FINITE ELEMENT METHOD

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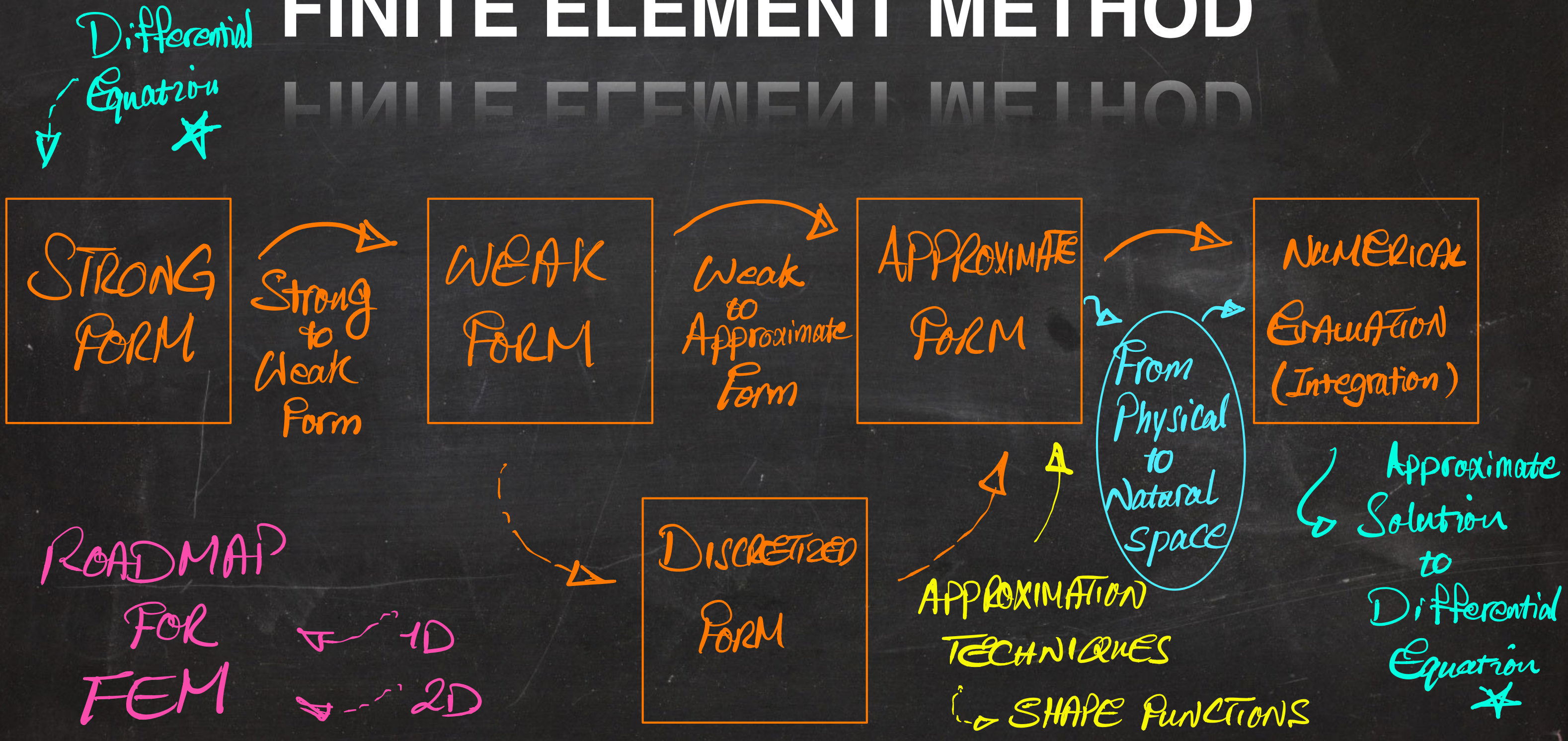
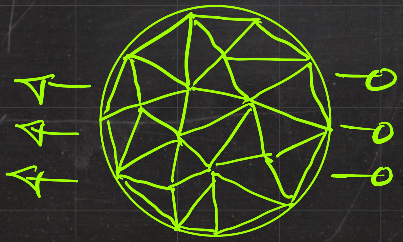


UNDERSTANDING FEM VIA AN ANALOGY (A BRUTAL SIMPLIFICATION)

Approximations in FEM

- → Solution Approximation ⇝ inherent to numerical techniques using computers
- → Equation Approximation ⇝ diff equation is transformed to weak form + space approximation
- → Input Approximation ⇝ space discretization



Discretization (Approximation)

- → DOMAIN (X)
- → SOLUTION (u)
- → TEST (w)

diff. Eq.

→ STRONG FORM

TO

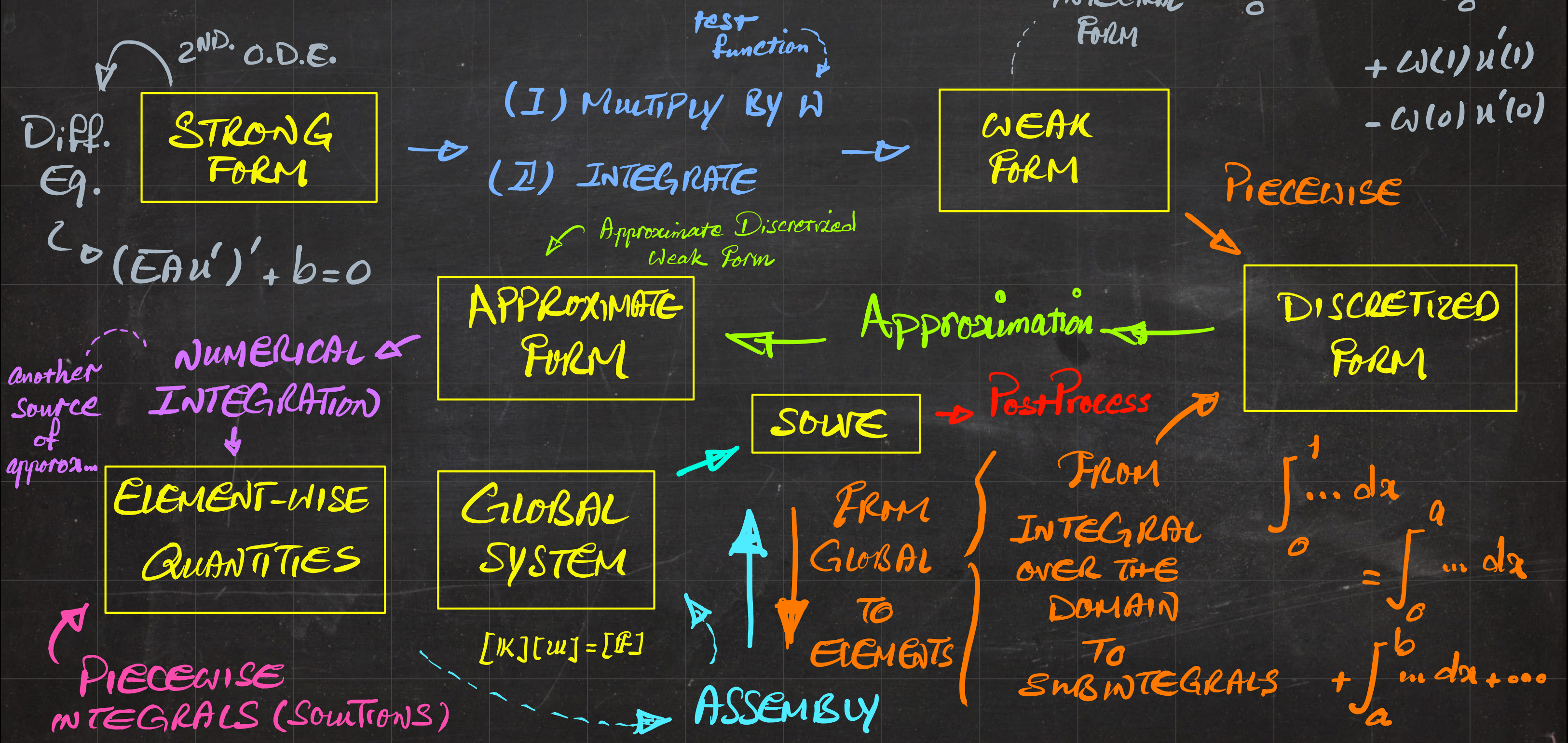
integral

→ WEAK FORM

1D FEM

Overview and Wrap-up

FINITE ELEMENT METHOD IN A NUTSHELL



2ND. O.D.E.

Diff. Eq.

STRONG FORM

$(EAu')' + b = 0$

test function

(I) MULTIPLY BY w

(II) INTEGRATE

WEAK FORM

INTEGRAL FORM

$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx + w(1)\,u'(1) - w(0)\,u'(0)$$

PIECEWISE

DISCRETIZED FORM

Approximation

Approximate Discretized Weak Form

APPROXIMATE FORM

NUMERICAL INTEGRATION

another source of approx...

ELEMENT-WISE QUANTITIES

GLOBAL SYSTEM

$[\mathbb{K}][u] = [\mathbb{F}]$

SOLVE

PostProcess

FROM GLOBAL TO ELEMENTS

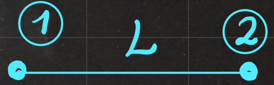
FROM INTEGRAL OVER THE DOMAIN TO SUBINTEGRALS

$$\int_0^1 \dots dx = \int_0^a \dots dx + \int_a^b \dots dx + \dots$$

ASSEMBLY

PIECEWISE INTEGRALS (SOLUTIONS)

FROM STRONG FORM TO ELEMENT STIFFNESS ← IN PHYSICAL SPACE NOW



$EAu''=0$ SUBJECT TO BCs

MULTIPLY BY TEST FUNCTION w

- DIRICHLET → u IS PRESCRIBED
- NEUMANN → u' IS PRESCRIBED

$$EAwu''=0 \leftarrow wu''=(wu')'-w'u'$$

$$EA\left[(wu')'-w'u'\right]=0 \Rightarrow EAw'u'=EA(wu')' \leftarrow \text{INTEGRATE}$$

$$\int_L EAw'u'\,dx=\int_L EA(wu')'\,dx=EAwu'\Big]_{①}^{②}=EAw^2u'^2-EAw^1u'^1$$

FROM STRONG FORM TO ELEMENT STIFFNESS ← IN PHYSICAL SPACE now

$EA\,u'' = 0$ SUBJECT TO BCs

$$EA \begin{bmatrix} \int_L N^{1'} N^{1'} dx & \int_L N^{1'} N^{2'} dx \\ \int_L N^{2'} N^{1'} dx & \int_L N^{2'} N^{2'} dx \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} = \begin{bmatrix} F^1 \\ F^2 \end{bmatrix}$$

$$\begin{bmatrix} K^{11} & K^{12} \\ K^{21} & K^{22} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} = \begin{bmatrix} F^1 \\ F^2 \end{bmatrix} \leftarrow K^{ij} = EA \int_L N^{i'} N^{j'} dx$$

$$K^{ij} = EA \int_L N^{i\prime} N^{j\prime} \, dx$$

← PHYSICAL

RECALL:

$$\int_{-1}^{1} g(\xi)\, d\xi = \sum_{gp=1}^{GPE} g(\xi_{gp})\, \alpha_{gp}$$

← NATURAL

$$= EA \int_{-1}^{1} \frac{\partial N^i}{\partial \xi} \frac{\partial N^j}{\partial \xi} J^{-1} \, d\xi$$

$$= EA \sum_{gp=1}^{GPE} \left\{ \left[\frac{\partial N^i}{\partial \xi} \frac{\partial N^j}{\partial \xi} J^{-1} \right]_{gp} \times \alpha_{gp} \right\}$$

← Loop over gp

⋮

END

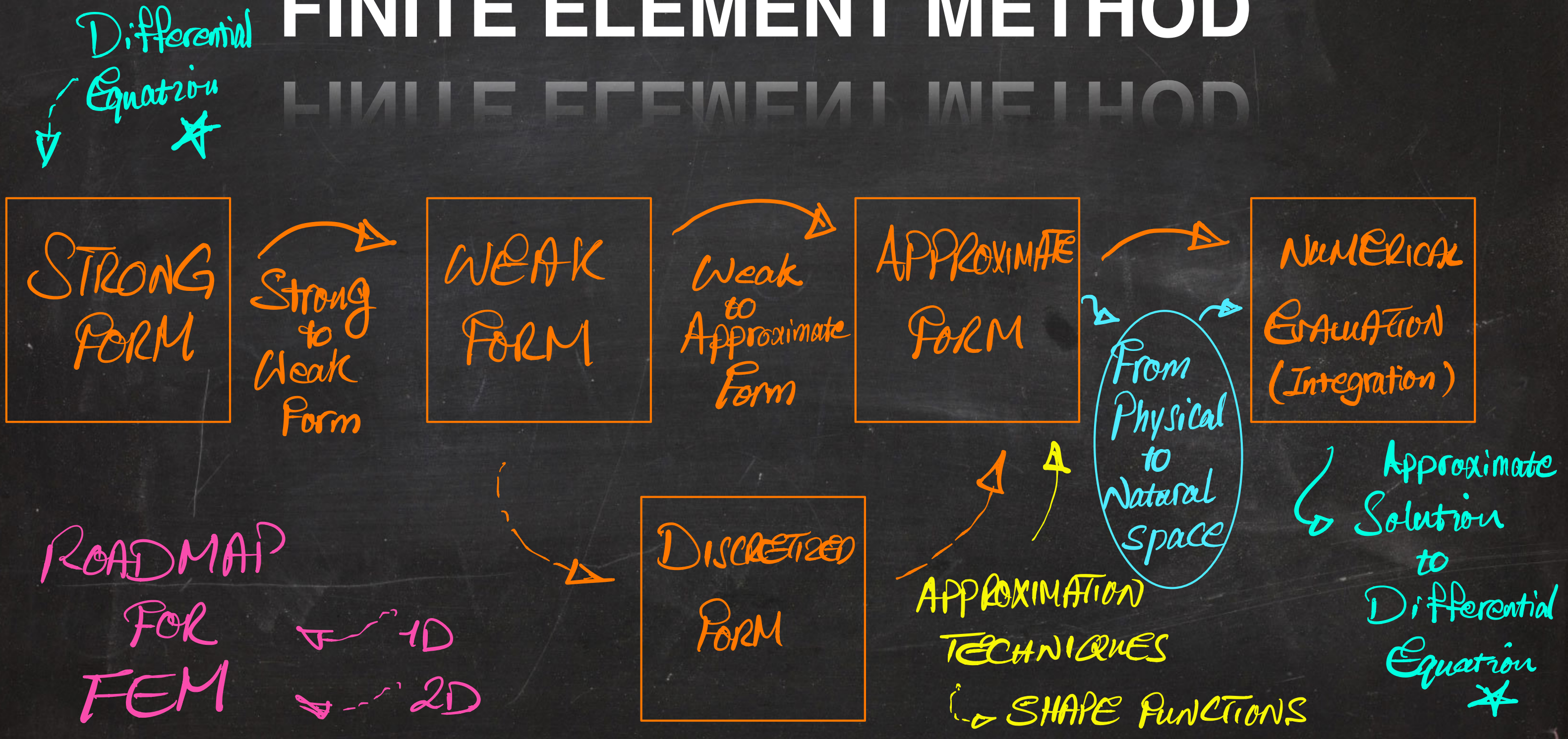
e.g. in MATLAB

WHAT YOU SEE IN THE CODE !

```
For gp=1:GPE
    ⋮
End
```

2D FEM

FINITE ELEMENT METHOD



MATHEMATICAL PRELIMINARIES

EINSTEIN SUMMATION CONVENTION

→ A little definition for notation convenience

→ A REPEATED INDEX TWICE MEANS SUMMATION OVER THAT INDEX

also, called "dummy index"

$$\sum_{i=1}^{3} u_i v_i = u_1 v_1 + u_2 v_2 + u_3 v_3 \equiv u_i v_i$$

← i is summation index

i: free index

$$\sum_{j=1}^{3} A_{ij} u_j \Rightarrow \begin{cases} i=1 \Rightarrow A_{11} u_1 + A_{12} u_2 + A_{13} u_3 \\ i=2 \Rightarrow A_{21} u_1 + A_{22} u_2 + A_{23} u_3 \\ i=3 \Rightarrow A_{31} u_1 + A_{32} u_2 + A_{33} u_3 \end{cases} \Rightarrow \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \equiv A_{ij} u_j$$

$1 \leqslant i \leqslant 3$

j: summation index

$$\mathbb{u}\cdot\mathbb{v} = u_x v_x + u_y v_y + u_z v_z = u_1 v_1 + u_2 v_2 + u_3 v_3 = \sum_{i=1}^{3} u_i v_i = u_i v_i$$

$\hookrightarrow$ Dot Product $(\mathbb{u}, \mathbb{v}) \rightsquigarrow$ SCALAR $\leftarrow u_i v_i \leftarrow \mathbb{u}\cdot\mathbb{v}$

Double Dot Product $(\mathbb{A}, \mathbb{B}) \rightsquigarrow$ SCALAR $\leftarrow A_{ij} B_{ij} \leftarrow \mathbb{A}:\mathbb{B}$

$\mathbb{u} \otimes \mathbb{v}$ Dyadic Product $(\mathbb{u}, \mathbb{v}) \rightsquigarrow$ MATRIX $\leftarrow u_i v_j \leftarrow [\mathbb{u}\otimes\mathbb{v}]_{ij}$

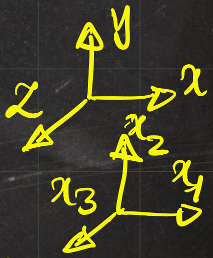
KRONECKER DELTA $\rightsquigarrow \delta_{ij} = \Phi_i \cdot \Phi_j \Rightarrow \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$

$\mathbb{u}\cdot\mathbb{v} = u_i v_i$ $\quad$ $[\mathbb{A}\cdot\mathbb{B}]_{ik} = A_{ij} B_{jk}$ $\quad$ $[\mathbb{u}\otimes\mathbb{v}]_{ij} = u_i v_j$

$\delta_{ij} = \boldsymbol{e}_i\cdot\boldsymbol{e}_j$ $\quad$ $[\mathbb{A}\cdot\mathbb{u}]_i = A_{ij} u_j$ $\quad$ $\mathbb{A}:\mathbb{B} = A_{ij} B_{ij}$

$\mathbb{E}$ ↝ FOURTH-ORDER TENSOR (ARRAY) ↝ $3\times3\times3\times3 = 81$ Components

$$\hookrightarrow \overset{\text{2nd.}}{\mathbb{S}} = \overset{\text{4th.}}{\mathbb{E}} : \overset{\text{2nd.}}{\boldsymbol{\varepsilon}} \rightsquigarrow [\mathbb{S}]_{ij} = [\mathbb{E}]_{ijkl}\,[\boldsymbol{\varepsilon}]_{kl}$$



