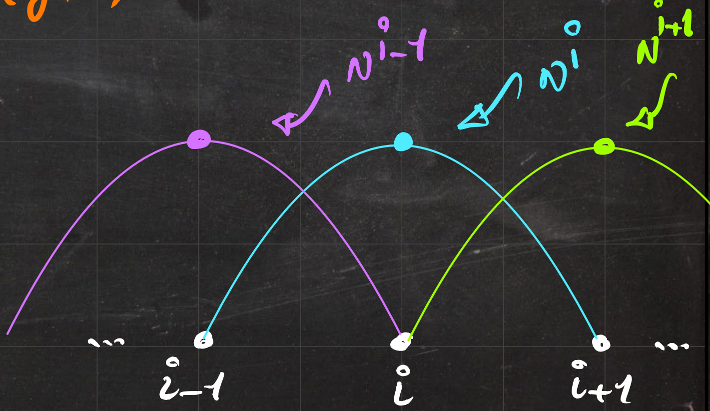
⇝ NEARLY IDENTICAL FOR 2D 3D

$$N^i(x) \rightsquigarrow \begin{cases} N^i = 1 \ @ \ x^j \ (j=i) \\ N^i = 0 \ @ \ x^j \ (j \neq i) \end{cases}$$

LINEAR HAT FUNCTIONS



QUADRATIC HAT FUNCTIONS



SHAPE FUNCTIONS (HAT FUNCTIONS , TENT FUNCTIONS)

↳ A powerful tool for approximations ⟿ SYSTEMATIC

⟿ NEARLY IDENTICAL FOR 2D / 3D

$$N^i(x) \rightsquigarrow \begin{cases} N^i = 1 \ @ \ x^j \ (j = i) \\ N^i = 0 \ @ \ x^j \ (j \neq i) \end{cases}$$

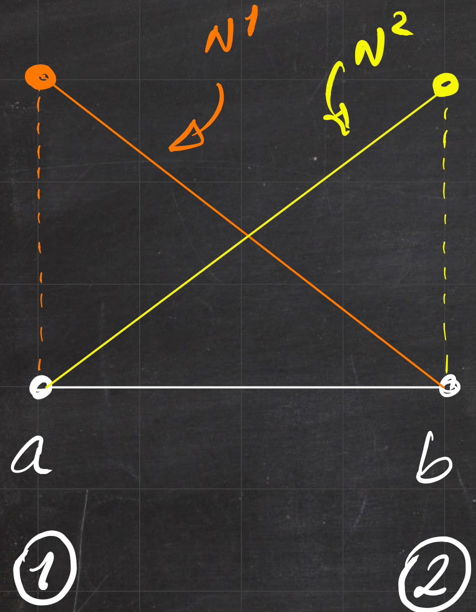
NODES PER ELEMENT ⟶ NPE

$$u \simeq \sum_{i=1}^{NPE} N^i u^i \longrightarrow \begin{cases} u = N^1 u^1 + N^2 u^2 \\ u = N^1 u^1 + N^2 u^2 + N^3 u^3 \\ u = N^1 u^1 + N^2 u^2 + N^3 u^3 + N^4 u^4 \end{cases}$$

linear approximation

quadratic approximation

cubic approximation



$$N^1 = \frac{x-b}{a-b}$$

$$N^2 = \frac{x-a}{b-a}$$

LINEAR
SHAPE
FUNCTIONS

LINEAR APPROXIMATION

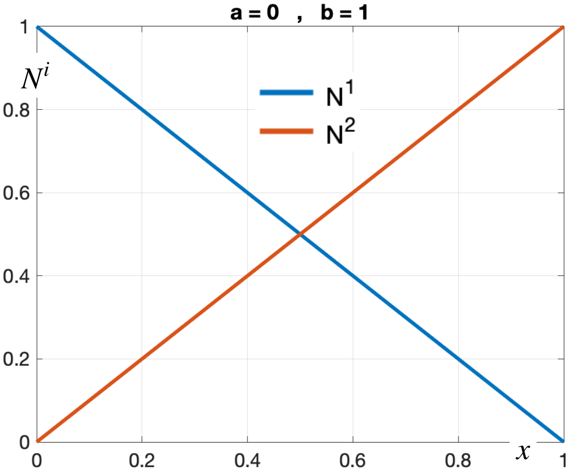
QUADRATIC APPROXIMATION

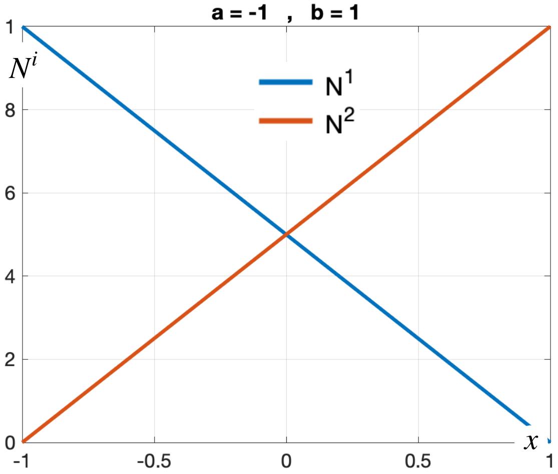
CUBIC APPROXIMATION

1D Linear Shape Functions

$$N^1 = \frac{[x - b]}{[a - b]}$$

$$N^2 = \frac{[x - a]}{[b - a]}$$



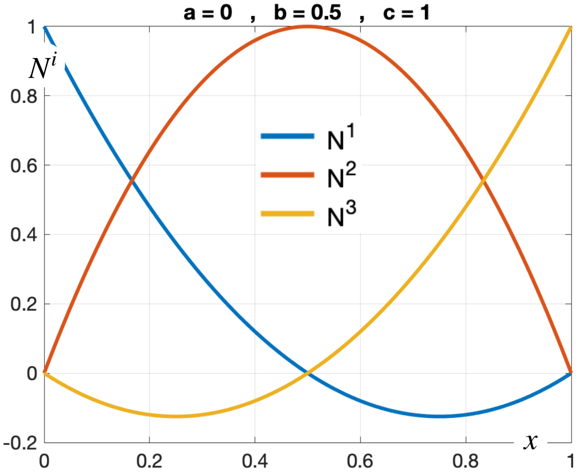


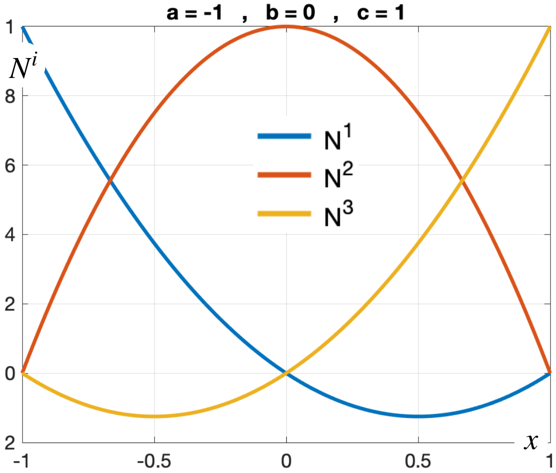
1D Quadratic Shape Functions

$$N^1 = \frac{[x-b][x-c]}{[a-b][a-c]}$$

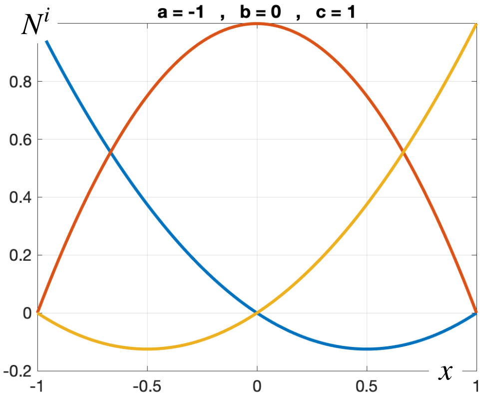
$$N^2 = \frac{[x-a][x-c]}{[b-a][b-c]}$$

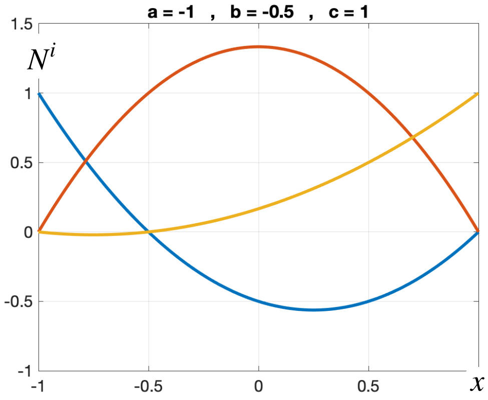
$$N^3 = \frac{[x-a][x-b]}{[c-a][c-b]}$$

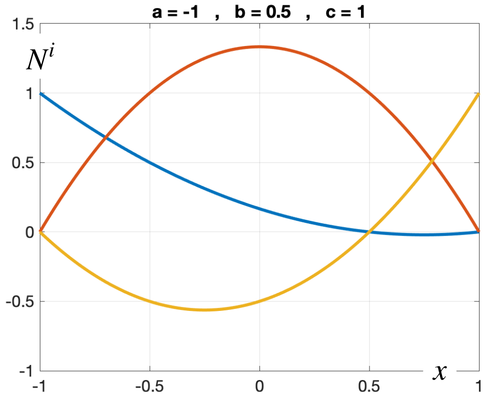




1D Quadratic Shape Functions



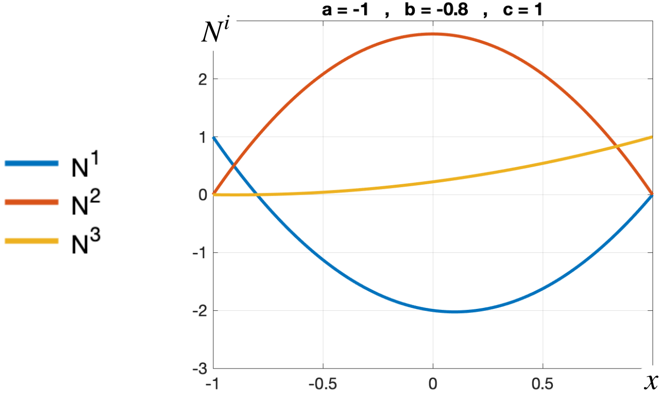


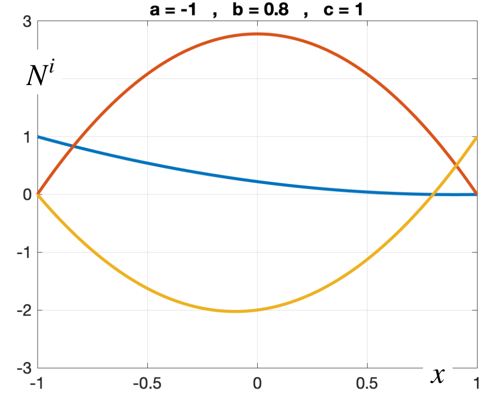


$$N^1 = \frac{[x-b][x-c]}{[a-b][a-c]}$$

$$N^2 = \frac{[x-a][x-c]}{[b-a][b-c]}$$

$$N^3 = \frac{[x-a][x-b]}{[c-a][c-b]}$$





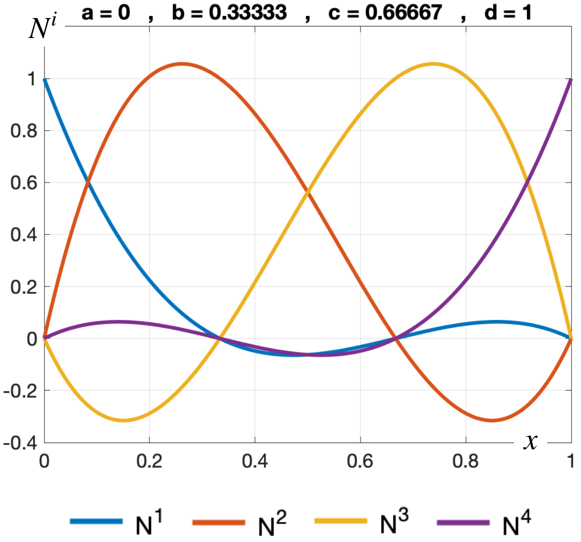
1D Cubic Shape Functions

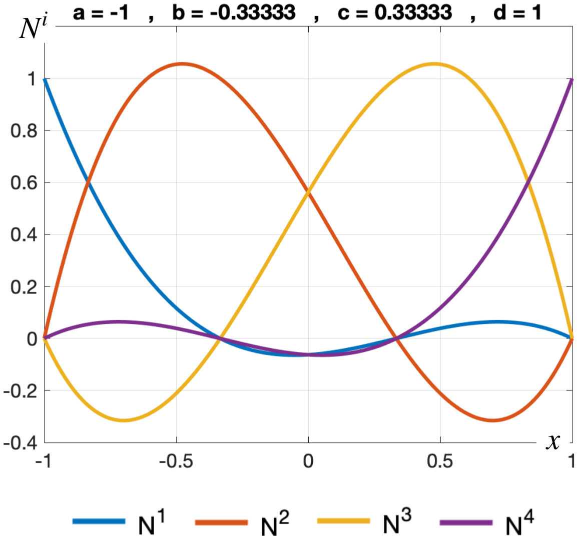
$$N^1 = \frac{[x-b][x-c][x-d]}{[a-b][a-c][a-d]}$$

$$N^2 = \frac{[x-a][x-c][x-d]}{[b-a][b-c][b-d]}$$

$$N^3 = \frac{[x-a][x-b][x-d]}{[c-a][c-b][c-d]}$$

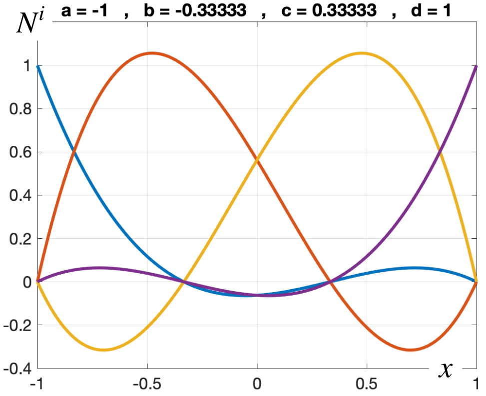
$$N^4 = \frac{[x-a][x-b][x-c]}{[d-a][d-b][d-c]}$$