INSTEAD OF x, y, z ↝ $1, 2, 3$ ↝ x_1, x_2, x_3 $\quad$ $\boldsymbol{e}_x \sim \boldsymbol{e}_1$

SCALAR → 0 , VECTOR → 1 , MATRIX → 2

GRADIENT INCREASES THE ORDER BY 1

DIVERGENCE REDUCES THE ORDER BY 1

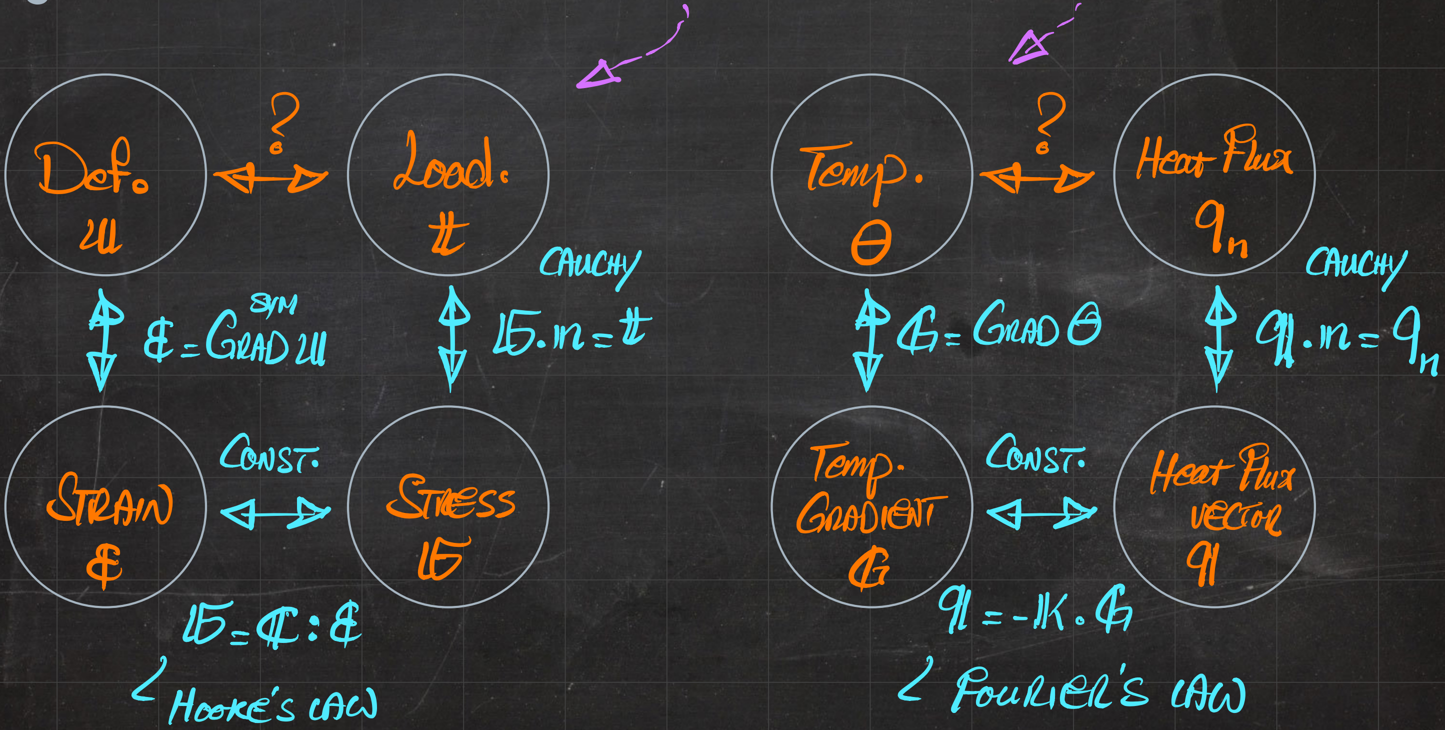
$$\mathrm{GRAD}\,\phi = \begin{bmatrix} \partial\phi/\partial x_1 \\ \partial\phi/\partial x_2 \\ \partial\phi/\partial x_3 \end{bmatrix}$$

$$\mathrm{GRAD}\,\mathbf{u} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

$$\mathrm{DIV}\,\mathbf{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

$$\mathrm{DIV}\,\mathbf{A} = \begin{bmatrix} \frac{\partial A_{11}}{\partial x_1} + \frac{\partial A_{12}}{\partial x_2} + \frac{\partial A_{13}}{\partial x_3} \\ \frac{\partial A_{21}}{\partial x_1} + \frac{\partial A_{22}}{\partial x_2} + \frac{\partial A_{23}}{\partial x_3} \\ \frac{\partial A_{31}}{\partial x_1} + \frac{\partial A_{32}}{\partial x_2} + \frac{\partial A_{33}}{\partial x_3} \end{bmatrix}$$

Big Picture of Mechanics (Mechanical Problems & Thermal Problems)



$$\mathbb{E} = \text{GRAD}^{\text{SYM}}\,\mathbb{u}$$

$$\mathbb{S}\cdot\mathbb{n} = \mathbb{t}$$

$$\mathbb{S} = \mathbb{C} : \mathbb{E}$$

↙ HOOKE'S LAW

$$\mathbb{G} = \text{GRAD}\,\theta$$

$$\mathbb{q}\cdot\mathbb{n} = q_n$$

$$\mathbb{q} = -\mathbb{K}\cdot\mathbb{G}$$

↙ FOURIER'S LAW

STRONG FORM (GENERIC FORM) $\rightsquigarrow$ $\mathrm{Div}\,\boldsymbol{\sigma} + \mathbf{b} = \mathbf{0}$, $\mathrm{Div}\,\mathbf{q} + c = 0$

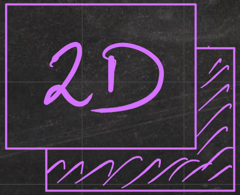
$$\frac{\partial \sigma_{jk}}{\partial x_k} + b_j = 0$$

$$\frac{\partial q_i}{\partial x_i} + c = 0$$

$$\begin{cases} \dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{xy}}{\partial y} + b_x = 0 \\[2ex] \dfrac{\partial \sigma_{yx}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} + b_y = 0 \end{cases}$$

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + c = 0$$

2D $\rightsquigarrow$ Plane Strain, Plane Stress $\Big\}$ $\boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\varepsilon}$



2D Finite Element Library

two-dimensional finite elements library

- two-dimensional 4-noded quadrilateral element (D2QU4N)

  a.k.a. bilinear quadrilateral element

- two-dimensional 9-noded quadrilateral element (D2QU9N)

  a.k.a. Lagrange biquadratic quadrilateral element

- two-dimensional 8-noded quadrilateral element (D2QU8N)

  a.k.a. serendipity biquadratic quadrilateral element

- two-dimensional 3-noded triangular element (D2TR3N)

  a.k.a. constant strain triangle

- two-dimensional 6-noded triangular element (D2TR6N)

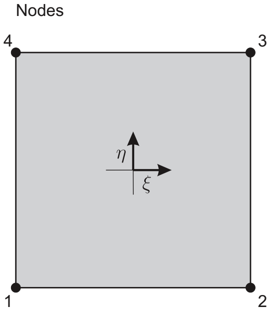
  a.k.a. quadratic triangle

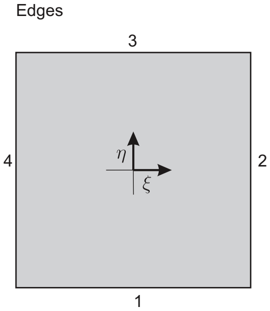
- two-dimensional quadrature rule

2D Finite Element Library

D2QU4N

bilinear quadrilateral element





Node Number	Coordinates	
	ξ	η
1	-1	-1
2	1	-1
3	1	1
4	-1	1

$$N^1 = \frac{1}{4}\left(1-\xi\right)\left(1-\eta\right)$$

$$N^2 = \frac{1}{4}\left(1+\xi\right)\left(1-\eta\right)$$

$$N^3 = \frac{1}{4}\left(1+\xi\right)\left(1+\eta\right)$$

$$N^4 = \frac{1}{4}\left(1-\xi\right)\left(1+\eta\right)$$

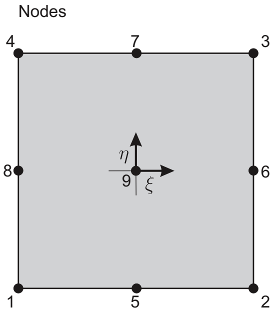
$$N^1_{,\xi} = -\frac{1}{4}\left(1-\eta\right) \qquad N^1_{,\eta} = -\frac{1}{4}\left(1-\xi\right)$$

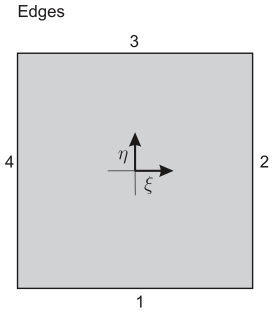
$$N^2_{,\xi} = +\frac{1}{4}\left(1-\eta\right) \qquad N^2_{,\eta} = -\frac{1}{4}\left(1+\xi\right)$$

$$N^3_{,\xi} = +\frac{1}{4}\left(1+\eta\right) \qquad N^3_{,\eta} = +\frac{1}{4}\left(1+\xi\right)$$

$$N^4_{,\xi} = -\frac{1}{4}\left(1+\eta\right) \qquad N^4_{,\eta} = +\frac{1}{4}\left(1-\xi\right)$$

D2QU9N Lagrange biquadratic quadrilateral element





Node Number	Coordinates	
	ξ	η
1	-1	-1
2	1	-1
3	1	1
4	-1	1
5	0	-1
6	1	0
7	0	1
8	-1	0
9	0	0

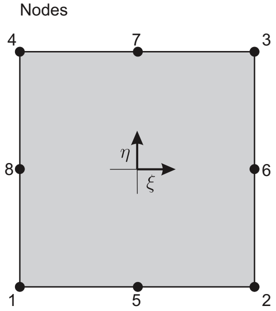
$$
\begin{aligned}
N^1 &= +\frac{1}{4}\left(1-\xi\right)\xi\left(1-\eta\right)\eta\\
N^2 &= -\frac{1}{4}\left(1+\xi\right)\xi\left(1-\eta\right)\eta\\
N^3 &= +\frac{1}{4}\left(1+\xi\right)\xi\left(1+\eta\right)\eta\\
N^4 &= -\frac{1}{4}\left(1-\xi\right)\xi\left(1+\eta\right)\eta\\
N^5 &= -\frac{1}{2}\left(1-\xi\right)\left(1+\xi\right)\left(1-\eta\right)\eta\\
N^6 &= +\frac{1}{2}\left(1+\xi\right)\xi\left(1-\eta\right)\left(1+\eta\right)\\
N^7 &= +\frac{1}{2}\left(1-\xi\right)\left(1+\xi\right)\left(1+\eta\right)\eta\\
N^8 &= -\frac{1}{2}\left(1-\xi\right)\xi\left(1-\eta\right)\left(1+\eta\right)\\
N^9 &= \left(1-\xi\right)\left(1+\xi\right)\left(1-\eta\right)\left(1+\eta\right)
\end{aligned}
$$

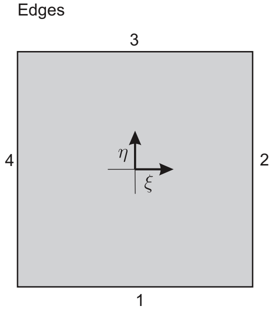
$$
\begin{aligned}
N^1_{,\xi} &= +\frac{1}{4}\left(1-2\xi\right)\left(1-\eta\right)\eta\\
N^2_{,\xi} &= -\frac{1}{4}\left(1+2\xi\right)\left(1-\eta\right)\eta\\
N^3_{,\xi} &= +\frac{1}{4}\left(1+2\xi\right)\left(1+\eta\right)\eta\\
N^4_{,\xi} &= -\frac{1}{4}\left(1-2\xi\right)\left(1+\eta\right)\eta\\
N^5_{,\xi} &= \xi\eta\left(1-\eta\right)\\
N^6_{,\xi} &= \frac{1}{2}\left(1+2\xi\right)\left(1-\eta\right)\left(1+\eta\right)\\
N^7_{,\xi} &= -\xi\eta\left(1+\eta\right)\\
N^8_{,\xi} &= -\frac{1}{2}\left(1-2\xi\right)\left(1-\eta\right)\left(1+\eta\right)\\
N^9_{,\xi} &= -2\xi\left(1-\eta\right)\left(1+\eta\right)
\end{aligned}
$$

$$
\begin{aligned}
N^1_{,\eta} &= +\frac{1}{4}\left(1-\xi\right)\xi\left(1-2\eta\right)\\
N^2_{,\eta} &= -\frac{1}{4}\left(1+\xi\right)\xi\left(1-2\eta\right)\\
N^3_{,\eta} &= +\frac{1}{4}\left(1+\xi\right)\xi\left(1+2\eta\right)\\
N^4_{,\eta} &= -\frac{1}{4}\left(1-\xi\right)\xi\left(1+2\eta\right)\\
N^5_{,\eta} &= \frac{1}{2}\left(1-\xi\right)\left(1+\xi\right)\left(2\eta-1\right)\\
N^6_{,\eta} &= -\left(1+\xi\right)\xi\eta\\
N^7_{,\eta} &= \frac{1}{2}\left(1-\xi\right)\left(1+\xi\right)\left(1+2\eta\right)\\
N^8_{,\eta} &= \left(1-\xi\right)\xi\eta\\
N^9_{,\eta} &= -2\left(1-\xi\right)\left(1+\xi\right)\eta
\end{aligned}
$$

D2QU8N

serendipity biquadratic quadrilateral element





Node Number	Coordinates	
	ξ	η
1	-1	-1
2	1	-1
3	1	1
4	-1	1
5	0	-1
6	1	0
7	0	1
8	-1	0

$$N^1 = -\frac{1}{4}\left(1-\xi\right)\left(1-\eta\right)\left(1+\xi+\eta\right)$$

$$N^2 = -\frac{1}{4}\left(1+\xi\right)\left(1-\eta\right)\left(1-\xi+\eta\right)$$

$$N^3 = -\frac{1}{4}\left(1+\xi\right)\left(1+\eta\right)\left(1-\xi-\eta\right)$$

$$N^4 = -\frac{1}{4}\left(1-\xi\right)\left(1+\eta\right)\left(1+\xi-\eta\right)$$

$$N^5 = \frac{1}{2}\left(1-\xi\right)\left(1+\xi\right)\left(1-\eta\right)$$

$$N^6 = \frac{1}{2}\left(1+\xi\right)\left(1+\eta\right)\left(1-\eta\right)$$

$$N^7 = \frac{1}{2}\left(1-\xi\right)\left(1+\xi\right)\left(1+\eta\right)$$

$$N^8 = \frac{1}{2}\left(1-\xi\right)\left(1+\eta\right)\left(1-\eta\right)$$

$$N^1_{,\xi} = +\frac{1}{4}\left(1-\eta\right)\left(2\xi+\eta\right)$$

$$N^2_{,\xi} = +\frac{1}{4}\left(1-\eta\right)\left(2\xi-\eta\right)$$

$$N^3_{,\xi} = +\frac{1}{4}\left(1+\eta\right)\left(2\xi+\eta\right)$$

$$N^4_{,\xi} = +\frac{1}{4}\left(1+\eta\right)\left(2\xi-\eta\right)$$

$$N^5_{,\xi} = -\xi\left(1-\eta\right)$$

$$N^6_{,\xi} = +\frac{1}{2}\left(1-\eta\right)\left(1+\eta\right)$$

$$N^7_{,\xi} = -\xi\left(1+\eta\right)$$

$$N^8_{,\xi} = -\frac{1}{2}\left(1-\eta\right)\left(1+\eta\right)$$

$$N^1_{,\eta} = +\frac{1}{4}\left(1-\xi\right)\left(\xi+2\eta\right)$$

$$N^2_{,\eta} = +\frac{1}{4}\left(1+\xi\right)\left(-\xi+2\eta\right)$$

$$N^3_{,\eta} = +\frac{1}{4}\left(1+\xi\right)\left(\xi+2\eta\right)$$

$$N^4_{,\eta} = +\frac{1}{4}\left(1-\xi\right)\left(-\xi+2\eta\right)$$

$$N^5_{,\eta} = -\frac{1}{2}\left(1-\xi\right)\left(1+\xi\right)$$

$$N^6_{,\eta} = -\left(1+\xi\right)\eta$$

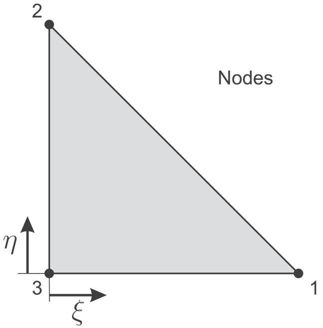
$$N^7_{,\eta} = +\frac{1}{2}\left(1-\xi\right)\left(1+\xi\right)$$

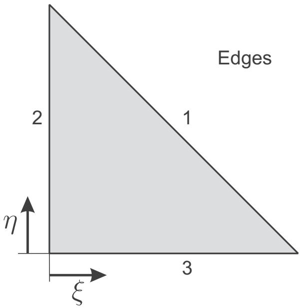
$$N^8_{,\eta} = -\left(1-\xi\right)\eta$$

2D Finite Element Library

D2TR3N

constant strain triangle (CST)





Node Number	Coordinates	
	ξ	η
1	1	0
2	0	1
3	0	0

$$N^1 = \xi \qquad N^1_{,\xi} = 1 \qquad N^1_{,\eta} = 0$$

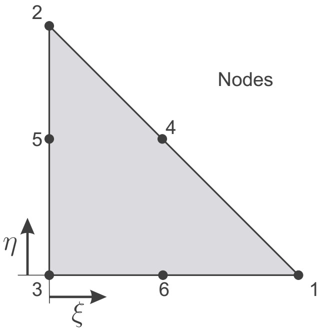
$$N^2 = \eta \qquad N^2_{,\xi} = 0 \qquad N^2_{,\eta} = 1$$

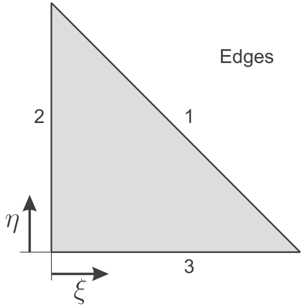
$$N^3 = (1 - \xi - \eta) \qquad N^3_{,\xi} = -1 \qquad N^3_{,\eta}(\xi, \eta) = -1$$