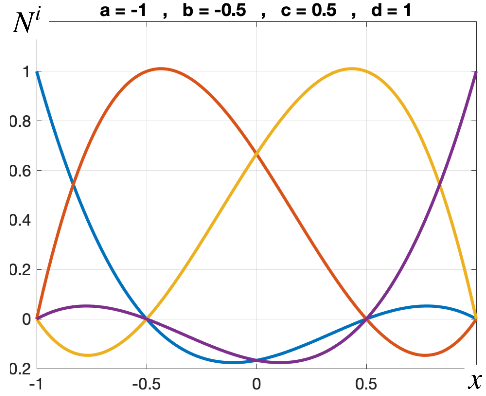


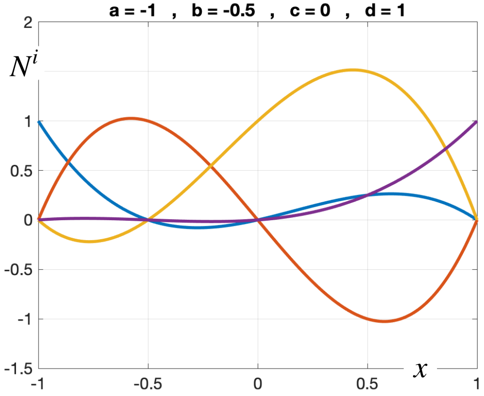


1D Cubic Shape Functions

Numerical Analysis
and High-level
Applications Lab
NAHAL







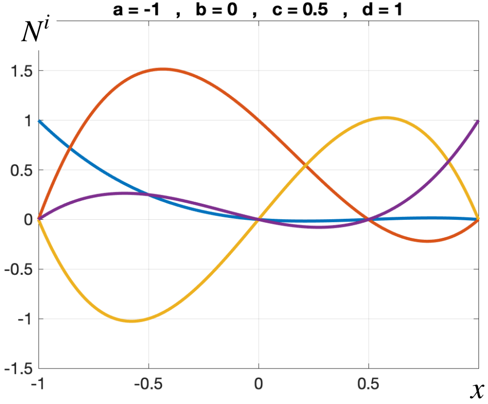
$$N^1 = \frac{[x-b][x-c][x-d]}{[a-b][a-c][a-d]}$$

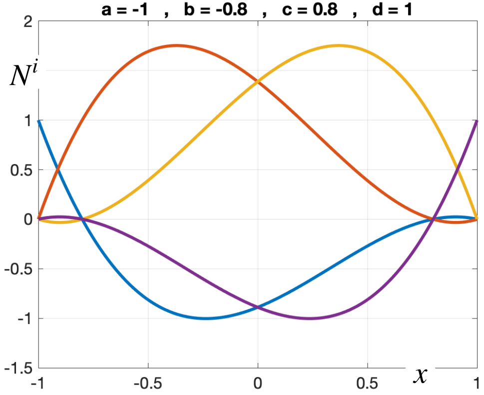
$$N^2 = \frac{[x-a][x-c][x-d]}{[b-a][b-c][b-d]}$$

$$N^3 = \frac{[x-a][x-b][x-d]}{[c-a][c-b][c-d]}$$

$$N^4 = \frac{[x-a][x-b][x-c]}{[d-a][d-b][d-c]}$$

🟦 N^1
🟧 N^2
🟨 N^3
🟪 N^4





APPROXIMATION: UNDERSTANDING VIA EXAMPLES

(I)

u^1 $\quad u(0) = 0$

u^2 $\quad u(1) = 1$

$u(0.5) = ? \approx 0.5$

$u(0.8) = ? \approx 0.8$

$$u = N^1 u^1 + N^2 u^2$$

$$= [1-x]\,\overset{0}{u^1} + x\,\overset{1}{u^2}$$

$$= x \Rightarrow u(x) = x \checkmark$$

(II)

u^1 $\quad u(0) = 1$

u^0 $\quad u(1.6) = 5$

$u(0.8) = ? \approx 3$

$u(1) = ? \approx 3.5$

$$u = N^1 \overset{1}{u^1} + N^2 \overset{5}{u^2}$$

$$= \frac{[x-1.6]}{-1.6}\,u^1 + \frac{[x-0]}{1.6}\,u^2$$

$$\Rightarrow u(x) = 2.5x + 1$$

(III)

u^1 $\quad u(0) = 1$

u^2 $\quad u(0.5) = 2$

u^3 $\quad u(1) = 4$

$u(0.8) = ?$

$$N^1 = \frac{[x-0.5][x-1]}{0.5}$$

$$N^2 = \frac{[x-0][x-1]}{-0.25}$$

$$N^3 = \frac{[x-0][x-0.5]}{0.5}$$

$$u = N^1 u^1 + N^2 u^2 + N^3 u^3$$

$$= 2[x^2 - 1.5x + 0.5] - 8[x^2 - x] + 8[x^2 - 0.5x]$$

$$\Rightarrow u(x) = 2x^2 + x + 1$$

UNDERSTANDING THE SIGNIFICANCE OF WEAK FORM BY EXAMPLE

$$\int_0^1 w'u'dx = \int_0^1 w\,dx$$

$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

- → 1-PIECE LINEAR APPROXIMATION
- → 2-PIECE LINEAR (UNIFORM) APPROXIMATION
- → 1-PIECE QUADRATIC APPROXIMATION

$$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$$

REMEMBER HOW WE DID BEFORE

$w = C_1 x + C_2 \qquad u = D_1 x + D_2$

$\ldots \int_0^1 w'u'dx = \int_0^1 w\,dx \ldots \; \forall w$

$\ldots \Rightarrow D_1 \,\&\, D_2$ ✓

UNDERSTANDING THE SIGNIFICANCE OF WEAK FORM BY EXAMPLE

$$\int_0^1 w'u'dx = \int_0^1 w\,dx$$

$u''+1=0 \qquad 0\leq x\leq 1$

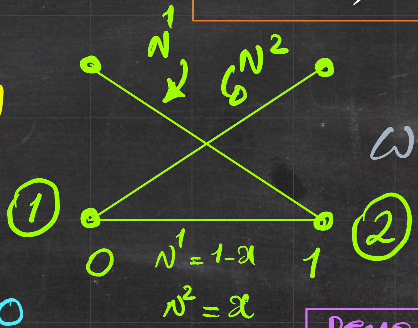
D: $u(0)=0$ ← prescribed

N: $u'(1)=0$ ←

→ 1-PIECE LINEAR APPROXIMATION

$w = N^1 \cancel{w^1} + N^2 w^2 \qquad u = N^1 \cancel{u^1} + N^2 u^2$

$w^1 = 0$ ← $w|_D = 0 \qquad u^1 = 0$ ← $u(0)=0$



$N^1 = 1-x$

$N^2 = x$

w : ARBITRARY, CONTINUOUS, $w|_D = 0$

$w = N^2 w^2 = x w^2 \qquad u = N^2 u^2 = x u^2 \Rightarrow u = \frac{1}{2}x$ ✓

$$\int_0^1 w^2 u^2\,dx = \int_0^1 x w^2\,dx \rightarrow w^2 u^2 = w^2 \frac{1}{2} \quad \forall w^2 \Rightarrow u^2 = \frac{1}{2}$$