2D Finite Element Library

D2TR6N

quadratic triangle





Node Number	Coordinates	
	ξ	η
1	1	0
2	0	1
3	0	0
4	1/2	1/2
5	0	1/2
6	1/2	0

$N^1 = \xi(2\xi - 1)$

$N^1_{,\xi} = -1 + 4\xi$

$N^1_{,\eta} = 0$

$N^2 = \eta(2\eta - 1)$

$N^2_{,\xi} = 0$

$N^2_{,\eta} = -1 + 4\eta$

$N^3 = (1 - \xi - \eta)(1 - 2\xi - 2\eta)$

$N^3_{,\xi} = -3 + 4\xi + 4\eta$

$N^3_{,\eta} = -3 + 4\xi + 4\eta$

$N^4 = 4\xi\eta$

$N^4_{,\xi} = 4\eta$

$N^4_{,\eta} = 4\xi$

$N^5 = 4\eta(1 - \xi - \eta)$

$N^5_{,\xi} = -4\eta$

$N^5_{,\eta} = -4(-1 + 2\eta + \xi)$

$N^6 = 4\xi(1 - \xi - \eta)$

$N^6_{,\xi} = -4(-1 + \eta + 2\xi)$

$N^6_{,\eta} = -4\xi$

two-dimensional quadrature rule i

Triangular Elements Gauss Point Rule

$$\int_0^1 \int_0^{1-\eta} \{\bullet\} \; \mathrm{d}\xi \; \mathrm{d}\eta \approx \frac{1}{2} \sum_{i=1}^{N_{\text{Gauss Points}}} \alpha_i \; \{\bullet\}|_{\text{Gauss Point}^i}$$

Gauss Point Number	Coordinates		Weight Factor
	ξ	η	α
1	1/3	1/3	1

Gauss Point Number	Coordinates		Weight Factor
	ξ	η	α
1	1/6	1/6	1/3
2	4/6	1/6	1/3
3	1/6	4/6	1/3

two-dimensional quadrature rule ii

Quadrilateral Elements Gauss Point Rule

$$\int_{-1}^{1}\int_{-1}^{1}\{\bullet\}\ \mathrm{d}\xi\ \mathrm{d}\eta \approx \sum_{i=1}^{N_{\text{Gauss Points}}} \alpha_i\ \{\bullet\}|_{\text{Gauss Point}^i}$$

Gauss Point Number	Coordinates		Weight Factor
	ξ	η	α
1	0	0	2×2

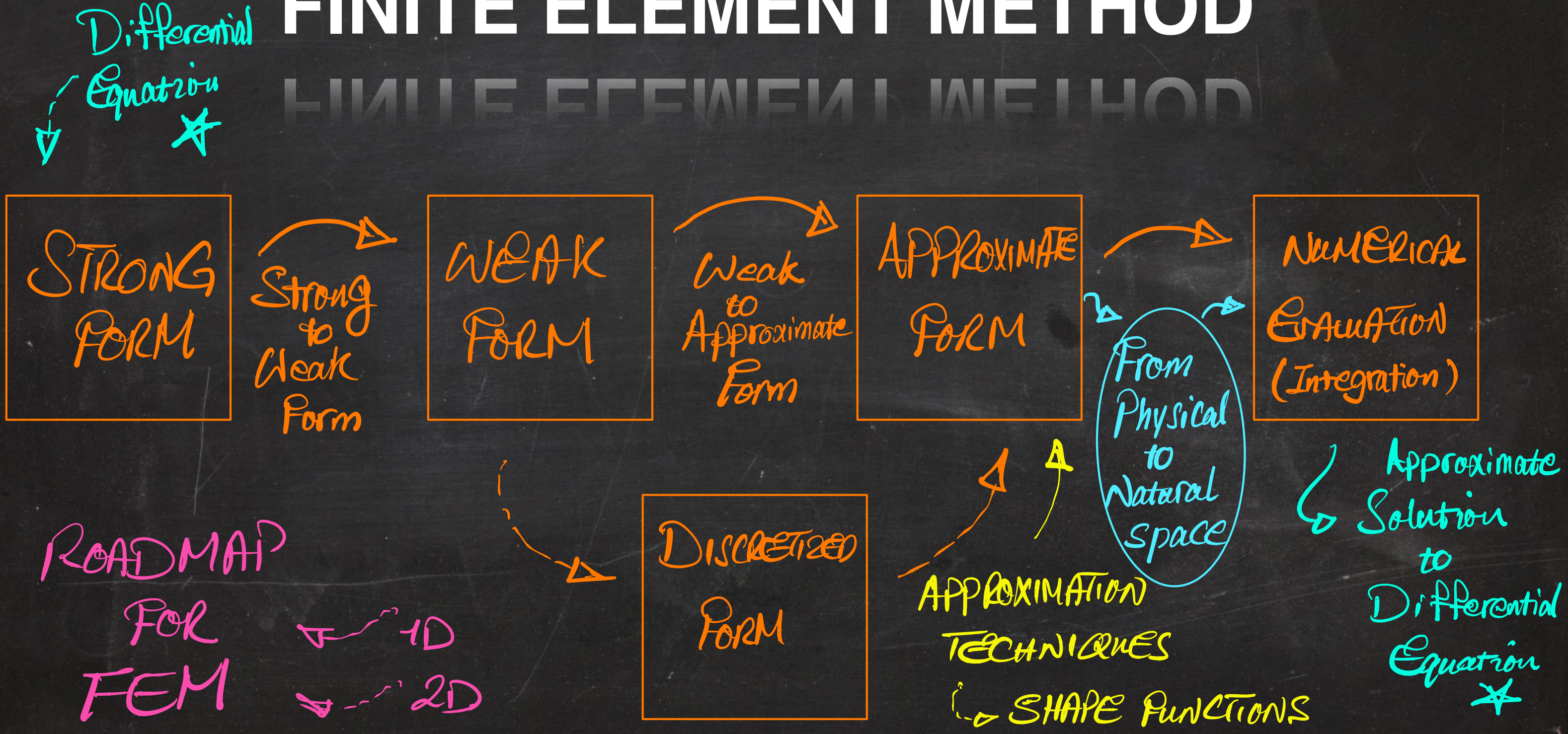
Gauss Point Number	Coordinates		Weight Factor
	ξ	η	α
1	$-1/\sqrt{3}$	$-1/\sqrt{3}$	1×1
2	$+1/\sqrt{3}$	$-1/\sqrt{3}$	1×1
3	$+1/\sqrt{3}$	$+1/\sqrt{3}$	1×1
4	$-1/\sqrt{3}$	$+1/\sqrt{3}$	1×1

2D Finite Element Library

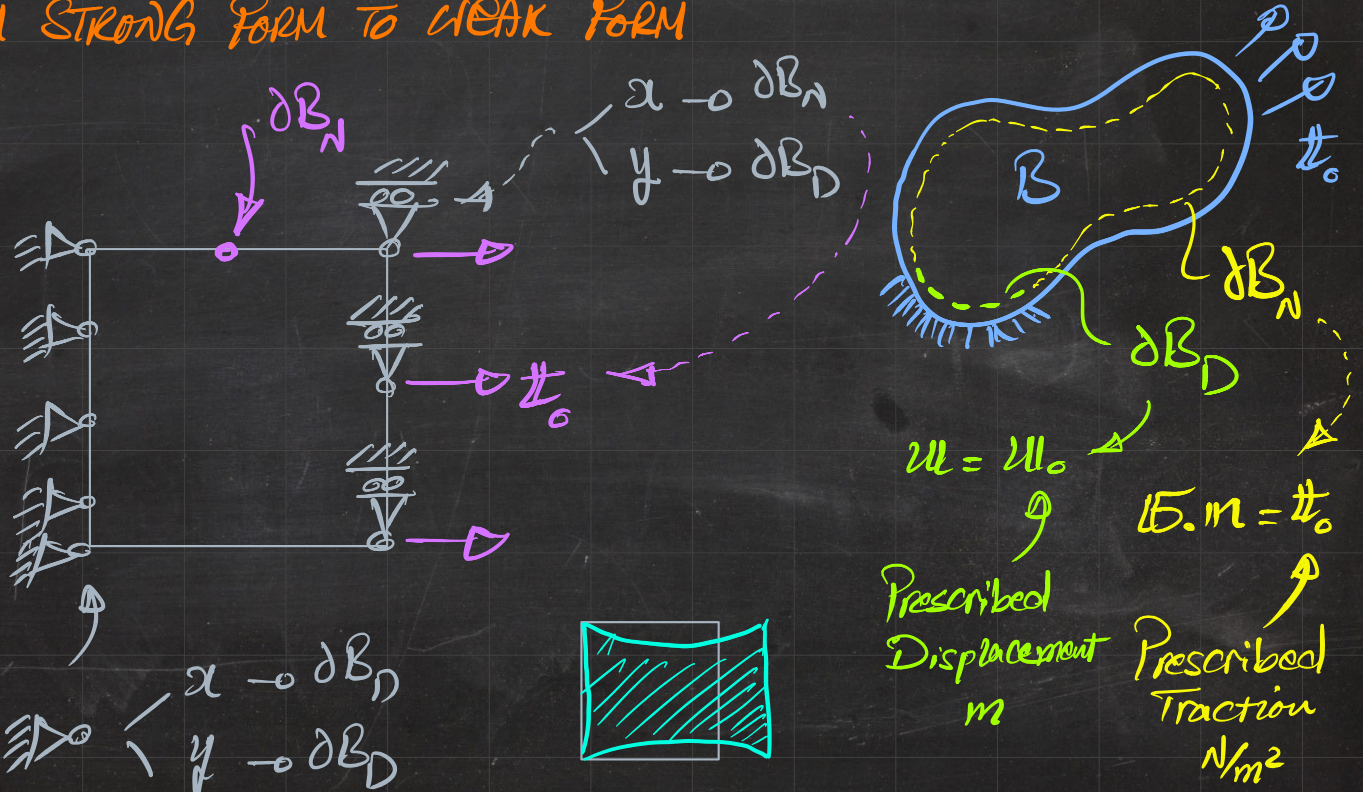
two-dimensional quadrature rule iii

Gauss Point Number	Coordinates		Weight Factor
	ξ	η	α
1	$-\sqrt{3/5}$	$-\sqrt{3/5}$	$5/9 \times 5/9$
2	$+\sqrt{3/5}$	$-\sqrt{3/5}$	$5/9 \times 5/9$
3	$\sqrt{3/5}$	$\sqrt{3/5}$	$5/9 \times 5/9$
4	$-\sqrt{3/5}$	$\sqrt{3/5}$	$5/9 \times 5/9$
5	0	$-\sqrt{3/5}$	$5/9 \times 8/9$
6	$+\sqrt{3/5}$	0	$5/9 \times 8/9$
7	0	$\sqrt{3/5}$	$5/9 \times 8/9$
8	$-\sqrt{3/5}$	0	$5/9 \times 8/9$
9	0	0	$8/9 \times 8/9$

FINITE ELEMENT METHOD



FROM STRONG FORM TO WEAK FORM



FROM STRONG FORM TO WEAK FORM

$\mathrm{Div}\,\boldsymbol{\sigma} = 0$ in $\mathcal{B}$ subject to BCs

↳ STRONG FORM IN THE ABSENCE OF BODY FORCES

$\boldsymbol{u} = \boldsymbol{u}_0$ at $\partial\mathcal{B}_D$

$\boldsymbol{\sigma}\cdot\boldsymbol{n} = \boldsymbol{t}_0$ at $\partial\mathcal{B}_N$

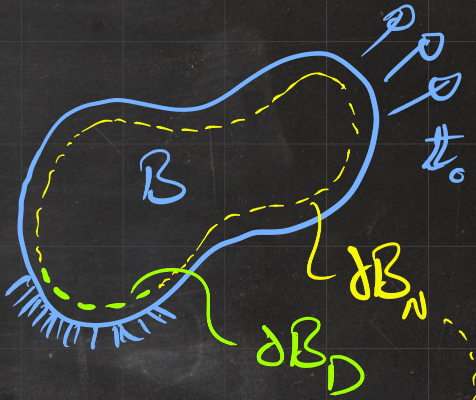
dot

$\boldsymbol{w}\cdot\mathrm{Div}\,\boldsymbol{\sigma} = 0$ SCALAR

↳ TEST FUNCTION

↳ $\forall\,\boldsymbol{w}$

$\boldsymbol{w}|_{\partial\mathcal{B}_D} = 0$



$\boldsymbol{u} = \boldsymbol{u}_0$

Prescribed Displacement

m

$\boldsymbol{\sigma}\cdot\boldsymbol{n} = \boldsymbol{t}_0$

Prescribed Traction

N/m²

FROM STRONG FORM TO WEAK FORM

$u = u_0$ at $\partial\mathcal{B}_D$

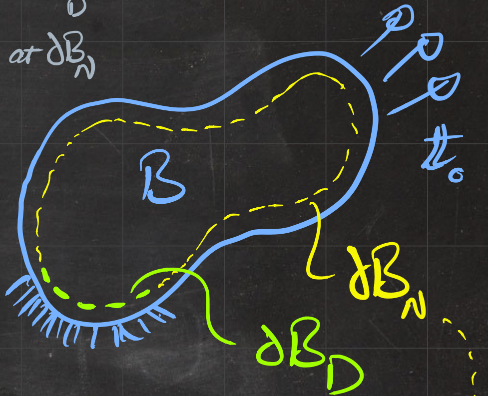
$\mathbf{P}\cdot n = t_0$ at $\partial\mathcal{B}_N$

$\text{Div}\,\mathbf{P} = 0$ in $\mathcal{B}$ subject to BCs

$$w \cdot \text{Div}\,\mathbf{P} = 0 \qquad \forall w\,, \quad w|_{\partial\mathcal{B}_D} = 0$$

$$\int_{\mathcal{B}} [\text{Grad}\, w] : \mathbf{P}\, dA = \int_{\mathcal{B}} \text{Div}(w\cdot\mathbf{P})\, dA$$

$$\boxed{\int_{\mathcal{B}} [\text{Grad}\, w] : \mathbf{P}\, dA = \int_{\partial\mathcal{B}_N} w\cdot t_0\, dL}$$



$u = u_0$

Prescribed Displacement

m

$\mathbf{P}\cdot n = t_0$

Prescribed Traction

N/m^2

FROM WEAK FORM TO APPROXIMATE FORM

$$\boxed{\int_{\mathcal{B}} [\text{Grad}\,\mathbf{w}] : \mathbf{P}\, dA = \int_{\partial\mathcal{B}_N} \mathbf{w} \cdot \mathbf{t}_0\, dL}$$

$$\mathbf{w}^j \otimes \text{Grad}\, N^j$$

$$\mathbf{P} = \mathbb{E} : \boldsymbol{\varepsilon}$$

$$\Downarrow$$

$$\mathbf{P} = \mathbb{E} : [\text{Grad}\,\mathbf{u}]$$

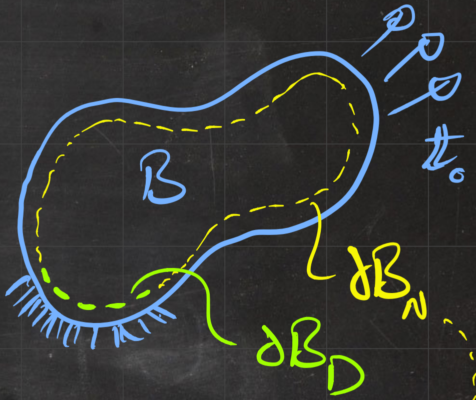
$$\text{Grad}\,\mathbf{u} = \mathbf{u}^i \otimes \text{Grad}\, N^i$$

$$\mathbf{w} = N^j \mathbf{w}^j$$

$$\text{Grad}\,\mathbf{w} = \mathbf{w}^j \otimes \text{Grad}\, N^j$$

$$\mathbf{u} = N^i \mathbf{u}^i$$

$$= N^1 \mathbf{u}^1 + N^2 \mathbf{u}^2 + \dots$$



$$\mathbf{u} = \mathbf{u}_0$$

Prescribed Displacement

m

$$\mathbf{P} \cdot \mathbf{n} = \mathbf{t}_0$$

Prescribed Traction

N/m^2

FROM WEAK FORM TO APPROXIMATE FORM

$$\boxed{\int_{\mathcal{B}} [\text{Grad}\, \boldsymbol{w}] : \boldsymbol{\sigma}\, dA = \int_{\partial \mathcal{B}_N} \boldsymbol{w} \cdot \boldsymbol{t}_0 \, dL}$$

$$\boldsymbol{w}^j \otimes \text{Grad}\, N^j$$

$$\boldsymbol{\sigma} = \mathbb{E} : \boldsymbol{\varepsilon}$$

$$\Downarrow$$

$$\boldsymbol{\sigma} = \mathbb{E} : [\text{Grad}\, \boldsymbol{u}]$$

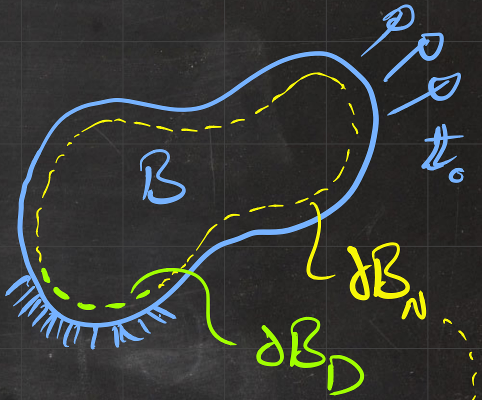
$$\Downarrow$$

$$\boldsymbol{\sigma} = \mathbb{E} : [\boldsymbol{u}^i \otimes \text{Grad}\, N^i]$$

$$\boldsymbol{w} = N^j \boldsymbol{w}^j$$

$$\downarrow$$

$$\text{Grad}\, \boldsymbol{w} = \boldsymbol{w}^j \otimes \text{Grad}\, N^j$$



$$\boldsymbol{u} = \boldsymbol{u}_0$$

Prescribed Displacement

m

$$\boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{t}_0$$

Prescribed Traction

N/m^2

FROM WEAK FORM TO APPROXIMATE FORM

$$\boxed{\int_B [\mathrm{Grad}\,\mathbb{w}] : \mathbb{S}\, dA = \int_{\partial B_N} \mathbb{w} \cdot \mathbb{t}_0 \, dL}$$

$$[\mathbb{K}^{ij}]_{ac} = \int_B [N^i_{,x}]_b \, E_{abcd} \, [N^j_{,x}]_d \, dA$$

$$\boxed{[\mathbb{K}^{ij}]_{ac} \, [\mathbb{u}^j]_c = [\mathbb{F}^i]_a}$$

$\mathbb{K}^{ij}$ STIFFNESS BETWEEN NODES i & j

↳ 2x2 matrix

$$\{i, j\} \in \{1, 2, \ldots, NPE\}$$

$$[\mathbb{K}^{ij}] = \begin{bmatrix} K^{ij}_{11} & K^{ij}_{12} \\ K^{ij}_{21} & K^{ij}_{22} \end{bmatrix}$$

APPROXIMATE FORM

$$\boxed{[\mathbb{K}^{ij}]_{ac}\,[u^{j}]_{c} = [\mathbb{F}^{i}]_{a}}$$

$$\hookrightarrow [\mathbb{K}][\mathbb{u}] = [\mathbb{F}]$$

$$\{i, j\} \in \{1, 2, \dots, NPE\}$$

$$[\mathbb{K}^{ij}]_{ac} = \int_B [N^{i}_{,x}]_b \, E_{abcd} \, [N^{j}_{,x}]_d \, dA$$

$\mathbb{K}^{ij} \rightarrow$ STIFFNESS BETWEEN NODES i & j

$\hookrightarrow 2 \times 2$ matrix

$$[\mathbb{K}^{ij}] = \begin{bmatrix} K^{ij}_{11} & K^{ij}_{12} \\ K^{ij}_{21} & K^{ij}_{22} \end{bmatrix}$$

UNDERSTANDING $[\mathbb{K}^{ij}]_{ac}$ ⟵ $\mathbb{K}^{ij}_{ac}$ ⟵ SUPERSCRIPTS ⟵ NODES

⟵ SUBSCRIPTS ⟵ DIRECTIONS

$$[\mathbb{K}^{ij}]_{ac}$$

$$\downarrow$$

$$K^{ij}_{ac} = \frac{\delta F^i_a}{\delta u^j_c}$$

$$[\mathbb{K}^{ij}]_{ac} = \int_{\mathcal{B}} [N^i_{,a}]_b \, E_{abcd} \, [N^j_{,a}]_d \, dA$$

↪ STIFFNESS BETWEEN

DIRECTION a of NODE i

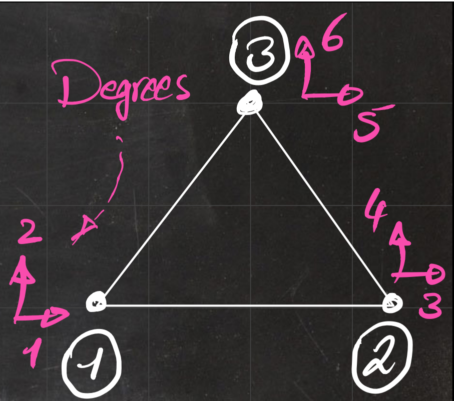
&

DIRECTION c of NODE j

$$\mathbb{K}_{\Delta} = \begin{bmatrix} \mathbb{K}^{11} & \mathbb{K}^{12} & \mathbb{K}^{13} \\ \mathbb{K}^{21} & \mathbb{K}^{22} & \mathbb{K}^{23} \\ \mathbb{K}^{31} & \mathbb{K}^{32} & \mathbb{K}^{33} \end{bmatrix} \longleftarrow 6 \times 6$$

$$\hookrightarrow \underset{3}{\underset{\uparrow}{\text{NoN}}} \times \underset{2}{\underset{\uparrow}{\text{PD}}}$$

$$\mathbb{K}_{\Delta} = \left[\begin{array}{cc|cc|cc} K^{11}_{11} & K^{11}_{12} & K^{12}_{11} & K^{12}_{12} & K^{13}_{11} & K^{13}_{12} \\ K^{11}_{21} & K^{11}_{22} & K^{12}_{21} & K^{12}_{22} & K^{13}_{21} & K^{13}_{22} \\ \hline K^{21}_{11} & K^{21}_{12} & K^{22}_{11} & K^{22}_{12} & K^{23}_{11} & K^{23}_{12} \\ K^{21}_{21} & K^{21}_{22} & K^{22}_{21} & K^{22}_{22} & K^{23}_{21} & K^{23}_{22} \\ \hline K^{31}_{11} & K^{31}_{12} & K^{32}_{11} & K^{32}_{12} & K^{33}_{11} & K^{33}_{12} \\ K^{31}_{21} & K^{31}_{22} & K^{32}_{21} & K^{32}_{22} & K^{33}_{21} & K^{33}_{22} \end{array} \right]$$



$1 \rightarrow \text{NODE}^1_x$

$2 \rightarrow \text{NODE}^1_y$

$3 \rightarrow \text{NODE}^2_x$

$4 \rightarrow \text{NODE}^2_y$

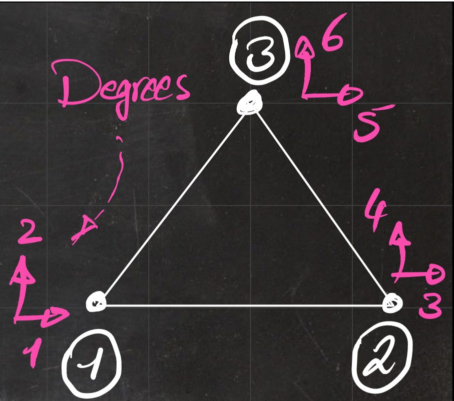
$5 \rightarrow \text{NODE}^3_x$

$6 \rightarrow \text{NODE}^3_y$

$$\mathbb{K}_{\Delta} = \begin{bmatrix} \mathbb{K}^{11} & \mathbb{K}^{12} & \mathbb{K}^{13} \\ \mathbb{K}^{21} & \mathbb{K}^{22} & \mathbb{K}^{23} \\ \mathbb{K}^{31} & \mathbb{K}^{32} & \mathbb{K}^{33} \end{bmatrix} \longleftarrow 6\times6$$

↳ NoN x PD

$\uparrow 3 \quad \uparrow 2$



$$\mathbb{K}_{\Delta} = \begin{array}{c|cc|cc|cc|c} & 1_x & 1_y & 2_x & 2_y & 3_x & 3_y & \\ \hline & K^{11}_{11} & K^{11}_{12} & K^{12}_{11} & K^{12}_{12} & K^{13}_{11} & K^{13}_{12} & 1_x \\ & K^{11}_{21} & K^{11}_{22} & K^{12}_{21} & K^{12}_{22} & K^{13}_{21} & K^{13}_{22} & 1_y \\ \hline & K^{21}_{11} & K^{21}_{12} & K^{22}_{11} & K^{22}_{12} & K^{23}_{11} & K^{23}_{12} & 2_x \\ & K^{21}_{21} & K^{21}_{22} & K^{22}_{21} & K^{22}_{22} & K^{23}_{21} & K^{23}_{22} & 2_y \\ \hline & K^{31}_{11} & K^{31}_{12} & K^{32}_{11} & K^{32}_{12} & K^{33}_{11} & K^{33}_{12} & 3_x \\ & K^{31}_{21} & K^{31}_{22} & K^{32}_{21} & K^{32}_{22} & K^{33}_{21} & K^{33}_{22} & 3_y \end{array}$$

1 → NODE^1_x

2 → NODE^1_y

3 → NODE^2_x

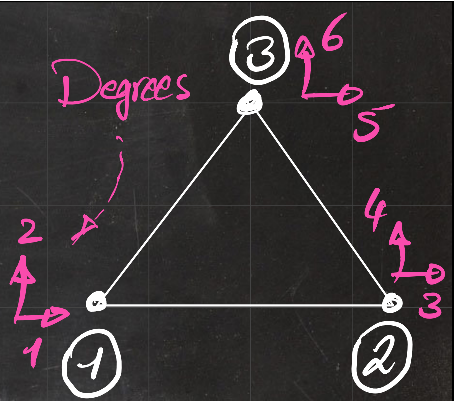
4 → NODE^2_y

5 → NODE^3_x

6 → NODE^3_y

$$\underset{\Delta}{\mathbb{K}} = \begin{bmatrix} \mathbb{K}^{11} & \mathbb{K}^{12} & \mathbb{K}^{13} \\ \mathbb{K}^{21} & \mathbb{K}^{22} & \mathbb{K}^{23} \\ \mathbb{K}^{31} & \mathbb{K}^{32} & \mathbb{K}^{33} \end{bmatrix} \leftarrow 6\times6$$

$$\hookrightarrow \underset{\overset{\uparrow}{3}}{\text{NoN}} \times \underset{\overset{\uparrow}{2}}{\text{PD}}$$



$$\underset{\Delta}{\mathbb{K}} = \begin{array}{cccccccc} & 1_x & 1_y & 2_x & 2_y & 3_x & 3_y & \\ \left[\right. & K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} & \left.\right] 1_x \\ & K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} & 1_y \\ & K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} & 2_x \\ & K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} & 2_y \\ & K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} & 3_x \\ & K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} & 3_y \end{array}$$

$1 \rightarrow \text{NODE}^{1}{}_{x}$

$2 \rightarrow \text{NODE}^{1}{}_{Y}$

$3 \rightarrow \text{NODE}^{2}{}_{x}$

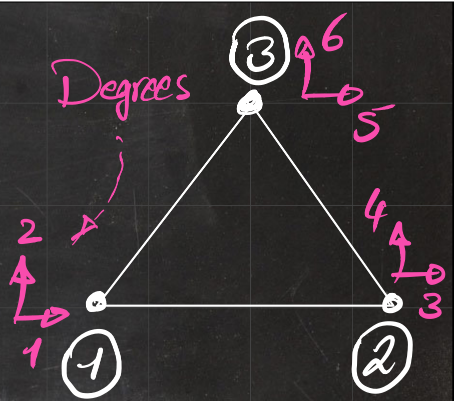
$4 \rightarrow \text{NODE}^{2}{}_{Y}$

$5 \rightarrow \text{NODE}^{3}{}_{x}$

$6 \rightarrow \text{NODE}^{3}{}_{y}$

$$\mathbb{K}_{\Delta} = \begin{bmatrix} \mathbb{K}^{11} & \mathbb{K}^{12} & \mathbb{K}^{13} \\ \mathbb{K}^{21} & \mathbb{K}^{22} & \mathbb{K}^{23} \\ \mathbb{K}^{31} & \mathbb{K}^{32} & \mathbb{K}^{33} \end{bmatrix} \leftarrow 6\times 6$$

$\hookrightarrow \underset{\uparrow 3}{\text{NoN}} \times \underset{\uparrow 2}{\text{PD}}$



$$\mathbb{K}_{\Delta} = \begin{array}{cccccc|c} 1_x & 1_y & 2_x & 2_y & 3_x & 3_y & \\ K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} & 1_x \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} & 1_y \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} & 2_x \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} & 2_y \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} & 3_x \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} & 3_y \end{array}$$

$1 \to \text{NODE}^{1}_{x}$

$2 \to \text{NODE}^{1}_{Y}$

$3 \to \text{NODE}^{2}_{x}$

$4 \to \text{NODE}^{2}_{y}$

$5 \to \text{NODE}^{3}_{x}$

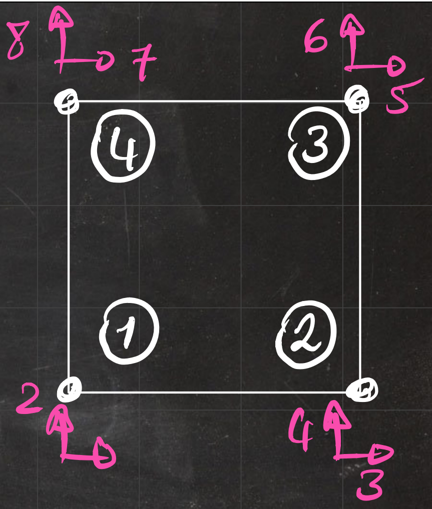
$6 \to \text{NODE}^{3}_{y}$

$$\mathbb{K}_{\square} = \begin{bmatrix} \mathbb{K}^{11} & \mathbb{K}^{12} & \mathbb{K}^{13} & \mathbb{K}^{14} \\ \mathbb{K}^{21} & \mathbb{K}^{22} & \mathbb{K}^{23} & \mathbb{K}^{24} \\ \mathbb{K}^{31} & \mathbb{K}^{32} & \mathbb{K}^{33} & \mathbb{K}^{34} \\ \mathbb{K}^{41} & \mathbb{K}^{42} & \mathbb{K}^{43} & \mathbb{K}^{44} \end{bmatrix} \rightarrow 8 \times 8$$

D2QU4N

↳ NoN x PD

↳ 4 ↳ 2



1, 2 → $\text{NODE}^1_{x,y}$

3, 4 → $\text{NODE}^2_{x,y}$

5, 6 → $\text{NODE}^3_{x,y}$

7, 8 → $\text{NODE}^4_{x,y}$

$$\mathbb{K}^{ij} = \begin{bmatrix} K^{ij}_{11} & K^{ij}_{12} \\ K^{ij}_{21} & K^{ij}_{22} \end{bmatrix} = \begin{bmatrix} K^{ij}_{xx} & K^{ij}_{xy} \\ K^{ij}_{yx} & K^{ij}_{yy} \end{bmatrix}$$

D2TR3N ← $[\mathbb{K}]_{6\times6}$

D2TR6N ← $[\mathbb{K}]_{12\times12}$

D2QU4N ← $[\mathbb{K}]_{8\times8}$

D2QU8N ← $[\mathbb{K}]_{16\times16}$

D2QU9N ← $[\mathbb{K}]_{18\times18}$

PROBLEMS TO ADDRESS

- ↳ INTEGRAL ↳ GAUSS QUADRATURE RULE
- ↳ $f(x)$ ↳ $x \to \$$
- ↳ E_{abcd} ↳ ?

$$[\mathbb{K}^{ij}]_{ac} = \int_{B} [N^{i}_{,x}]_{b}\, E_{abcd}\, [N^{j}_{,x}]_{d}\, dA$$

D2TR3N ← $[\mathbb{K}]_{6x6}$

D2TR6N ← $[\mathbb{K}]_{12x12}$

D2QU4N ← $[\mathbb{K}]_{8x8}$

D2QU8N ← $[\mathbb{K}]_{16x16}$

D2QU9N ← $[\mathbb{K}]_{18x18}$

$$E_{abcd} = \frac{E}{2[1+\nu]}\left[\delta_{ad}\,\delta_{bc}+\delta_{ac}\,\delta_{bd}\right] + \frac{E\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

CONSTITUTIVE TENSOR

4th.O.

2x2x2x2 = 16 COMPONENTS

Young's Modulus

ν: Poisson's Ratio

\nu

δ: Kronecker Delta

$$[\mathbb{K}^{ij}]_{ac} = \int_B [N^i_{,x}]_b\, E_{abcd}\, [N^j_{,x}]_d\, dA$$

D2TR3N $\leftarrow [\mathbb{K}]_{6\times 6}$

D2TR6N $\leftarrow [\mathbb{K}]_{12\times 12}$

D2QU4N $\leftarrow [\mathbb{K}]_{8\times 8}$

D2QU8N $\leftarrow [\mathbb{K}]_{16\times 16}$

D2QU9N $\leftarrow [\mathbb{K}]_{18\times 18}$

$$E_{abcd} = \frac{E}{2[1+\nu]}\left[\delta_{ad}\delta_{bc} + \delta_{ac}\delta_{bd}\right] + \frac{E\nu}{1-\nu^2}\delta_{ab}\delta_{cd}$$

$$\nu = \frac{\nu_{3D}}{1-\nu_{3D}}$$

PLANE STRAIN

$$E_{1111} = \frac{E}{[1+\nu]} + \frac{E\nu}{1-\nu^2} = \frac{E[1-\nu]+E\nu}{1-\nu^2} = \frac{E}{1-\nu^2}$$

a b c d

$$[\mathbb{K}^{ij}]_{ac} = \int_{\mathcal{B}} [\overset{i}{N}_{,\alpha}]_b \, E_{abcd} \, [\overset{j}{N}_{,\alpha}]_d \, dA$$

$$[\mathbb{K}^{ij}]_{ac} = \int_{B} [N^{i}_{,x}]_{b} \, E_{abcd} \, [N^{j}_{,x}]_{d} \, dA$$

$$E_{abcd} = \frac{E}{2[1+\nu]}\left[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}\right] + \frac{E\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

$$E_{1111} = \frac{E}{[1-\nu^2]} \qquad E_{1112} = 0 \qquad E_{1121} = 0 \qquad E_{1122} = \frac{E\nu}{[1-\nu^2]}$$

$$E_{1211} = 0 \qquad E_{1212} = \frac{E}{2[1+\nu]} \qquad E_{1221} = \frac{E}{2[1+\nu]} \qquad E_{1222} = 0$$

$$E_{2111} = 0 \qquad E_{2112} = \frac{E}{2[1+\nu]} \qquad E_{2121} = \frac{E}{2[1+\nu]} \qquad E_{2122} = 0$$

$$E_{2211} = \frac{E\nu}{[1-\nu^2]} \qquad E_{2212} = 0 \qquad E_{2221} = 0 \qquad E_{2222} = \frac{E}{[1-\nu^2]}$$

$$[\mathbb{K}^{ij}]_{ac} = \int_{\mathcal{B}} [N^i_{,\mathbb{x}}]_b \, E_{abcd} \, [N^j_{,\mathbb{x}}]_d \, dA$$

$$N^i_{,\mathbb{x}} \rightarrow \begin{bmatrix} \dfrac{\partial N^i}{\partial x} \\[2ex] \dfrac{\partial N^i}{\partial y} \end{bmatrix}$$

$$\frac{\partial N^i}{\partial x} = \frac{\partial N^i}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial N^i}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$\frac{\partial N^i}{\partial y} = \frac{\partial N^i}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial N^i}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$x = x(\xi, \eta)$

$y = y(\xi, \eta)$

$\mathbb{x} = \mathbb{x}(\boldsymbol{\xi})$

$\xi = \xi(x, y)$

$\eta = \eta(x, y)$

$\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbb{x})$

$$N^i_{,\boldsymbol{x}}$$

$$[\mathbb{K}^{ij}]_{ac} = \int_{\mathcal{B}} [N^i_{,\boldsymbol{x}}]_b \, E_{abcd} \, [N^j_{,\boldsymbol{x}}]_d \, dA$$

$$x = x(\xi,\eta)$$

$$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{\xi}) \leftarrow \quad y = y(\xi,\eta)$$

$$\updownarrow$$

$$\underbrace{\begin{bmatrix} \dfrac{\partial N^i}{\partial x} \\[2ex] \dfrac{\partial N^i}{\partial y} \end{bmatrix}}_{N^i_{,\boldsymbol{x}}} = \begin{bmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \eta}{\partial x} \\[2ex] \dfrac{\partial \xi}{\partial y} & \dfrac{\partial \eta}{\partial y} \end{bmatrix} \underbrace{\begin{bmatrix} \dfrac{\partial N^i}{\partial \xi} \\[2ex] \dfrac{\partial N^i}{\partial \eta} \end{bmatrix}}_{N^i_{,\boldsymbol{\xi}}}$$

$$\frac{\partial N^i}{\partial \boldsymbol{x}} \leftarrow N^i_{,\boldsymbol{x}}$$

$$\frac{\partial N^i}{\partial \boldsymbol{\xi}} \leftarrow N^i_{,\boldsymbol{\xi}}$$

$$\xi = \xi(x,y)$$

$$\eta = \eta(x,y)$$

$$\swarrow$$

$$\boldsymbol{\xi} = \boldsymbol{\xi}(\boldsymbol{x})$$

$N^i_{,\boldsymbol{x}}$

$$[\mathbb{K}^{ij}]_{ac} = \int_{\mathcal{B}} [N^i_{,\boldsymbol{x}}]_b \, E_{abcd} \, [N^j_{,\boldsymbol{x}}]_d \, dA$$