REMEMBER HOW WE DID BEFORE

$w = C_1 x + C_2 \qquad u = D_1 x + D_2$

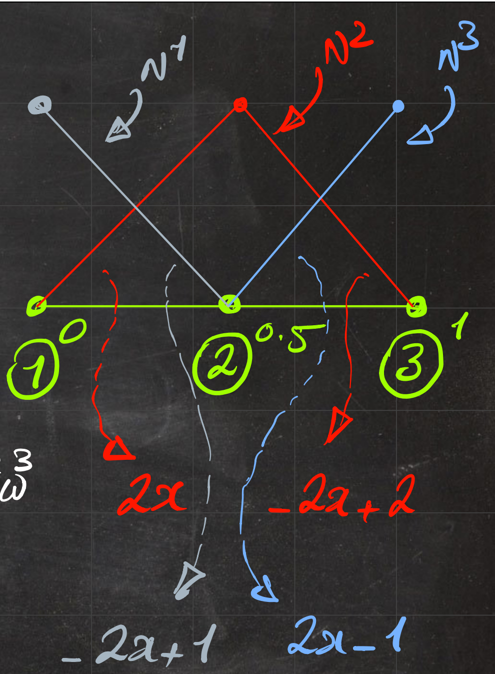
$\ldots \int_0^1 w'u'dx = \int_0^1 w\,dx \ldots \forall w$

$\ldots \Rightarrow D_1 \,\&\, D_2$ ✓

2- PIECE LINEAR UNIFORM APPROXIMATION

$$w = N^1 \cancelto{0}{w^1} + N^2 w^2 + N^3 w^3 \qquad u = N^1 \cancelto{0}{u^1} + N^2 u^2 + N^3 u^3$$

$$\Rightarrow w = N^2 w^2 + N^3 w^3 \qquad \Rightarrow u = N^2 u^2 + N^3 u^3$$



$$w^2\left[4u^2 - 2u^3 - \frac{1}{2}\right] + w^3\left[2u^3 - 2u^2 - \frac{1}{4}\right] = 0 \quad \forall w^2, w^3$$

$$\begin{cases} 4u^2 - 2u^3 - \frac{1}{2} = 0 \\ 2u^3 - 2u^2 - \frac{1}{4} = 0 \end{cases} \Rightarrow \begin{cases} u^2 = \frac{3}{8} \\ u^3 = \frac{1}{2} \end{cases} \Rightarrow u = u(x) \ ✓$$

$u'' + 1 = 0 \qquad 0 \leq x \leq 1$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

Summary:

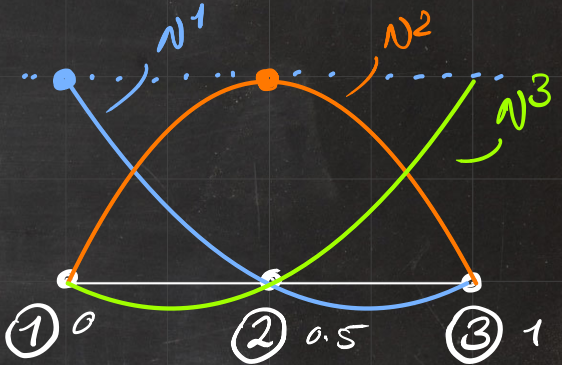
in the previous approach ⟶ we had
$$\begin{cases} u = \alpha_1 x + \beta_1 & 0 \leq x \leq \frac{1}{2} \\ u = \alpha_2 x + \beta_2 & \frac{1}{2} \leq x \leq 1 \end{cases}$$

$$\ldots \Rightarrow \text{WE CALCULATED } \alpha_1, \alpha_2, \beta_1, \beta_2 \longrightarrow \underbrace{\text{THEN COMPUTE NODAL VALUES}}_{\text{NECESSARY}}$$

in the current approach ⟶ we have
$$u = N^1 u^1 + N^2 u^2 + N^3 u^3 \quad \begin{array}{l} 0 \leq x \leq \frac{1}{2} \\ \frac{1}{2} \leq x \leq 1 \end{array}$$

$$\ldots \Rightarrow \text{WE CALCULATE } u^1, u^2, u^3 \longrightarrow \overbrace{\text{THEN COMPUTE}}^{\text{UNNECESSARY}} \quad \begin{cases} u = u(x) & 0 \leq x \leq \frac{1}{2} \\ u = u(x) & \frac{1}{2} \leq x \leq 1 \end{cases}$$

1 - PIECE QUADRATIC APPROXIMATION



$$w = N^1 \cancelto{0}{w^1} + N^2 w^2 + N^3 w^3 \qquad u = N^1 \cancelto{0}{u^1} + N^2 u^2 + N^3 u^3$$

$$\int_0^1 \left[[N^{2\prime}]^2 u^2 + N^{2\prime} N^{3\prime} u^3 - N^2 \right] dx = 0$$

$$\int_0^1 \left[N^{3\prime} N^{2\prime} u^2 + [N^{3\prime}]^2 u^3 - N^3 \right] dx = 0$$

$$\begin{bmatrix} \int_0^1 N^{2\prime} N^{2\prime} dx & \int_0^1 N^{2\prime} N^{3\prime} dx \\ \int_0^1 N^{2\prime} N^{3\prime} dx & \int_0^1 N^{3\prime} N^{3\prime} dx \end{bmatrix} \begin{bmatrix} u^2 \\ u^3 \end{bmatrix} = \begin{bmatrix} \int_0^1 N^2 dx \\ \int_0^1 N^3 dx \end{bmatrix}$$

$$N^1 = 2[x-0.5][x-1] = 2x^2 - 3x + 1$$

$$N^2 = -4[x-0][x-1] = -4x^2 + 4x$$

$$N^3 = 2[x-0][x-0.5] = 2x^2 - x$$

$$N^{1\prime} = 4x - 3$$

$$N^{2\prime} = -8x + 4$$

$$N^{3\prime} = 4x - 1$$

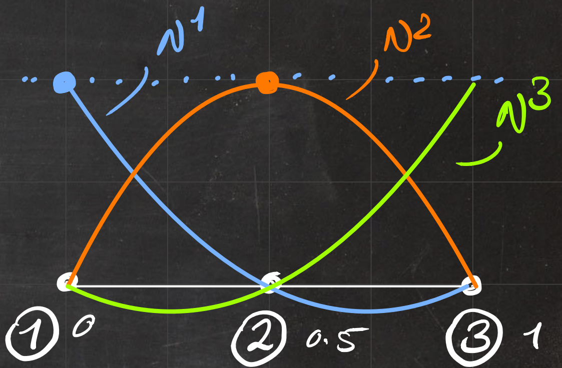
$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

1-PIECE QUADRATIC APPROXIMATION

$$w = N^1 \cancelto{0}{w^1} + N^2 w^2 + N^3 w^3 \qquad u = N^1 \cancelto{0}{u^1} + N^2 u^2 + N^3 u^3$$



$$\begin{bmatrix} 16/3 & -8/3 \\ -8/3 & 16/3 \end{bmatrix} \begin{bmatrix} u^2 \\ u^3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/6 \end{bmatrix}$$

$$\int_0^1 N^2 \, dx = \frac{2}{3}$$

$$\int_0^1 N^3 \, dx = \frac{1}{6}$$

$$N^1 = 2[x-0.5][x-1] = 2x^2 - 3x + 1$$

$$N^2 = -4[x-0][x-1] = -4x^2 + 4x$$

$$N^3 = 2[x-0][x-0.5] = 2x^2 - x$$

$$\int_0^1 N^{2\prime} N^{2\prime} \, dx = \frac{16}{3} \qquad \int_0^1 N^{2\prime} N^{3\prime} \, dx = -\frac{8}{3}$$

$$\int_0^1 N^{2\prime} N^{3\prime} \, dx = -\frac{8}{3} \qquad \int_0^1 N^{3\prime} N^{3\prime} \, dx = \frac{7}{3}$$

$$N^{1\prime} = 4x - 3$$

$$N^{2\prime} = -8x + 4$$

$$N^{3\prime} = 4x - 1$$

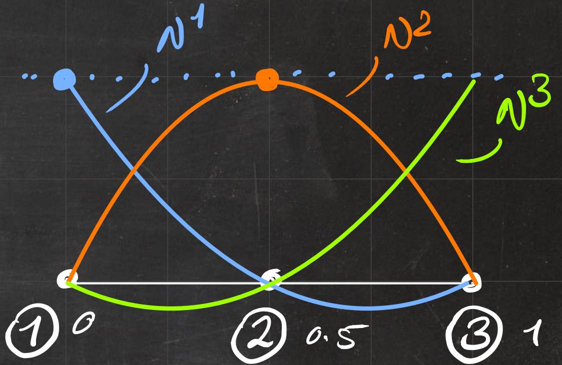
$u'' + 1 = 0 \qquad 0 \leq x \leq 1$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

1-PIECE QUADRATIC APPROXIMATION

$w = N^1 \cancelto{0}{w^1} + N^2 w^2 + N^3 w^3$ $\qquad$ $u = N^1 \cancelto{0}{u^1} + N^2 u^2 + N^3 u^3$



$$\begin{bmatrix} 16/3 & -8/3 \\ -8/3 & 16/3 \end{bmatrix}\begin{bmatrix} u^2 \\ u^3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/6 \end{bmatrix}$$

$$\Rightarrow \begin{cases} u^2 = 3/8 \\ u^3 = 1/2 \end{cases}$$

$$u = N^2 u^2 + N^3 u^3$$
$$= [-4x^2 + 4x]\, 3/8$$
$$+ [2x^2 - x]\, 1/2$$

this was analytical solution ↓

$$\Rightarrow u = -\frac{1}{2}x^2 + x$$

$N^1 = 2[x-0.5][x-1] = 2x^2 - 3x + 1$

$N^2 = -4[x-0][x-1] = -4x^2 + 4x$

$N^3 = 2[x-0][x-0.5] = 2x^2 - x$

$N^{1\prime} = 4x - 3$

$N^{2\prime} = -8x + 4$

$N^{3\prime} = 4x - 1$

$u'' + 1 = 0 \qquad 0 \leq x \leq 1$

D: $u(0) = 0$ ← prescribed

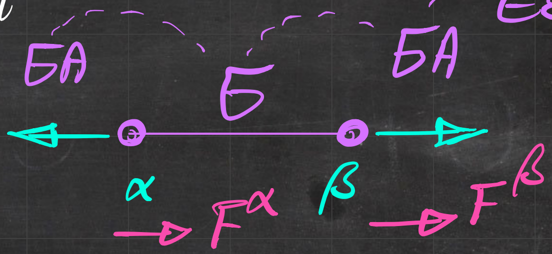
N: $u'(1) = 0$ ←

Consider the generic strong form $(EAu')' + b = 0 \quad \alpha \leqslant x \leqslant \beta$

Subject to BCs at α, β

↳ … ↩ weak from derivation process

$E\varepsilon = Eu'$



Corresponding weak form

$$\int_\alpha^\beta EA\, w' u'\, dx = \underbrace{EA\, u'(\beta)\, w(\beta)}_{F^\beta} - \underbrace{EA\, u'(\alpha)\, w(\alpha)}_{F^\alpha} + \underbrace{\overbrace{\int_\alpha^\beta b\, w\, dx}^{\text{NEGLECT!}}}_{\text{body forces over the domain}}$$

$F^{\alpha,\beta}$: EXTERNAL FORCES AT NODES α, β

Consider the generic strong form $(EAu')' + b = 0 \quad \alpha \leq x \leq \beta$

Subject to BCs at α, β

↳ ... ← weak from derivation process

Corresponding weak form

$F^{\alpha,\beta}$: EXTERNAL FORCES AT NODES α, β

$$\boxed{EA \int_{\alpha}^{\beta} w' u' \, dx = w(\beta) F^{\beta} + w(\alpha) F^{\alpha}}$$

Approximate using

1- PIECE LINEAR

1- PIECE QUADRATIC

$$\boxed{EA\int_{\alpha}^{\beta} w' u' \, dx = w(\beta) F^{\beta} + w(\alpha) F^{\alpha}}$$

$$[\mathbb{K}][u] = [F]$$

$$\hookrightarrow \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} \qquad \hookrightarrow \begin{bmatrix} F^1 \\ F^2 \end{bmatrix}$$

$$\mathbb{K} = EA \begin{bmatrix} \int_{\alpha}^{\beta} N^{1\prime} N^{1\prime} \, dx & \int_{\alpha}^{\beta} N^{1\prime} N^{2\prime} \, dx \\ \int_{\alpha}^{\beta} N^{2\prime} N^{1\prime} \, dx & \int_{\alpha}^{\beta} N^{2\prime} N^{2\prime} \, dx \end{bmatrix}$$

$$\hookrightarrow \mathbb{K} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Approximate using