$x = x(\xi,\eta)$

$y = y(\xi,\eta)$

$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{\xi})$

$$\left[\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{v}}\right]_{\alpha\beta} = \frac{\partial u_\alpha}{\partial v_\beta}$$

$\mathbb{J}^{-1}$

$$\begin{bmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \xi}{\partial y} \\[2ex] \dfrac{\partial \eta}{\partial x} & \dfrac{\partial \eta}{\partial y} \end{bmatrix} \quad \frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{x}}$$

$$N^i_{,\boldsymbol{x}} = \left[\; \dots \;\right] N^i_{,\boldsymbol{\xi}}$$

$\xi = \xi(x,y)$

$\eta = \eta(x,y)$

$\boldsymbol{\xi} = \boldsymbol{\xi}(\boldsymbol{x})$

$\mathbb{J}$

$$\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}} \;\text{2D}\; \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta} \\[2ex] \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \eta}{\partial x} \\[2ex] \dfrac{\partial \xi}{\partial y} & \dfrac{\partial \eta}{\partial y} \end{bmatrix}$$

$$N^{i}_{,\mathbf{x}}$$

$$[\mathbb{K}^{ij}]_{ac} = \int_{\mathcal{B}} [N^{i}_{,\mathbf{x}}]_{b} \, E_{abcd} \, [N^{j}_{,\mathbf{x}}]_{d} \, dA$$

$$x = x(\xi,\eta)$$

$$y = y(\xi,\eta)$$

$$\mathbf{x} = \mathbf{x}(\boldsymbol{\xi})$$

$$\begin{bmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \xi}{\partial y} \\[2ex] \dfrac{\partial \eta}{\partial x} & \dfrac{\partial \eta}{\partial y} \end{bmatrix} \leftarrow \mathbb{J}^{-1}$$

$$\mathbb{J} = \frac{\partial \mathbf{x}}{\partial \boldsymbol{\xi}}$$

$$\mathbb{J}^{-1} = \frac{\partial \boldsymbol{\xi}}{\partial \mathbf{x}}$$

$$N^{i}_{,\mathbf{x}} = \big[\ \cdots\ \big] N^{i}_{,\boldsymbol{\xi}}$$

$$\begin{bmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \eta}{\partial x} \\[2ex] \dfrac{\partial \xi}{\partial y} & \dfrac{\partial \eta}{\partial y} \end{bmatrix} = \mathbb{J}^{-t}$$

$$\xi = \xi(x,y)$$

$$\eta = \eta(x,y)$$

$$\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{x})$$

$$[\mathbb{K}^{ij}]_{ac} = \int_{\mathcal{B}} [N^{i}_{,\boldsymbol{x}}]_b \, E_{abcd} \, [N^{j}_{,\boldsymbol{x}}]_d \, dA$$

$$[\mathbb{K}^{ij}]_{ac} = \int_{\mathcal{B}} [\mathbb{J}^{-t} \cdot N^{i}_{,\boldsymbol{\xi}}]_b \, E_{abcd} \, [\mathbb{J}^{-t} \cdot N^{j}_{,\boldsymbol{\xi}}]_d \, dA$$

$$\mathbb{J} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}} \quad \hookrightarrow \boldsymbol{x} = \boldsymbol{x}(\boldsymbol{\xi}) \quad ✓ \quad \boldsymbol{x} = N^{s} \boldsymbol{x}^{s}$$

$$\hookrightarrow N^{s}(\xi, \eta)$$

$$[\mathbb{K}^{ij}]_{ac} = \int_{\mathcal{B}} [\mathbf{J}^{-t} \cdot N^{i}_{,\xi}]_{b} \, \mathbb{E}_{abcd} \, [\mathbf{J}^{-t} \cdot N^{j}_{,\xi}]_{d} \, dA$$

$$J_{11} = \frac{\partial x}{\partial \xi} = x^{1} \frac{\partial N^{1}}{\partial \xi} + x^{2} \frac{\partial N^{2}}{\partial \xi} + \cdots + x^{NPE} \frac{\partial N^{NPE}}{\partial \xi}$$

$$J_{12} = \frac{\partial x}{\partial \eta} = x^{1} \frac{\partial N^{1}}{\partial \eta} + x^{2} \frac{\partial N^{2}}{\partial \eta} + \cdots + x^{NPE} \frac{\partial N^{NPE}}{\partial \eta}$$

$$J_{21} = \frac{\partial y}{\partial x} = y^{1} \frac{\partial N^{1}}{\partial \xi} + y^{2} \frac{\partial N^{2}}{\partial \xi} + \cdots + y^{NPE} \frac{\partial N^{NPE}}{\partial \xi}$$

$$J_{22} = \frac{\partial y}{\partial \eta} = y^{1} \frac{\partial N^{1}}{\partial \eta} + y^{2} \frac{\partial N^{2}}{\partial \eta} + \cdots + y^{NPE} \frac{\partial N^{NPE}}{\partial \eta}$$

$$[\mathbb{K}^{ij}]_{ac} = \int_{\mathcal{B}} [\mathbb{J}^{-t} \cdot N^{i}_{,\xi}]_{b} \, \mathbb{E}_{abcd} \, [\mathbb{J}^{-t} \cdot N^{j}_{,\xi}]_{d} \, dA$$

$$\underbrace{\begin{bmatrix} J_{11} & J_{21} \\ J_{12} & J_{22} \end{bmatrix}}_{\mathbb{J}^{t}} = \underbrace{\begin{bmatrix} \dfrac{\partial N^1}{\partial \xi} & \dfrac{\partial N^2}{\partial \xi} & \dots & \dfrac{\partial N^{NPE}}{\partial \xi} \\[2ex] \dfrac{\partial N^1}{\partial \eta} & \dfrac{\partial N^2}{\partial \eta} & \dots & \dfrac{\partial N^{NPE}}{\partial \eta} \end{bmatrix}}_{2 \times NPE} \underbrace{\begin{bmatrix} x^1 & y^1 \\ x^2 & y^2 \\ \vdots & \vdots \\ x^{NPE} & y^{NPE} \end{bmatrix}}_{NPE \times 2}$$

$$K^{ij}_{ac} = \sum_{gp=1}^{GPE} [\underline{J}^{-t} \cdot N^{i}_{,\xi}]_b \, E_{abcd} \, [\underline{J}^{-t} \cdot N^{j}_{,\xi}]_d \, \mathrm{Det}\,\underline{J} \times \alpha_{gp} \times \frac{1}{2}$$

IF TRIANGLE

JACOBIAN $\frac{\partial x}{\partial \xi}$ → $\underline{J} = \begin{bmatrix} x^1 \dots x^{NPE} \\ y^1 \dots y^{NPE} \end{bmatrix} \begin{bmatrix} N^1_{,\xi} & N^1_{,\eta} \\ \vdots & \vdots \\ N^{NPE}_{,\xi} & N^{NPE}_{,\eta} \end{bmatrix}$

$$E_{abcd} = \frac{E}{2[1+\nu]} [\delta_{ad}\delta_{bc} + \delta_{ac}\delta_{bd}] + \frac{E\nu}{1-\nu^2}\delta_{ab}\delta_{cd}$$

$$\left\{ \begin{array}{llll} E_{1111} = \frac{E}{[1-\nu^2]} & E_{1112} = 0 & E_{1121} = 0 & E_{1122} = \frac{E\nu}{[1-\nu^2]} \\ E_{1211} = 0 & E_{1212} = \frac{E}{2[1+\nu]} & E_{1221} = \frac{E}{2[1+\nu]} & E_{1222} = 0 \\ E_{2111} = 0 & E_{2112} = \frac{E}{2[1+\nu]} & E_{2121} = \frac{E}{2[1+\nu]} & E_{2122} = 0 \\ E_{2211} = \frac{E\nu}{[1-\nu^2]} & E_{2212} = 0 & E_{2221} = 0 & E_{2222} = \frac{E}{[1-\nu^2]} \end{array} \right.$$

FINITE ELEMENT METHOD

2D FEM

formulation summary
& understanding via examples

$$K^{i\,j}_{a\,c}$$ *stiffness between direction "a" of node "i" & direction "c" of node "j"* $$\equiv \frac{\delta F^{i}_{a}}{\delta u^{j}_{c}}$$

Quadrilateral Elements

$$K^{i\,j}_{a\,c} = \sum_{\mathrm{gp}\,=\,1}^{\mathrm{GPE}} \left[\boldsymbol{J}^{-T} \cdot N^{i}_{,\boldsymbol{\xi}}\right]_{b} E_{a\,b\,c\,d} \left[\boldsymbol{J}^{-T} \cdot N^{j}_{,\boldsymbol{\xi}}\right]_{d} \mathrm{Det}\boldsymbol{J}\,\alpha_{gp}$$

Triangular Elements

$$K^{i\,j}_{a\,c} = \sum_{\mathrm{gp}\,=\,1}^{\mathrm{GPE}} \left[\boldsymbol{J}^{-T} \cdot N^{i}_{,\boldsymbol{\xi}}\right]_{b} E_{a\,b\,c\,d} \left[\boldsymbol{J}^{-T} \cdot N^{j}_{,\boldsymbol{\xi}}\right]_{d} \mathrm{Det}\boldsymbol{J}\,\alpha_{gp} \times \frac{1}{2}$$

Quadrilateral Elements

$$K^{\,i\,j}_{\,a\,c} = \sum_{\mathrm{gp}\,=\,1}^{\mathrm{GPE}} \left[\boldsymbol{J}^{-T} \cdot N^{i}_{,\boldsymbol{\xi}}\right]_b E_{\,a\,b\,c\,d} \left[\boldsymbol{J}^{-T} \cdot N^{j}_{,\boldsymbol{\xi}}\right]_d \,\mathrm{Det}\boldsymbol{J}\,\alpha_{gp}$$

Triangular Elements

$$K^{\,i\,j}_{\,a\,c} = \sum_{\mathrm{gp}\,=\,1}^{\mathrm{GPE}} \left[\boldsymbol{J}^{-T} \cdot N^{i}_{,\boldsymbol{\xi}}\right]_b E_{\,a\,b\,c\,d} \left[\boldsymbol{J}^{-T} \cdot N^{j}_{,\boldsymbol{\xi}}\right]_d \,\mathrm{Det}\boldsymbol{J}\,\alpha_{gp} \times \frac{1}{2}$$

$$E_{\,a\,b\,c\,d} = \frac{E}{2[1+\nu]}\left[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}\right] + \frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

Quadrilateral Elements

$$K^{\,i\,j}_{\,a\,c} = \sum_{\text{gp}\,=\,1}^{\text{GPE}} \left[\boldsymbol{J}^{-T}\cdot N^{i}_{,\boldsymbol{\xi}}\right]_b E_{a\,b\,c\,d} \left[\boldsymbol{J}^{-T}\cdot N^{j}_{,\boldsymbol{\xi}}\right]_d \,\mathrm{Det}\boldsymbol{J}\,\alpha_{gp}$$

Triangular Elements

$$K^{\,i\,j}_{\,a\,c} = \sum_{\text{gp}\,=\,1}^{\text{GPE}} \left[\boldsymbol{J}^{-T}\cdot N^{i}_{,\boldsymbol{\xi}}\right]_b E_{a\,b\,c\,d} \left[\boldsymbol{J}^{-T}\cdot N^{j}_{,\boldsymbol{\xi}}\right]_d \,\mathrm{Det}\boldsymbol{J}\,\alpha_{gp} \times \frac{1}{2}$$

$$E_{a\,b\,c\,d} = \frac{E}{2[1+\nu]}\left[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}\right] + \frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

2D plane strain

2D plane strain
constitutive tensor components

$$E_{abcd} = \frac{E}{2[1+\nu]}[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}] + \frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

$$\begin{aligned}
E_{1111} &= \frac{E}{1-\nu^2} & E_{1112} &= 0 & E_{1121} &= 0 & E_{1122} &= \frac{E\,\nu}{1-\nu^2} \\
E_{1211} &= 0 & E_{1212} &= \frac{E}{2[1+\nu]} & E_{1221} &= \frac{E}{2[1+\nu]} & E_{1222} &= 0 \\
E_{2111} &= 0 & E_{2112} &= \frac{E}{2[1+\nu]} & E_{2121} &= \frac{E}{2[1+\nu]} & E_{2122} &= 0 \\
E_{2211} &= \frac{E\,\nu}{1-\nu^2} & E_{2212} &= 0 & E_{2221} &= 0 & E_{2222} &= \frac{E}{1-\nu^2}
\end{aligned}$$

$$K^{i\,j}_{a\,c}$$ *stiffness between direction "a" of node "i" & direction "c" of node "j"*

Quadrilateral Elements

$$K^{i\,j}_{a\,c} = \sum_{\text{gp}\,=\,1}^{\text{GPE}} \left[\boldsymbol{J}^{-T} \cdot N^{i}_{,\boldsymbol{\xi}}\right]_b E_{a\,b\,c\,d} \left[\boldsymbol{J}^{-T} \cdot N^{j}_{,\boldsymbol{\xi}}\right]_d \,\mathrm{Det}\boldsymbol{J}\,\alpha_{gp}$$

Triangular Elements

$$K^{i\,j}_{a\,c} = \sum_{\text{gp}\,=\,1}^{\text{GPE}} \left[\boldsymbol{J}^{-T} \cdot N^{i}_{,\boldsymbol{\xi}}\right]_b E_{a\,b\,c\,d} \left[\boldsymbol{J}^{-T} \cdot N^{j}_{,\boldsymbol{\xi}}\right]_d \,\mathrm{Det}\boldsymbol{J}\,\alpha_{gp} \times \frac{1}{2}$$

$$\boldsymbol{J} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\text{NPE}} \\ y^1 & y^2 & \dots & y^{\text{NPE}} \end{bmatrix} \begin{bmatrix} N^1_{\xi} & N^1_{\eta} \\ N^2_{\xi} & N^2_{\eta} \\ \vdots & \vdots \\ N^{\text{NPE}}_{\xi} & N^{\text{NPE}}_{\eta} \end{bmatrix}$$

$$E_{a\,b\,c\,d} = \frac{E}{2[1+\nu]}\left[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}\right] + \frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

plane strain

$$K^{i\,j}_{a\,c}$$ *stiffness between direction "a" of node "i" & direction "c" of node "j"*

Quadrilateral Elements

$$K^{i\,j}_{a\,c} = \sum_{\mathrm{gp}\,=\,1}^{\mathrm{GPE}} \left[\boldsymbol{J}^{-T} \cdot N^{i}_{,\boldsymbol{\xi}}\right]_b E_{a\,b\,c\,d} \left[\boldsymbol{J}^{-T} \cdot N^{j}_{,\boldsymbol{\xi}}\right]_d \mathrm{Det}\boldsymbol{J}\,\alpha_{gp}$$

Triangular Elements

$$K^{i\,j}_{a\,c} = \sum_{\mathrm{gp}\,=\,1}^{\mathrm{GPE}} \left[\boldsymbol{J}^{-T} \cdot N^{i}_{,\boldsymbol{\xi}}\right]_b E_{a\,b\,c\,d} \left[\boldsymbol{J}^{-T} \cdot N^{j}_{,\boldsymbol{\xi}}\right]_d \mathrm{Det}\boldsymbol{J}\,\alpha_{gp} \times \frac{1}{2}$$

$$\boldsymbol{J} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\mathrm{NPE}} \\ y^1 & y^2 & \dots & y^{\mathrm{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\mathrm{NPE}}_\xi & N^{\mathrm{NPE}}_\eta \end{bmatrix}$$

$$E_{a\,b\,c\,d} = \frac{E}{2[1+\nu]}\left[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}\right] + \frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

plane strain

$$K^{\,i\,j}_{\,a\,c}$$ *stiffness between direction "a" of node "i" & direction "c" of node "j"*

Quadrilateral Elements

$$K^{\,i\,j}_{\,a\,c} = \sum_{\mathrm{gp}\,=\,1}^{\mathrm{GPE}} \left[\boldsymbol{J}^{-T} \cdot N^{i}_{,\boldsymbol{\xi}}\right]_{b} E_{\,a\,b\,c\,d} \left[\boldsymbol{J}^{-T} \cdot N^{j}_{,\boldsymbol{\xi}}\right]_{d} \,\mathrm{Det}\boldsymbol{J}\; \alpha_{gp}$$

Triangular Elements

$$K^{\,i\,j}_{\,a\,c} = \sum_{\mathrm{gp}\,=\,1}^{\mathrm{GPE}} \left[\boldsymbol{J}^{-T} \cdot N^{i}_{,\boldsymbol{\xi}}\right]_{b} E_{\,a\,b\,c\,d} \left[\boldsymbol{J}^{-T} \cdot N^{j}_{,\boldsymbol{\xi}}\right]_{d} \,\mathrm{Det}\boldsymbol{J}\; \alpha_{gp} \times \frac{1}{2}$$

plane strain

$$E_{\,a\,b\,c\,d} = \frac{E}{2[1+\nu]}\left[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}\right] + \frac{E\,\nu}{1-\nu^{2}}\,\delta_{ab}\,\delta_{cd}$$

$$\boldsymbol{J} = \begin{bmatrix} x^{1} & x^{2} & \dots & x^{\mathrm{NPE}} \\ y^{1} & y^{2} & \dots & y^{\mathrm{NPE}} \end{bmatrix} \begin{bmatrix} N^{1}_{\xi} & N^{1}_{\eta} \\ N^{2}_{\xi} & N^{2}_{\eta} \\ \vdots & \vdots \\ N^{\mathrm{NPE}}_{\xi} & N^{\mathrm{NPE}}_{\eta} \end{bmatrix}$$

$$\begin{aligned}
E_{1111} &= \frac{E}{1-\nu^{2}} & E_{1112} &= 0 & E_{1121} &= 0 & E_{1122} &= \frac{E\,\nu}{1-\nu^{2}} \\
E_{1211} &= 0 & E_{1212} &= \frac{E}{2[1+\nu]} & E_{1221} &= \frac{E}{2[1+\nu]} & E_{1222} &= 0 \\
E_{2111} &= 0 & E_{2112} &= \frac{E}{2[1+\nu]} & E_{2121} &= \frac{E}{2[1+\nu]} & E_{2122} &= 0 \\
E_{2211} &= \frac{E\,\nu}{1-\nu^{2}} & E_{2212} &= 0 & E_{2221} &= 0 & E_{2222} &= \frac{E}{1-\nu^{2}}
\end{aligned}$$

2D Finite Element Library

two-dimensional finite elements library

- two-dimensional 4-noded quadrilateral element (D2QU4N)

  a.k.a. bilinear quadrilateral element

- two-dimensional 9-noded quadrilateral element (D2QU9N)

  a.k.a. Lagrange biquadratic quadrilateral element

- two-dimensional 8-noded quadrilateral element (D2QU8N)

  a.k.a. serendipity biquadratic quadrilateral element

- two-dimensional 3-noded triangular element (D2TR3N)

  a.k.a. constant strain triangle

- two-dimensional 6-noded triangular element (D2TR6N)

  a.k.a. quadratic triangle

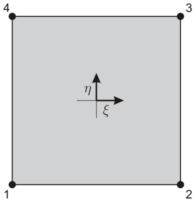
- two-dimensional quadrature rule

2D Finite Element Library

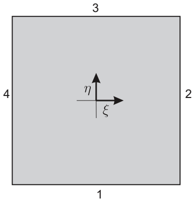
D2QU4N

bilinear quadrilateral element

Nodes



Edges



Node Number	Coordinates	
	ξ	η
1	-1	-1
2	1	-1
3	1	1
4	-1	1

$$N^1 = \frac{1}{4}\left(1-\xi\right)\left(1-\eta\right)$$

$$N^2 = \frac{1}{4}\left(1+\xi\right)\left(1-\eta\right)$$

$$N^3 = \frac{1}{4}\left(1+\xi\right)\left(1+\eta\right)$$

$$N^4 = \frac{1}{4}\left(1-\xi\right)\left(1+\eta\right)$$

$$N^1_{,\xi} = -\frac{1}{4}\left(1-\eta\right) \qquad N^1_{,\eta} = -\frac{1}{4}\left(1-\xi\right)$$

$$N^2_{,\xi} = +\frac{1}{4}\left(1-\eta\right) \qquad N^2_{,\eta} = -\frac{1}{4}\left(1+\xi\right)$$

$$N^3_{,\xi} = +\frac{1}{4}\left(1+\eta\right) \qquad N^3_{,\eta} = +\frac{1}{4}\left(1+\xi\right)$$

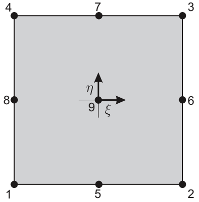
$$N^4_{,\xi} = -\frac{1}{4}\left(1+\eta\right) \qquad N^4_{,\eta} = +\frac{1}{4}\left(1-\xi\right)$$

2D Finite Element Library

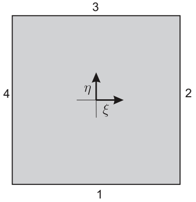
D2QU9N

Lagrange biquadratic quadrilateral element

Nodes



Edges

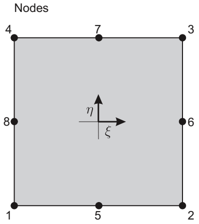


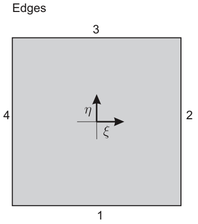
Node Number	Coordinates	
	ξ	η
1	-1	-1
2	1	-1
3	1	1
4	-1	1
5	0	-1
6	1	0
7	0	1
8	-1	0
9	0	0

$$
\begin{aligned}
N^1 &= +\frac{1}{4}\left(1-\xi\right)\xi\left(1-\eta\right)\eta & N^1_{,\xi} &= +\frac{1}{4}\left(1-2\xi\right)\left(1-\eta\right)\eta & N^1_{,\eta} &= +\frac{1}{4}\left(1-\xi\right)\xi\left(1-2\eta\right)\\
N^2 &= -\frac{1}{4}\left(1+\xi\right)\xi\left(1-\eta\right)\eta & N^2_{,\xi} &= -\frac{1}{4}\left(1+2\xi\right)\left(1-\eta\right)\eta & N^2_{,\eta} &= -\frac{1}{4}\left(1+\xi\right)\xi\left(1-2\eta\right)\\
N^3 &= +\frac{1}{4}\left(1+\xi\right)\xi\left(1+\eta\right)\eta & N^3_{,\xi} &= +\frac{1}{4}\left(1+2\xi\right)\left(1+\eta\right)\eta & N^3_{,\eta} &= +\frac{1}{4}\left(1+\xi\right)\xi\left(1+2\eta\right)\\
N^4 &= -\frac{1}{4}\left(1-\xi\right)\xi\left(1+\eta\right)\eta & N^4_{,\xi} &= -\frac{1}{4}\left(1-2\xi\right)\left(1+\eta\right)\eta & N^4_{,\eta} &= -\frac{1}{4}\left(1-\xi\right)\xi\left(1+2\eta\right)\\
N^5 &= -\frac{1}{2}\left(1-\xi\right)\left(1+\xi\right)\left(1-\eta\right)\eta & N^5_{,\xi} &= \xi\eta\left(1-\eta\right) & N^5_{,\eta} &= \frac{1}{2}\left(1-\xi\right)\left(1+\xi\right)\left(2\eta-1\right)\\
N^6 &= +\frac{1}{2}\left(1+\xi\right)\xi\left(1-\eta\right)\left(1+\eta\right) & N^6_{,\xi} &= \frac{1}{2}\left(1+2\xi\right)\left(1-\eta\right)\left(1+\eta\right) & N^6_{,\eta} &= -\left(1+\xi\right)\xi\eta\\
N^7 &= +\frac{1}{2}\left(1-\xi\right)\left(1+\xi\right)\left(1+\eta\right)\eta & N^7_{,\xi} &= -\xi\eta\left(1+\eta\right) & N^7_{,\eta} &= \frac{1}{2}\left(1-\xi\right)\left(1+\xi\right)\left(1+2\eta\right)\\
N^8 &= -\frac{1}{2}\left(1-\xi\right)\xi\left(1-\eta\right)\left(1+\eta\right) & N^8_{,\xi} &= -\frac{1}{2}\left(1-2\xi\right)\left(1-\eta\right)\left(1+\eta\right) & N^8_{,\eta} &= \left(1-\xi\right)\xi\eta\\
N^9 &= \left(1-\xi\right)\left(1+\xi\right)\left(1-\eta\right)\left(1+\eta\right) & N^9_{,\xi} &= -2\xi\left(1-\eta\right)\left(1+\eta\right) & N^9_{,\eta} &= -2\left(1-\xi\right)\left(1+\xi\right)\eta
\end{aligned}
$$

2D Finite Element Library

D2QU8N serendipity biquadratic quadrilateral element





Node Number	Coordinates	
	ξ	η
1	-1	-1
2	1	-1
3	1	1
4	-1	1
5	0	-1
6	1	0
7	0	1
8	-1	0

$$N^1 = -\frac{1}{4}(1-\xi)(1-\eta)(1+\xi+\eta) \qquad N^1_{,\xi} = +\frac{1}{4}(1-\eta)(2\xi+\eta) \qquad N^1_{,\eta} = +\frac{1}{4}(1-\xi)(\xi+2\eta)$$

$$N^2 = -\frac{1}{4}(1+\xi)(1-\eta)(1-\xi+\eta) \qquad N^2_{,\xi} = +\frac{1}{4}(1-\eta)(2\xi-\eta) \qquad N^2_{,\eta} = +\frac{1}{4}(1+\xi)(-\xi+2\eta)$$

$$N^3 = -\frac{1}{4}(1+\xi)(1+\eta)(1-\xi-\eta) \qquad N^3_{,\xi} = +\frac{1}{4}(1+\eta)(2\xi+\eta) \qquad N^3_{,\eta} = +\frac{1}{4}(1+\xi)(\xi+2\eta)$$

$$N^4 = -\frac{1}{4}(1-\xi)(1+\eta)(1+\xi-\eta) \qquad N^4_{,\xi} = +\frac{1}{4}(1+\eta)(2\xi-\eta) \qquad N^4_{,\eta} = +\frac{1}{4}(1-\xi)(-\xi+2\eta)$$

$$N^5 = \frac{1}{2}(1-\xi)(1+\xi)(1-\eta) \qquad N^5_{,\xi} = -\xi(1-\eta) \qquad N^5_{,\eta} = -\frac{1}{2}(1-\xi)(1+\xi)$$

$$N^6 = \frac{1}{2}(1+\xi)(1+\eta)(1-\eta) \qquad N^6_{,\xi} = +\frac{1}{2}(1-\eta)(1+\eta) \qquad N^6_{,\eta} = -(1+\xi)\eta$$

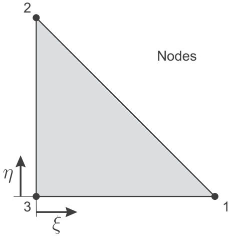
$$N^7 = \frac{1}{2}(1-\xi)(1+\xi)(1+\eta) \qquad N^7_{,\xi} = -\xi(1+\eta) \qquad N^7_{,\eta} = +\frac{1}{2}(1-\xi)(1+\xi)$$

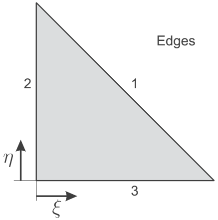
$$N^8 = \frac{1}{2}(1-\xi)(1+\eta)(1-\eta) \qquad N^8_{,\xi} = -\frac{1}{2}(1-\eta)(1+\eta) \qquad N^8_{,\eta} = -(1-\xi)\eta$$

2D Finite Element Library

D2TR3N

constant strain triangle (CST)





Node Number	Coordinates	
	ξ	η
1	1	0
2	0	1
3	0	0

$$N^1 = \xi \qquad N^1_{,\xi} = 1 \qquad N^1_{,\eta} = 0$$

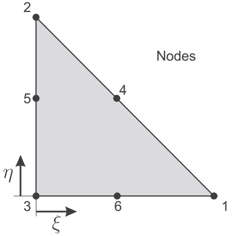
$$N^2 = \eta \qquad N^2_{,\xi} = 0 \qquad N^2_{,\eta} = 1$$

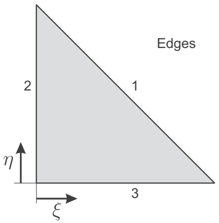
$$N^3 = (1 - \xi - \eta) \qquad N^3_{,\xi} = -1 \qquad N^3_{,\eta}(\xi, \eta) = -1$$

2D Finite Element Library

D2TR6N

quadratic triangle





Node Number	Coordinates	
	ξ	η
1	1	0
2	0	1
3	0	0
4	1/2	1/2
5	0	1/2
6	1/2	0

$$N^1 = \xi(2\xi - 1) \qquad N^1_{,\xi} = -1 + 4\xi \qquad N^1_{,\eta} = 0$$

$$N^2 = \eta(2\eta - 1) \qquad N^2_{,\xi} = 0 \qquad N^2_{,\eta} = -1 + 4\eta$$

$$N^3 = (1 - \xi - \eta)(1 - 2\xi - 2\eta) \qquad N^3_{,\xi} = -3 + 4\xi + 4\eta \qquad N^3_{,\eta} = -3 + 4\xi + 4\eta$$

$$N^4 = 4\xi\eta \qquad N^4_{,\xi} = 4\eta \qquad N^4_{,\eta} = 4\xi$$

$$N^5 = 4\eta(1 - \xi - \eta) \qquad N^5_{,\xi} = -4\eta \qquad N^5_{,\eta} = -4(-1 + 2\eta + \xi)$$

$$N^6 = 4\xi(1 - \xi - \eta) \qquad N^6_{,\xi} = -4(-1 + \eta + 2\xi) \qquad N^6_{,\eta} = -4\xi$$

two-dimensional quadrature rule i

Triangular Elements Gauss Point Rule

$$\int_0^1 \int_0^{1-\eta} \{\bullet\} \; \mathrm{d}\xi \; \mathrm{d}\eta \approx \frac{1}{2} \sum_{i=1}^{N_{\text{Gauss Points}}} \alpha_i \; \{\bullet\}\big|_{\text{Gauss Point}^i}$$

Gauss Point Number	Coordinates		Weight Factor
	ξ	η	α
1	1/3	1/3	1

Gauss Point Number	Coordinates		Weight Factor
	ξ	η	α
1	1/6	1/6	1/3
2	4/6	1/6	1/3
3	1/6	4/6	1/3

2D Finite Element Library

two-dimensional quadrature rule ii

Quadrilateral Elements Gauss Point Rule

$$\int_{-1}^{1}\int_{-1}^{1}\{\bullet\}\ \mathrm{d}\xi\ \mathrm{d}\eta \approx \sum_{i=1}^{N_{\text{Gauss Points}}} \alpha_i\ \{\bullet\}|_{\text{Gauss Point}^i}$$

Gauss Point Number	Coordinates		Weight Factor
	ξ	η	α
1	0	0	2×2

Gauss Point Number	Coordinates		Weight Factor
	ξ	η	α
1	$-1/\sqrt{3}$	$-1/\sqrt{3}$	1×1
2	$+1/\sqrt{3}$	$-1/\sqrt{3}$	1×1
3	$+1/\sqrt{3}$	$+1/\sqrt{3}$	1×1
4	$-1/\sqrt{3}$	$+1/\sqrt{3}$	1×1

two-dimensional quadrature rule iii

Gauss Point Number	Coordinates		Weight Factor
	ξ	η	α
1	$-\sqrt{3/5}$	$-\sqrt{3/5}$	$5/9 \times 5/9$
2	$+\sqrt{3/5}$	$-\sqrt{3/5}$	$5/9 \times 5/9$
3	$\sqrt{3/5}$	$\sqrt{3/5}$	$5/9 \times 5/9$
4	$-\sqrt{3/5}$	$\sqrt{3/5}$	$5/9 \times 5/9$
5	0	$-\sqrt{3/5}$	$5/9 \times 8/9$
6	$+\sqrt{3/5}$	0	$5/9 \times 8/9$
7	0	$\sqrt{3/5}$	$5/9 \times 8/9$
8	$-\sqrt{3/5}$	0	$5/9 \times 8/9$
9	0	0	$8/9 \times 8/9$

Understanding Jacobian

$$\boldsymbol{J} = \frac{\partial x}{\partial \xi}$$

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Understanding Jacobian

$$\boldsymbol{J} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}}$$

$$\mathrm{d}\boldsymbol{x} = \boldsymbol{J}\,\mathrm{d}\boldsymbol{\xi}$$

Understanding Jacobian

$$\boldsymbol{J} = \frac{\partial x}{\partial \xi}$$

linear mapping
from natural space
to physical space

$$\mathrm{d}x = \boldsymbol{J}\,\mathrm{d}\xi$$

Understanding Jacobian

$$\boldsymbol{J} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\text{NPE}} \\ y^1 & y^2 & \dots & y^{\text{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\text{NPE}}_\xi & N^{\text{NPE}}_\eta \end{bmatrix}$$

linear mapping
from natural space
to physical space

$$\mathrm{d}\boldsymbol{x} = \boldsymbol{J}\,\mathrm{d}\boldsymbol{\xi}$$

Understanding Jacobian

$$\boldsymbol{J} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\mathrm{NPE}} \\ y^1 & y^2 & \dots & y^{\mathrm{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\mathrm{NPE}}_\xi & N^{\mathrm{NPE}}_\eta \end{bmatrix}$$

linear mapping
from natural space
to physical space

$$\mathrm{d}\boldsymbol{x} = \boldsymbol{J}\,\mathrm{d}\boldsymbol{\xi} \qquad J = \mathrm{Det}\boldsymbol{J} = \frac{\mathrm{d}A_{\boldsymbol{x}}}{\mathrm{d}A_{\boldsymbol{\xi}}}$$

Understanding Jacobian

$$\boldsymbol{J} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}} = \begin{bmatrix} x^1 & x^2 & \ldots & x^{\mathrm{NPE}} \\ y^1 & y^2 & \ldots & y^{\mathrm{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\mathrm{NPE}}_\xi & N^{\mathrm{NPE}}_\eta \end{bmatrix}$$

linear mapping
from natural space
to physical space

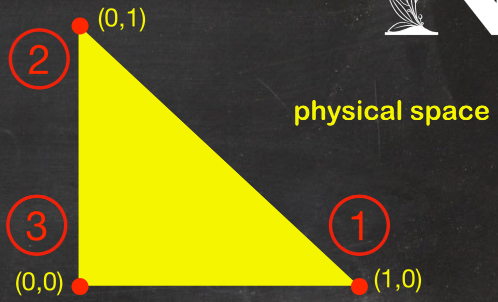
$$\mathrm{d}\boldsymbol{x} = \boldsymbol{J}\,\mathrm{d}\boldsymbol{\xi} \qquad J = \mathrm{Det}\boldsymbol{J} = \frac{\mathrm{d}A_{\boldsymbol{x}}}{\mathrm{d}A_{\boldsymbol{\xi}}}$$

often, the (scalar) determinant
of the Jacobian matrix is also
referred to as Jacobian

…

the scalar Jacobian is a linear
mapping between the area
elements from the natural
space to the physical space

Example (D2TR3N) … constant strain triangle

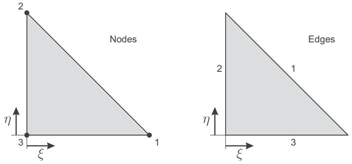


pay attention to numbering of the nodes
!

$$\boldsymbol{J} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\text{NPE}} \\ y^1 & y^2 & \dots & y^{\text{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\text{NPE}}_\xi & N^{\text{NPE}}_\eta \end{bmatrix}$$

$$J = \text{Det}\boldsymbol{J} = \frac{\mathrm{d}A_{\boldsymbol{x}}}{\mathrm{d}A_{\boldsymbol{\xi}}}$$

natural space



Node Number	Coordinates	
	ξ	η
1	1	0
2	0	1
3	0	0

$N^1 = \xi$

$N^2 = \eta$

$N^3 = (1 - \xi - \eta)$

$N^1_{,\xi} = 1$ $N^1_{,\eta} = 0$

$N^2_{,\xi} = 0$ $N^2_{,\eta} = 1$

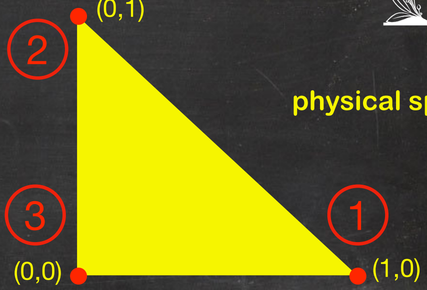
$N^3_{,\xi} = -1$ $N^3_{,\eta}(\xi,\eta) = -1$