1- PIECE LINEAR

1- PIECE QUADRATIC

$$\boxed{EA\int_{\alpha}^{\beta} w' u' \, dx = w(\beta) F^{\beta} + w(\alpha) F^{\alpha}}$$

$$\mathbb{K} = \frac{EA}{3L}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}$$

$$[\mathbb{K}][\mathbb{u}] = [\mathbb{F}]$$

$$\begin{bmatrix} u^1 \\ u^2 \\ u^3 \end{bmatrix} \qquad \begin{bmatrix} F^1 \\ F^2 \\ F^3 \end{bmatrix}$$

$$\mathbb{K} = EA\begin{bmatrix} \int_{\alpha}^{\beta} N^{1\prime} N^{1\prime} dx & \int_{\alpha}^{\beta} N^{1\prime} N^{2\prime} dx & \int_{\alpha}^{\beta} N^{1\prime} N^{3\prime} dx \\ \int_{\alpha}^{\beta} N^{2\prime} N^{1\prime} dx & \int_{\alpha}^{\beta} N^{2\prime} N^{2\prime} dx & \int_{\alpha}^{\beta} N^{2\prime} N^{3\prime} dx \\ \int_{\alpha}^{\beta} N^{3\prime} N^{1\prime} dx & \int_{\alpha}^{\beta} N^{3\prime} N^{2\prime} dx & \int_{\alpha}^{\beta} N^{3\prime} N^{3\prime} dx \end{bmatrix}$$

GENERAL STRUCTURE OF STIFFNESS MATRIX FOR 1D FINITE ELEMENTS

$$\mathbb{K} = EA \begin{bmatrix} NPE \times NPE \\ \\ \end{bmatrix}$$

NPE: Node Per Element

$[NPE \times PD] \times [NPE \times PD]$

↳

	1D	2D
LINEAR TRUSS	2x2	4x4
QUADR. TRUSS	3x3	6x6

$$\mathbb{K} = EA \begin{bmatrix} \int_\alpha^\beta N^{1\prime} N^{1\prime} \, dx & \int_\alpha^\beta N^{1\prime} N^{2\prime} \, dx \\ \int_\alpha^\beta N^{2\prime} N^{1\prime} \, dx & \int_\alpha^\beta N^{2\prime} N^{2\prime} \, dx \end{bmatrix}$$

← LINEAR TRUSS ELEMENT

$$\hookrightarrow \quad K^{ij} = EA \int_\alpha^\beta N^{i\prime} N^{j\prime} \, dx$$

GENERAL STRUCTURE OF STIFFNESS MATRIX FOR 1D FINITE ELEMENTS

$$\mathbb{K} = EA \begin{bmatrix} \text{NPE} \times \text{NPE} \end{bmatrix}$$

NPE: Node Per Element

$\rightarrow$ [NPE x PD] x [NPE x PD]

	1D	2D
LINEAR TRUSS	2x2	4x4
QUADR. TRUSS	3x3	6x6

$$\mathbb{K} = EA \begin{bmatrix} \int_\alpha^\beta N^{1\prime} N^{1\prime}\, dx & \int_\alpha^\beta N^{1\prime} N^{2\prime}\, dx & \int_\alpha^\beta N^{1\prime} N^{3\prime}\, dx \\ \int_\alpha^\beta N^{2\prime} N^{1\prime}\, dx & \int_\alpha^\beta N^{2\prime} N^{2\prime}\, dx & \int_\alpha^\beta N^{2\prime} N^{3\prime}\, dx \\ \int_\alpha^\beta N^{3\prime} N^{1\prime}\, dx & \int_\alpha^\beta N^{3\prime} N^{2\prime}\, dx & \int_\alpha^\beta N^{3\prime} N^{3\prime}\, dx \end{bmatrix}$$

$\leftarrow$ QUADRATIC TRUSS ELEMENT

$$\rightarrow K^{ij} = EA \int_\alpha^\beta N^{i\prime} N^{j\prime}\, dx$$

GENERAL STRUCTURE OF STIFFNESS MATRIX FOR 1D FINITE ELEMENTS

$$\mathbb{K} = EA\begin{bmatrix} NPE \times NPE \end{bmatrix}$$

NPE: Node Per Element

$[NPE \times PD] \times [NPE \times PD]$ 2D

	1D	2D
LINEAR TRUSS	2x2	4x4
QUADR. TRUSS	3x3	6x6

$N^i = N^i(x)$ $\quad N^j = N^j(x)$

EVALUATE THIS INTEGRAL

$$K^{ij} = EA\int_\alpha^\beta N^{i\prime} N^{j\prime}\, dx = EA\int_\alpha^\beta f(x)\, dx$$

$N^{i\prime} = N^{i\prime}(x)$ $\quad N^{j\prime} = N^{j\prime}(x)$

NUMERICALLY

NUMERICAL INTEGRATION :

$$\int_a^b f(x)\,dx = ?$$

f(x) ✓ LINEAR

$$f(x) = \frac{f(b)-f(a)}{b-a}[x-a] + f(a)$$

$$\int_a^b f(x)\,dx = \frac{f(b)-f(a)}{b-a}\frac{1}{2}x^2\Big]_a^b - \frac{f(b)-f(a)}{b-a}ax\Big]_a^b + f(a)x\Big]_a^b$$

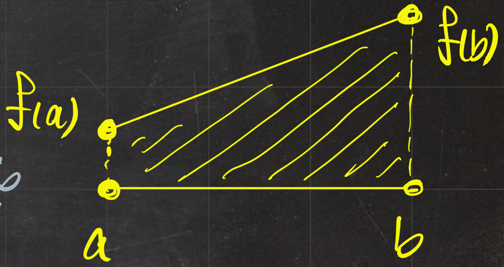
$$= \frac{1}{2}\frac{f(b)-f(a)}{b-a}[b^2-a^2] - \frac{f(b)-f(a)}{b-a}a[b-a] + f(a)[b-a]$$

$$= \frac{b+a}{2}[f(b)-f(a)] - a[f(b)-f(a)] + f(a)[b-a]$$

NUMERICAL INTEGRATION :

$$\int_a^b f(x)\,dx = ?$$

$f(x) \rightarrow$ LINEAR



$$f(x) = \frac{f(b)-f(a)}{b-a}[x-a] + f(a)$$

$$\int_a^b f(x)\,dx = \frac{f(b)-f(a)}{b-a}\frac{1}{2}x^2\Big]_a^b - \frac{f(b)-f(a)}{b-a}ax\Big]_a^b + f(a)x\Big]_a^b$$

$$= \frac{b+a}{2}[f(b)-f(a)] - a[f(b)-f(a)] + f(a)[b-a]$$

$$= \frac{b-a}{2}[f(b)-f(a)] + f(a)[b-a] = \frac{b-a}{2}[f(b)+f(a)]$$

NUMERICAL INTEGRATION :

$$\int_a^b f(x)\,dx = ?$$

f(x) → LINEAR

$$f(x) = \frac{f(b)-f(a)}{b-a}[x-a] + f(a)$$

$$\int_a^b f(x)\,dx = \frac{f(b)-f(a)}{b-a}\,\frac{1}{2}x^2\Big]_a^b - \frac{f(b)-f(a)}{b-a}ax\Big]_a^b + f(a)x\Big]_a^b$$

$$= \frac{b-a}{2}\left[f(b)+f(a)\right]$$

$$= \frac{f(b)+f(a)}{2}\left[b-a\right]$$

NUMERICAL INTEGRATION :