Example (D2TR3N) … constant strain triangle

$$\boldsymbol{J} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\boldsymbol{J} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\mathrm{NPE}} \\ y^1 & y^2 & \dots & y^{\mathrm{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\mathrm{NPE}}_\xi & N^{\mathrm{NPE}}_\eta \end{bmatrix}$$

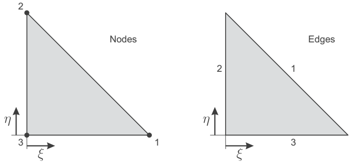
$$J = \mathrm{Det}\boldsymbol{J} = \frac{\mathrm{d}A_{\boldsymbol{x}}}{\mathrm{d}A_{\boldsymbol{\xi}}}$$



physical space

pay attention to numbering of the nodes !

natural space



Node Number	Coordinates	
	ξ	η
1	1	0
2	0	1
3	0	0

$N^1 = \xi$ $\qquad N^1_{,\xi} = 1 \qquad N^1_{,\eta} = 0$

$N^2 = \eta$ $\qquad N^2_{,\xi} = 0 \qquad N^2_{,\eta} = 1$

$N^3 = (1 - \xi - \eta)$ $\qquad N^3_{,\xi} = -1 \qquad N^3_{,\eta}(\xi, \eta) = -1$

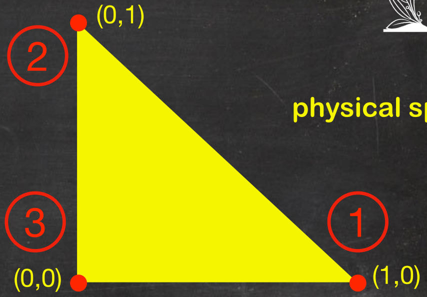
Example (D2TR3N) … constant strain triangle

$$\boldsymbol{J} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$J = \mathrm{Det}\boldsymbol{J} = \frac{\mathrm{d}A_{\boldsymbol{x}}}{\mathrm{d}A_{\boldsymbol{\xi}}} = 1$$

$$\boldsymbol{J} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\mathrm{NPE}} \\ y^1 & y^2 & \dots & y^{\mathrm{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\mathrm{NPE}}_\xi & N^{\mathrm{NPE}}_\eta \end{bmatrix}$$

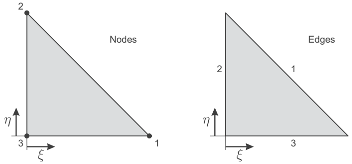
$$J = \mathrm{Det}\boldsymbol{J} = \frac{\mathrm{d}A_{\boldsymbol{x}}}{\mathrm{d}A_{\boldsymbol{\xi}}}$$



physical space

pay attention to numbering of the nodes
!

natural space



Node Number	Coordinates	
	ξ	η
1	1	0
2	0	1
3	0	0

$N^1 = \xi$

$N^2 = \eta$

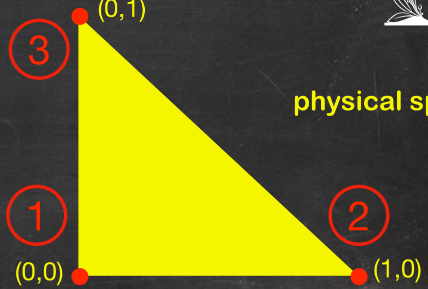
$N^3 = (1 - \xi - \eta)$

$N^1_{,\xi} = 1$ $\quad N^1_{,\eta} = 0$

$N^2_{,\xi} = 0$ $\quad N^2_{,\eta} = 1$

$N^3_{,\xi} = -1$ $\quad N^3_{,\eta}(\xi,\eta) = -1$

Example (D2TR3N) … constant strain triangle



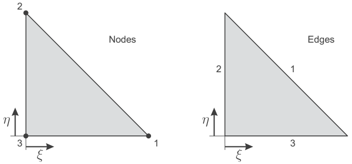
physical space

pay attention to
numbering of
the nodes
!

$$\boldsymbol{J} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\text{NPE}} \\ y^1 & y^2 & \dots & y^{\text{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\text{NPE}}_\xi & N^{\text{NPE}}_\eta \end{bmatrix}$$

$$J = \text{Det}\boldsymbol{J} = \frac{\mathrm{d}A_{\boldsymbol{x}}}{\mathrm{d}A_{\boldsymbol{\xi}}}$$

natural space



Node Number	Coordinates	
	ξ	η
1	1	0
2	0	1
3	0	0

$N^1 = \xi$

$N^2 = \eta$

$N^3 = (1 - \xi - \eta)$

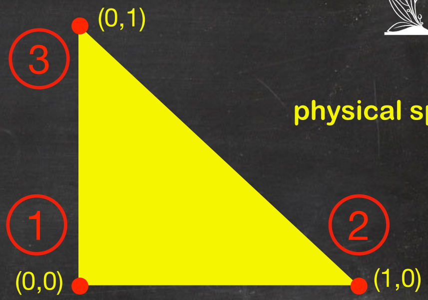
$N^1_{,\xi} = 1$ $\quad N^1_{,\eta} = 0$

$N^2_{,\xi} = 0$ $\quad N^2_{,\eta} = 1$

$N^3_{,\xi} = -1$ $\quad N^3_{,\eta}(\xi, \eta) = -1$

Example (D2TR3N) … constant strain triangle

$$\boldsymbol{J} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$



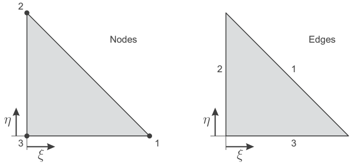
physical space

pay attention to numbering of the nodes
!

$$\boldsymbol{J} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\text{NPE}} \\ y^1 & y^2 & \dots & y^{\text{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\text{NPE}}_\xi & N^{\text{NPE}}_\eta \end{bmatrix}$$

$$J = \text{Det}\boldsymbol{J} = \frac{\mathrm{d}A_x}{\mathrm{d}A_\xi}$$

natural space



Node Number	Coordinates	
	ξ	η
1	1	0
2	0	1
3	0	0

$N^1 = \xi$ $\qquad N^1_{,\xi} = 1 \qquad N^1_{,\eta} = 0$

$N^2 = \eta$ $\qquad N^2_{,\xi} = 0 \qquad N^2_{,\eta} = 1$

$N^3 = (1 - \xi - \eta)$ $\qquad N^3_{,\xi} = -1 \qquad N^3_{,\eta}(\xi, \eta) = -1$

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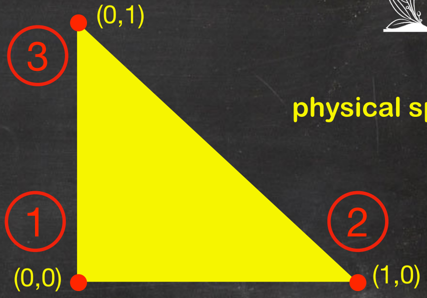
Example (D2TR3N) … constant strain triangle

$$\boldsymbol{J} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

$$J = \mathrm{Det}\boldsymbol{J} = \frac{\mathrm{d}A_{\boldsymbol{x}}}{\mathrm{d}A_{\boldsymbol{\xi}}} = 1$$

$$\boldsymbol{J} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\mathrm{NPE}} \\ y^1 & y^2 & \dots & y^{\mathrm{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\mathrm{NPE}}_\xi & N^{\mathrm{NPE}}_\eta \end{bmatrix}$$

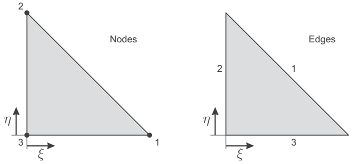
$$J = \mathrm{Det}\boldsymbol{J} = \frac{\mathrm{d}A_{\boldsymbol{x}}}{\mathrm{d}A_{\boldsymbol{\xi}}}$$



physical space

pay attention to numbering of the nodes !

natural space



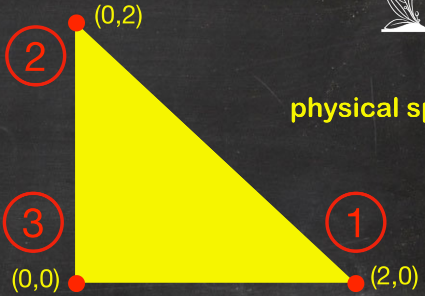
Node Number	Coordinates	
	ξ	η
1	1	0
2	0	1
3	0	0

$N^1 = \xi$ $\qquad N^1_{,\xi} = 1 \qquad N^1_{,\eta} = 0$

$N^2 = \eta$ $\qquad N^2_{,\xi} = 0 \qquad N^2_{,\eta} = 1$

$N^3 = (1 - \xi - \eta)$ $\qquad N^3_{,\xi} = -1 \qquad N^3_{,\eta}(\xi, \eta) = -1$

Example (D2TR3N) … constant strain triangle



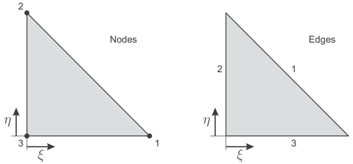
physical space

pay attention to
numbering of
the nodes
!

$$\boldsymbol{J} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\rm NPE} \\ y^1 & y^2 & \dots & y^{\rm NPE}\end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\rm NPE}_\xi & N^{\rm NPE}_\eta \end{bmatrix}$$

$$J = \mathrm{Det}\boldsymbol{J} = \frac{\mathrm{d}A_{\boldsymbol{x}}}{\mathrm{d}A_{\boldsymbol{\xi}}}$$

natural space



Node Number	Coordinates	
	ξ	η
1	1	0
2	0	1
3	0	0

$N^1 = \xi$

$N^2 = \eta$

$N^3 = (1 - \xi - \eta)$

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$N^2_{,\xi} = 0$ $N^2_{,\eta} = 1$

$N^3_{,\xi} = -1$ $N^3_{,\eta}(\xi, \eta) = -1$

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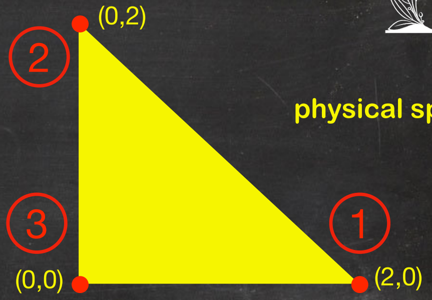
Example (D2TR3N) … constant strain triangle

$$\boldsymbol{J} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$



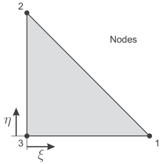
physical space

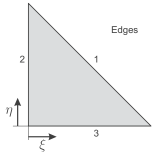
pay attention to numbering of the nodes !

$$\boldsymbol{J} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\text{NPE}} \\ y^1 & y^2 & \dots & y^{\text{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\text{NPE}}_\xi & N^{\text{NPE}}_\eta \end{bmatrix}$$

$$J = \text{Det}\boldsymbol{J} = \frac{\text{d}A_{\boldsymbol{x}}}{\text{d}A_{\boldsymbol{\xi}}}$$

natural space





Node Number	Coordinates	
	ξ	η
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2	0	1
3	0	0

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$N^2 = \eta$ $\qquad N^2_{,\xi} = 0 \qquad N^2_{,\eta} = 1$

$N^3 = (1 - \xi - \eta)$ $\qquad N^3_{,\xi} = -1 \qquad N^3_{,\eta}(\xi, \eta) = -1$

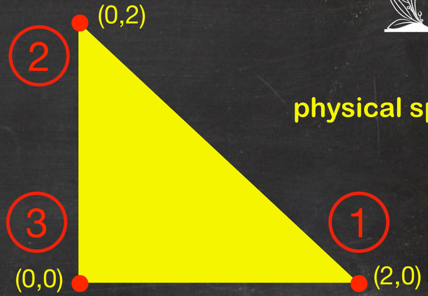
Example (D2TR3N) … constant strain triangle

$$\boldsymbol{J} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$J = \text{Det}\boldsymbol{J} = \frac{\mathrm{d}A_x}{\mathrm{d}A_\xi} = 4$$

$$\boldsymbol{J} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\text{NPE}} \\ y^1 & y^2 & \dots & y^{\text{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\text{NPE}}_\xi & N^{\text{NPE}}_\eta \end{bmatrix}$$

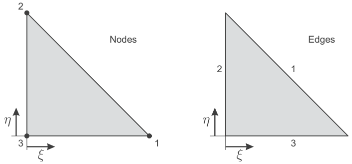
$$J = \text{Det}\boldsymbol{J} = \frac{\mathrm{d}A_x}{\mathrm{d}A_\xi}$$



physical space

pay attention to numbering of the nodes !

natural space



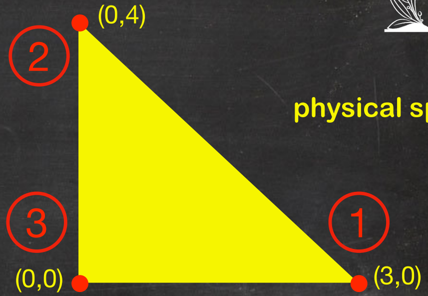
Node Number	Coordinates	
	ξ	η
1	1	0
2	0	1
3	0	0

$N^1 = \xi$ $\quad N^1_{,\xi} = 1 \quad N^1_{,\eta} = 0$

$N^2 = \eta$ $\quad N^2_{,\xi} = 0 \quad N^2_{,\eta} = 1$

$N^3 = (1 - \xi - \eta)$ $\quad N^3_{,\xi} = -1 \quad N^3_{,\eta}(\xi, \eta) = -1$

Example (D2TR3N) … constant strain triangle

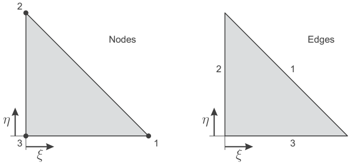


pay attention to numbering of the nodes !

$$\boldsymbol{J} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\text{NPE}} \\ y^1 & y^2 & \dots & y^{\text{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\text{NPE}}_\xi & N^{\text{NPE}}_\eta \end{bmatrix}$$

$$J = \text{Det}\boldsymbol{J} = \frac{\mathrm{d}A_{\boldsymbol{x}}}{\mathrm{d}A_{\boldsymbol{\xi}}}$$

natural space



Node Number	Coordinates	
	ξ	η
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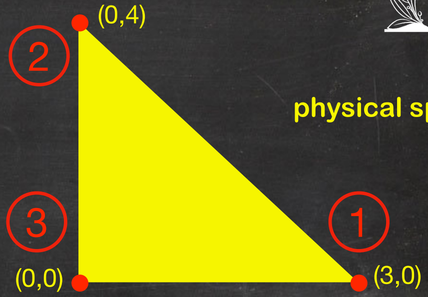
$N^1 = \xi$ $\quad N^1_{,\xi} = 1 \quad N^1_{,\eta} = 0$

$N^2 = \eta$ $\quad N^2_{,\xi} = 0 \quad N^2_{,\eta} = 1$

$N^3 = (1 - \xi - \eta)$ $\quad N^3_{,\xi} = -1 \quad N^3_{,\eta}(\xi, \eta) = -1$

Example (D2TR3N) … constant strain triangle

$$\boldsymbol{J} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$$



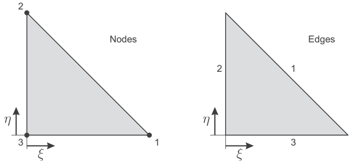
physical space

pay attention to numbering of the nodes !

$$\boldsymbol{J} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\mathrm{NPE}} \\ y^1 & y^2 & \dots & y^{\mathrm{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\mathrm{NPE}}_\xi & N^{\mathrm{NPE}}_\eta \end{bmatrix}$$

$$J = \mathrm{Det}\boldsymbol{J} = \frac{\mathrm{d}A_{\boldsymbol{x}}}{\mathrm{d}A_{\boldsymbol{\xi}}}$$

natural space



Node Number	Coordinates	
	ξ	η
1	1	0
2	0	1
3	0	0

$N^1 = \xi$ $\quad N^1_{,\xi} = 1 \quad N^1_{,\eta} = 0$

$N^2 = \eta$ $\quad N^2_{,\xi} = 0 \quad N^2_{,\eta} = 1$

$N^3 = (1 - \xi - \eta)$ $\quad N^3_{,\xi} = -1 \quad N^3_{,\eta}(\xi, \eta) = -1$

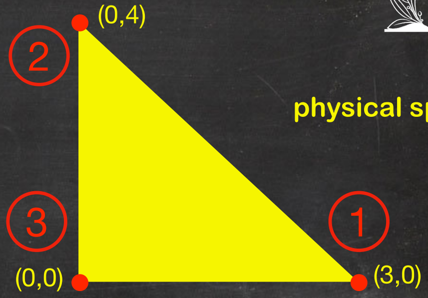
Example (D2TR3N) ... constant strain triangle

$$\boldsymbol{J} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$$

$$J = \mathrm{Det}\boldsymbol{J} = \frac{\mathrm{d}A_{\boldsymbol{x}}}{\mathrm{d}A_{\boldsymbol{\xi}}} = 12$$

$$\boldsymbol{J} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\mathrm{NPE}} \\ y^1 & y^2 & \dots & y^{\mathrm{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\mathrm{NPE}}_\xi & N^{\mathrm{NPE}}_\eta \end{bmatrix}$$

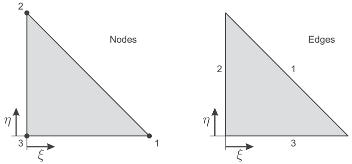
$$J = \mathrm{Det}\boldsymbol{J} = \frac{\mathrm{d}A_{\boldsymbol{x}}}{\mathrm{d}A_{\boldsymbol{\xi}}}$$



physical space

pay attention to numbering of the nodes !

natural space



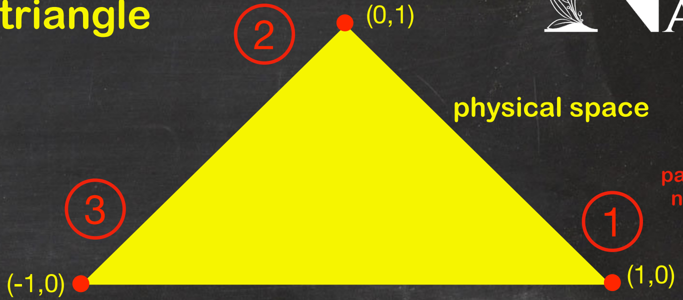
Node Number	Coordinates	
	ξ	η
1	1	0
2	0	1
3	0	0

$N^1 = \xi$ $\quad N^1_{,\xi} = 1 \quad N^1_{,\eta} = 0$

$N^2 = \eta$ $\quad N^2_{,\xi} = 0 \quad N^2_{,\eta} = 1$

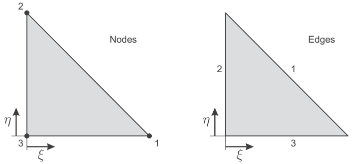
$N^3 = (1 - \xi - \eta)$ $\quad N^3_{,\xi} = -1 \quad N^3_{,\eta}(\xi, \eta) = -1$