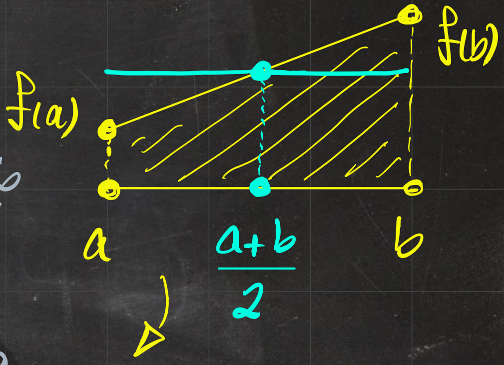
$$\int_a^b f(x)\,dx = ?$$

$f(x)$ ✓ LINEAR

$$f(x) = \frac{f(b)-f(a)}{b-a}[x-a] + f(a)$$

$$\int_a^b f(x)\,dx = \frac{f(b)-f(a)}{b-a}\,\frac{1}{2}x^2\Big]_a^b - \frac{f(b)-f(a)}{b-a}\,ax\Big]_a^b + f(a)x\Big]_a^b$$

$$= \underbrace{\frac{f(b)+f(a)}{2}}_{f\left(\frac{a+b}{2}\right)}\;\underbrace{[b-a]}_{L}$$



$$A = \frac{1}{2}[f(a)+f(b)][b-a]$$

NUMERICAL INTEGRATION :

$\int_a^b f(x)\, dx = ?$ f(x) ← LINEAR

$$\underbrace{\int_a^b f(x)\, dx}_{\text{EXACT INTEGRAL}} = \underbrace{[b-a]}_{\text{LENGTH}} \underbrace{f\left(\frac{a+b}{2}\right)}_{\text{EVALUATION OF } f(x) \text{ ONLY AT ONE POINT}}$$

← THIS IS EXACT

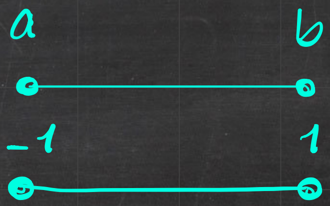
(not even an approximation)

NUMERICAL INTEGRATION :

$$\int_a^b f(x)\,dx = ?$$

f(x) ✓ QUADRATIC

$$\underbrace{\int_a^b f(x)\,dx}_{\text{EXACT INTEGRAL}} = \dots$$



$$g(\xi) = c_1\,\xi^2 + c_2\,\xi + c_3$$

$$\int_{-1}^{1} g(\xi)\,d\xi = ?$$

\xi

NUMERICAL INTEGRATION :

$$g(\xi) = c_1 \xi^2 + c_2 \xi + c_3$$

$$\int_{-1}^{1} g(\xi)\, d\xi = ?$$

NUMERICAL INTEGRATION :

$$g(\xi) = c_1 \xi^2 + c_2 \xi + c_3$$

$$\int_{-1}^{1} g(\xi)\, d\xi = \int_{-1}^{1} c_1 \xi^2 + c_2 \xi + c_3 \, d\xi$$

$$= \frac{1}{3} c_1 \xi^3 + \frac{1}{2} c_2 \xi^2 + c_3 \xi \Big]_{-1}^{1}$$

$$= \frac{2}{3} c_1 + 2 c_3$$

NUMERICAL INTEGRATION :

$$g(\xi) = c_1 \xi^2 + c_2 \xi + c_3$$

$$\int_{-1}^{1} g(\xi)\, d\xi = \frac{2}{3} c_1 + 2 c_3 \pm \frac{1}{\sqrt{3}} c_2$$

$$= \left[\frac{1}{3} c_1 + \frac{1}{\sqrt{3}} c_2 + c_3 \right] + \left[\frac{1}{3} c_1 - \frac{1}{\sqrt{3}} c_2 + c_3 \right]$$

$$= g\left(\frac{1}{\sqrt{3}}\right) + g\left(-\frac{1}{\sqrt{3}}\right)$$

NUMERICAL INTEGRATION :

GAUSS QUADRATURE FORMULA

↳ GAUSS POINTS

$$\int_{-1}^{1} g(\xi)\,d\xi = 2 \times g(0)$$

g: LINEAR

$$\int_{-1}^{1} g(\xi)\,d\xi = g\left(\frac{1}{\sqrt{3}}\right) + g\left(-\frac{1}{\sqrt{3}}\right)$$

g: QUADRATIC

$$\int_{-1}^{1} g(\xi)\,d\xi = g\left(\frac{1}{\sqrt{3}}\right) + g\left(-\frac{1}{\sqrt{3}}\right)$$

g: CUBIC

NUMERICAL INTEGRATION : ← QUADRATURE RULE ⟶ GAUSS POINTS

GPE ← NUMBER OF GAUSS POINTS PER ELEMENT

QUADRATURE POINTS

$$\int_{-1}^{1} g(\xi)\, d\xi = \sum_{i=1}^{GPE} \alpha_i \, g(\xi_i)$$

↑ weight factor

$$GPE \geqslant \frac{P+1}{2}$$

P ← ORDER OF POLYNOMIAL

	P	GPE	
LINEAR	1	1	$\rightarrow \xi_1 = 0 \ , \ \alpha_1 = 2$
QUADRATIC	2	2	$\rightarrow \xi_1 = -1/\sqrt{3} \ , \ \alpha_1 = 1$ $\xi_2 = 1/\sqrt{3} \ , \ \alpha_2 = 1$
CUBIC	3	2	
4th O.	4	3	$\rightarrow \xi_1 = -\sqrt{0.6} \ , \ \alpha_1 = 5/9$ $\xi_2 = 0 \ , \ \alpha_2 = 8/9$ $\xi_3 = \sqrt{0.6} \ , \ \alpha_3 = 5/9$
5th O.	5	3	

$$\sum_{i=1}^{GPE} \alpha_i = 2$$

NUMERICAL INTEGRATION :

$$\int_{-1}^{1} g(\xi)\, d\xi \overset{\text{GPE}}{=} \sum_{i=1} \alpha_i\, g(\xi_i)$$

$$\int_a^b f(x)\, dx = ? \quad \swarrow \text{ TRANSFORM TO } \int_{-1}^{1} g(\xi)\, d\xi$$

x_DOMAIN ↙ PHYSICAL SPACE

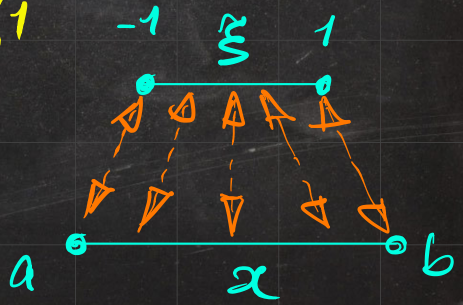
$a \leqslant x \leqslant b$

ξ_DOMAIN ↙ NATURAL SPACE

$-1 \leqslant \xi \leqslant 1$

MAPPING FROM PHYSICAL TO NATURAL SPACE

$$\hookrightarrow x \overset{?}{=} x(\xi) \Rightarrow f(x) = f(x(\xi)) = g(\xi)$$



NUMERICAL INTEGRATION :

$$\int_{-1}^{1} g(\xi)\, d\xi = \sum_{i=1}^{GPE} \alpha_i \, g(\xi_i)$$

$$\int_{a}^{b} f(x)\, dx = ? \leftarrow \text{TRANSFORM TO} \int_{-1}^{1} g(\xi)\, d\xi$$

$$J = \frac{\partial x}{\partial \xi}$$

Jacobian

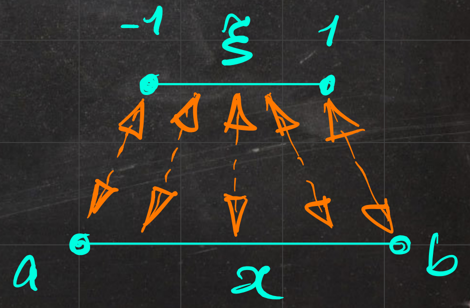
POLYNOMIAL ORDER

$$x \overset{?}{=} x(\xi) \Rightarrow x = \sum_{i=1}^{P+1} N^i x^i \quad \dots \quad dx = J(\xi)\, d\xi$$

$$N^i = N^i(\xi)$$

$$\text{Shape Functions} \rightarrow u = \sum N^i u^i \leftarrow \text{values}$$

$$u(x) \qquad N^i(x)$$



NUMERICAL INTEGRATION :

$$\int_{-1}^{1} g(\xi)\, d\xi = \sum_{i=1}^{GPE} \alpha_i\, g(\xi_i)$$

$$\int_a^b f(x)\, dx = ? \quad \leftarrow \quad \text{TRANSFORM TO} \quad \int_{-1}^{1} g(\xi)\, d\xi$$

$$J = \frac{\partial x}{\partial \xi}$$

Jacobian

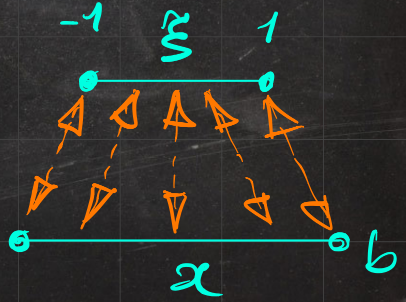
POLYNOMIAL ORDER

$$x \overset{?}{=} x(\xi) \Rightarrow x = \sum_{i=1}^{P+1} N^i x^i \quad \dots \quad dx = J(\xi)\, d\xi$$

$$N^i = N^i(\xi)$$

$$\int_a^b f(x)\, dx = \int_{-1}^{1} \underbrace{f(\xi)\, J(\xi)}_{g(\xi)}\, dx = \sum_{i=1}^{GPE} \dots$$

$f(x(\xi))$



NUMERICAL INTEGRATION :

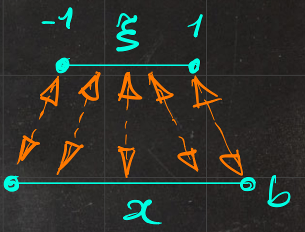
$$\int_{-1}^{1} g(\xi)\,d\xi = \sum_{i=1}^{GPE} \alpha_i\, g(\xi_i)$$

$$\int_a^b f(x)\,dx = \int_{-1}^{1} \underbrace{f(\xi)\,J(\xi)}_{g(\xi)}\,dx = \sum_{i=1}^{GPE} \dots$$

$f(x(\xi))$

$J = \frac{\partial x}{\partial \xi}$

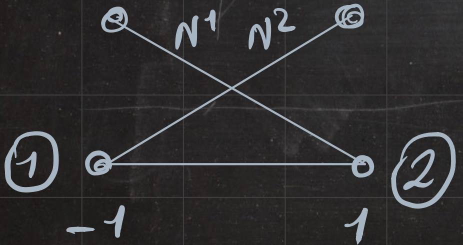
Jacobian



POLYNOMIAL ORDER

$$x \overset{?}{=} x(\xi) \Rightarrow x = \sum_{i=1}^{P+1} N^i x^i \quad \dots \quad dx = J(\xi)\,d\xi$$

$N^i = N^i(\xi)$



$$N^1 = -\frac{1}{2}[\xi - 1]$$

$$N^2 = \frac{1}{2}[\xi + 1]$$

$$N^1 = \frac{1}{2}\xi[\xi - 1]$$

$$N^2 = [1-\xi][1+\xi]$$

$$N^3 = \frac{1}{2}\xi[\xi + 1]$$

NUMERICAL INTEGRATION:

$$\int_{-1}^{1} g(\xi)\, d\xi = \sum_{i=1}^{GPE} \alpha_i\, g(\xi_i)$$

$$\int_a^b f(x)\, dx = \int_{-1}^{1} \underbrace{f(\xi)\, J(\xi)}_{g(\xi)}\, dx = \sum_{i=1}^{GPE} \dots$$

$f(x(\xi))$

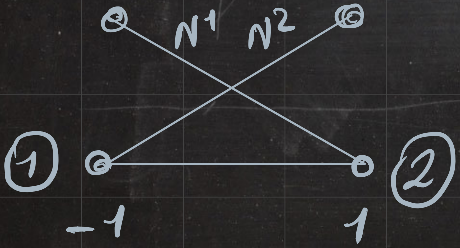
$$J = \frac{\partial x}{\partial \xi}$$

Jacobian

POLYNOMIAL ORDER

$$x \overset{?}{=} x(\xi) \Rightarrow x = \sum_{i=1}^{P+1} N^i x^i \quad \dots \quad dx = J(\xi)\, d\xi$$

$N^i = N^i(\xi)$



$$N^1 = -\frac{1}{2}[\xi - 1]$$

$$N^2 = \frac{1}{2}[\xi + 1]$$

EXAMPLE:

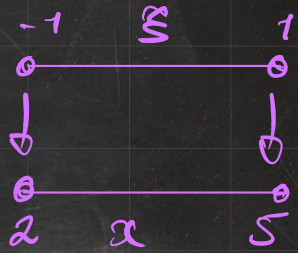
$$\int_2^5 [2x+1]\, dx = ?$$

NUMERICAL INTEGRATION :

$$\int_{-1}^{1} g(\xi)\, d\xi = \sum_{i=1}^{GPE} \alpha_i\, g(\xi_i)$$

$$\int_a^b f(x)\, dx = \int_{-1}^{1} \overbrace{\left[2[1.5\xi+3.5]+1\right]1.5}^{g(\xi)}\, d\xi$$

$$= 2 \times g(0) = 24$$



$$x = \sum_{i=1}^{2} N^i x^i = N^1 x^1 + N^2 x^2 = -\frac{1}{2}[\xi-1]\times 2 + \frac{1}{2}[\xi+1]\times 5 = 1.5\xi+3.5$$

$$x = 1.5\xi + 3.5$$

$$dx = \underbrace{1.5}_{J}\, d\xi$$

$$\ldots = x^2 + x\Big]_2^5 \quad ✓$$

$$= 25+5-4-2$$
$$= 24$$

EXAMPLE:

$$\int_2^5 [2x+1]\, dx = ?$$