Example (D2TR3N) … constant strain triangle



pay attention to numbering of the nodes
!

natural space



Node Number	Coordinates	
	ξ	η
1	1	0
2	0	1
3	0	0

$N^1 = \xi$

$N^2 = \eta$

$N^3 = (1 - \xi - \eta)$

$N^1_{,\xi} = 1$ $\quad N^1_{,\eta} = 0$

$N^2_{,\xi} = 0$ $\quad N^2_{,\eta} = 1$

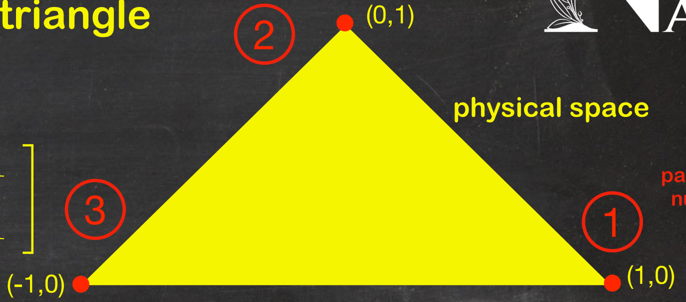
$N^3_{,\xi} = -1$ $\quad N^3_{,\eta}(\xi,\eta) = -1$

$$\boldsymbol{J} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\text{NPE}} \\ y^1 & y^2 & \dots & y^{\text{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\text{NPE}}_\xi & N^{\text{NPE}}_\eta \end{bmatrix}$$

$$J = \text{Det}\boldsymbol{J} = \frac{\text{d}A_{\boldsymbol{x}}}{\text{d}A_{\boldsymbol{\xi}}}$$

Example (D2TR3N) ... constant strain triangle

$$\boldsymbol{J} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$$

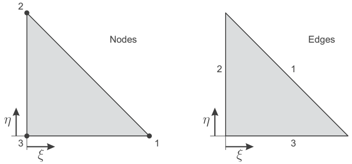


pay attention to numbering of the nodes
!

$$\boldsymbol{J} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\text{NPE}} \\ y^1 & y^2 & \dots & y^{\text{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\text{NPE}}_\xi & N^{\text{NPE}}_\eta \end{bmatrix}$$

$$J = \text{Det}\boldsymbol{J} = \frac{\mathrm{d}A_{\boldsymbol{x}}}{\mathrm{d}A_{\boldsymbol{\xi}}}$$

natural space



Node Number	Coordinates	
	ξ	η
1	1	0
2	0	1
3	0	0

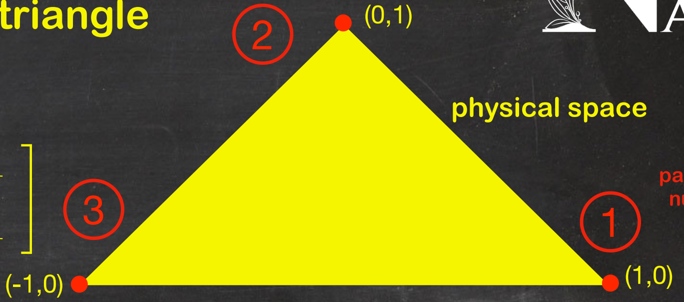
$N^1 = \xi$ $\qquad N^1_{,\xi} = 1 \qquad N^1_{,\eta} = 0$

$N^2 = \eta$ $\qquad N^2_{,\xi} = 0 \qquad N^2_{,\eta} = 1$

$N^3 = (1 - \xi - \eta)$ $\qquad N^3_{,\xi} = -1 \qquad N^3_{,\eta}(\xi, \eta) = -1$

Example (D2TR3N) … constant strain triangle

$$\boldsymbol{J} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$$



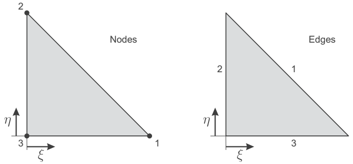
pay attention to numbering of the nodes
!

$$J = \text{Det}\boldsymbol{J} = \frac{\text{d}A_{\boldsymbol{x}}}{\text{d}A_{\boldsymbol{\xi}}} = 2$$

$$\boldsymbol{J} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\text{NPE}} \\ y^1 & y^2 & \dots & y^{\text{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\text{NPE}}_\xi & N^{\text{NPE}}_\eta \end{bmatrix}$$

$$J = \text{Det}\boldsymbol{J} = \frac{\text{d}A_{\boldsymbol{x}}}{\text{d}A_{\boldsymbol{\xi}}}$$

natural space



Node Number	Coordinates	
	ξ	η
1	1	0
2	0	1
3	0	0

$N^1 = \xi$

$N^2 = \eta$

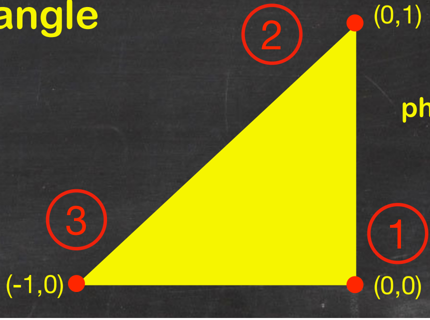
$N^3 = (1 - \xi - \eta)$

$N^1_{,\xi} = 1 \qquad N^1_{,\eta} = 0$

$N^2_{,\xi} = 0 \qquad N^2_{,\eta} = 1$

$N^3_{,\xi} = -1 \qquad N^3_{,\eta}(\xi,\eta) = -1$

Example (D2TR3N) … constant strain triangle

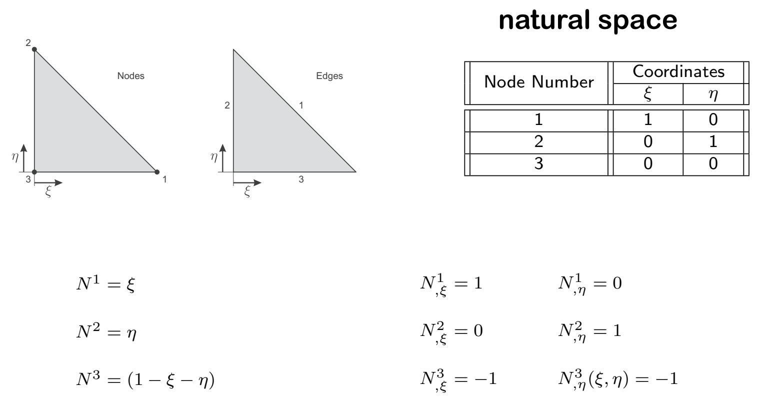


physical space

pay attention to numbering of the nodes !

$$\boldsymbol{J} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\text{NPE}} \\ y^1 & y^2 & \dots & y^{\text{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\text{NPE}}_\xi & N^{\text{NPE}}_\eta \end{bmatrix}$$

$$J = \text{Det}\boldsymbol{J} = \frac{\text{d}A_{\boldsymbol{x}}}{\text{d}A_{\boldsymbol{\xi}}}$$



natural space

Node Number	Coordinates	
	ξ	η
1	1	0
2	0	1
3	0	0

$N^1 = \xi$

$N^2 = \eta$

$N^3 = (1 - \xi - \eta)$

$N^1_{,\xi} = 1$ $\quad N^1_{,\eta} = 0$

$N^2_{,\xi} = 0$ $\quad N^2_{,\eta} = 1$

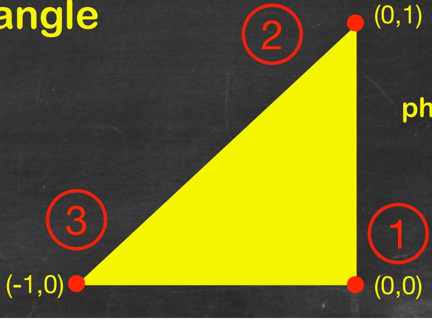
$N^3_{,\xi} = -1$ $\quad N^3_{,\eta}(\xi,\eta) = -1$

Example (D2TR3N) … constant strain triangle

$$\boldsymbol{J} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\boldsymbol{J} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\mathrm{NPE}} \\ y^1 & y^2 & \dots & y^{\mathrm{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\mathrm{NPE}}_\xi & N^{\mathrm{NPE}}_\eta \end{bmatrix}$$

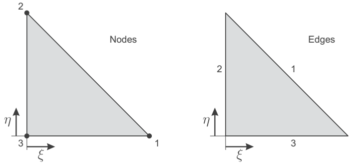
$$J = \mathrm{Det}\boldsymbol{J} = \frac{\mathrm{d}A_{\boldsymbol{x}}}{\mathrm{d}A_{\boldsymbol{\xi}}}$$



physical space

pay attention to numbering of the nodes
!

natural space



Node Number	Coordinates	
	ξ	η
1	1	0
2	0	1
3	0	0

$N^1 = \xi$

$N^2 = \eta$

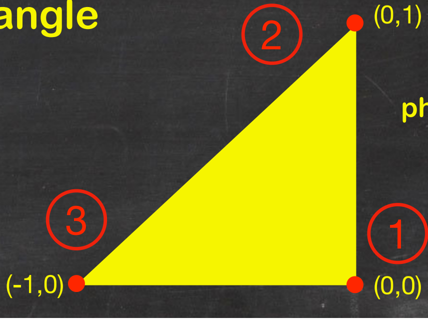
$N^3 = (1 - \xi - \eta)$

$N^1_{,\xi} = 1 \qquad N^1_{,\eta} = 0$

$N^2_{,\xi} = 0 \qquad N^2_{,\eta} = 1$

$N^3_{,\xi} = -1 \qquad N^3_{,\eta}(\xi,\eta) = -1$

Example (D2TR3N) … constant strain triangle



physical space

pay attention to numbering of the nodes !

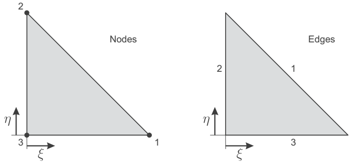
$$\boldsymbol{J} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$J = \mathrm{Det}\boldsymbol{J} = \frac{\mathrm{d}A_{\boldsymbol{x}}}{\mathrm{d}A_{\boldsymbol{\xi}}} = 1$$

$$\boldsymbol{J} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\mathrm{NPE}} \\ y^1 & y^2 & \dots & y^{\mathrm{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\mathrm{NPE}}_\xi & N^{\mathrm{NPE}}_\eta \end{bmatrix}$$

$$J = \mathrm{Det}\boldsymbol{J} = \frac{\mathrm{d}A_{\boldsymbol{x}}}{\mathrm{d}A_{\boldsymbol{\xi}}}$$

natural space



Node Number	Coordinates	
	ξ	η
1	1	0
2	0	1
3	0	0

$N^1 = \xi$ $\quad N^1_{,\xi} = 1 \quad N^1_{,\eta} = 0$

$N^2 = \eta$ $\quad N^2_{,\xi} = 0 \quad N^2_{,\eta} = 1$

$N^3 = (1 - \xi - \eta)$ $\quad N^3_{,\xi} = -1 \quad N^3_{,\eta}(\xi, \eta) = -1$

Example (D2TR3N) … constant strain triangle … stiffness

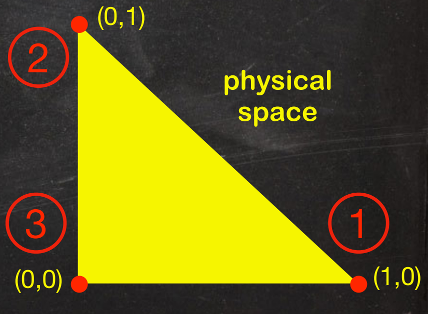
Triangular Elements

$$K^{ij}_{ac} = \sum_{\text{gp}=1}^{\text{GPE}} \left[\boldsymbol{J}^{-T} \cdot N^i_{,\boldsymbol{\xi}}\right]_b E_{abcd} \left[\boldsymbol{J}^{-T} \cdot N^j_{,\boldsymbol{\xi}}\right]_d \, \text{Det}\boldsymbol{J}\, \alpha_{gp} \times \frac{1}{2}$$

$$E_{abcd} = \frac{E}{2[1+\nu]}[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}] + \frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

plane strain

$$\boldsymbol{J} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\text{NPE}} \\ y^1 & y^2 & \dots & y^{\text{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\text{NPE}}_\xi & N^{\text{NPE}}_\eta \end{bmatrix}$$



Example (D2TR3N) … constant strain triangle … stiffness

Triangular Elements

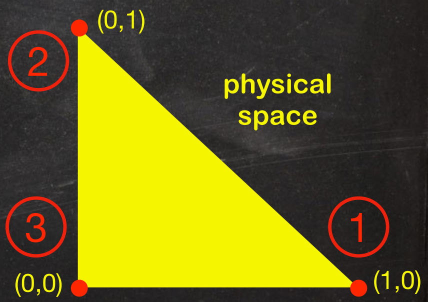
$$K^{\,i\,j}_{\,a\,c} = \sum_{\mathrm{gp}\,=\,1}^{\mathrm{GPE}} \left[\boldsymbol{J}^{-T} \cdot N^{i}_{,\boldsymbol{\xi}}\right]_b E_{a\,b\,c\,d} \left[\boldsymbol{J}^{-T} \cdot N^{j}_{,\boldsymbol{\xi}}\right]_d \mathrm{Det}\boldsymbol{J}\, \alpha_{gp} \times \frac{1}{2}$$

$$E_{a\,b\,c\,d} = \frac{E}{2[1+\nu]}[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}] + \frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

plane strain

$$\boldsymbol{J} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\mathrm{NPE}} \\ y^1 & y^2 & \dots & y^{\mathrm{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\mathrm{NPE}}_\xi & N^{\mathrm{NPE}}_\eta \end{bmatrix}$$

$$\boldsymbol{K} = \begin{bmatrix} \boldsymbol{K}^{11} & \boldsymbol{K}^{12} & \boldsymbol{K}^{13} \\ \boldsymbol{K}^{21} & \boldsymbol{K}^{22} & \boldsymbol{K}^{23} \\ \boldsymbol{K}^{31} & \boldsymbol{K}^{32} & \boldsymbol{K}^{33} \end{bmatrix}$$



Example (D2TR3N) … constant strain triangle … stiffness

Triangular Elements

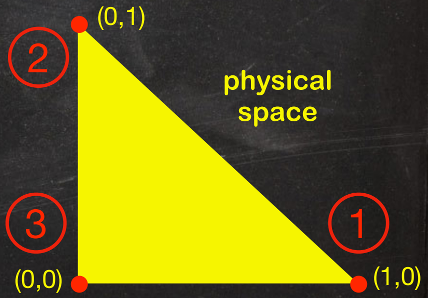
$$K^{\,i\,j}_{\,a\,c} = \sum_{\mathrm{gp}\,=\,1}^{\mathrm{GPE}} \left[\boldsymbol{J}^{-T}\cdot N^{i}_{,\boldsymbol{\xi}}\right]_b E_{a\,b\,c\,d} \left[\boldsymbol{J}^{-T}\cdot N^{j}_{,\boldsymbol{\xi}}\right]_d \mathrm{Det}\boldsymbol{J}\,\alpha_{gp} \times \frac{1}{2}$$

$$E_{a\,b\,c\,d} = \frac{E}{2[1+\nu]}[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}] + \frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

plane strain

$$\boldsymbol{K} = \begin{bmatrix} \boldsymbol{K}^{11} & \boldsymbol{K}^{12} & \boldsymbol{K}^{13} \\ \boldsymbol{K}^{21} & \boldsymbol{K}^{22} & \boldsymbol{K}^{23} \\ \boldsymbol{K}^{31} & \boldsymbol{K}^{32} & \boldsymbol{K}^{33} \end{bmatrix} = \begin{bmatrix} K^{11}_{11} & K^{11}_{12} & K^{12}_{11} & K^{12}_{12} & K^{13}_{11} & K^{13}_{12} \\ K^{11}_{21} & K^{11}_{22} & K^{12}_{21} & K^{12}_{22} & K^{13}_{21} & K^{13}_{22} \\ K^{21}_{11} & K^{21}_{12} & K^{22}_{11} & K^{22}_{12} & K^{23}_{11} & K^{23}_{12} \\ K^{21}_{21} & K^{21}_{22} & K^{22}_{21} & K^{22}_{22} & K^{23}_{21} & K^{23}_{22} \\ K^{31}_{11} & K^{31}_{12} & K^{32}_{11} & K^{32}_{12} & K^{33}_{11} & K^{33}_{12} \\ K^{31}_{21} & K^{31}_{22} & K^{32}_{21} & K^{32}_{22} & K^{33}_{21} & K^{33}_{22} \end{bmatrix}$$

$$\boldsymbol{J} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\mathrm{NPE}} \\ y^1 & y^2 & \dots & y^{\mathrm{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\mathrm{NPE}}_\xi & N^{\mathrm{NPE}}_\eta \end{bmatrix}$$



Example (D2TR3N) … constant strain triangle … stiffness

Triangular Elements

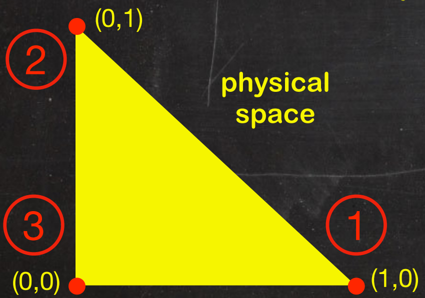
$$K^{ij}_{ac} = \sum_{\text{gp}=1}^{\text{GPE}} \left[\boldsymbol{J}^{-T} \cdot N^{i}_{,\boldsymbol{\xi}}\right]_b E_{abcd} \left[\boldsymbol{J}^{-T} \cdot N^{j}_{,\boldsymbol{\xi}}\right]_d \operatorname{Det}\boldsymbol{J}\, \alpha_{gp} \times \frac{1}{2}$$

$$E_{abcd} = \frac{E}{2[1+\nu]}[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}] + \frac{E\,\nu}{1-\nu^2}\delta_{ab}\,\delta_{cd}$$

plane strain

$$\boldsymbol{J} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\text{NPE}} \\ y^1 & y^2 & \dots & y^{\text{NPE}} \end{bmatrix} \begin{bmatrix} N^1_{\xi} & N^1_{\eta} \\ N^2_{\xi} & N^2_{\eta} \\ \vdots & \vdots \\ N^{\text{NPE}}_{\xi} & N^{\text{NPE}}_{\eta} \end{bmatrix}$$

$$\begin{array}{llll}
E_{1111} = \dfrac{E}{1-\nu^2} & E_{1112} = 0 & E_{1121} = 0 & E_{1122} = \dfrac{E\,\nu}{1-\nu^2} \\
E_{1211} = 0 & E_{1212} = \dfrac{E}{2[1+\nu]} & E_{1221} = \dfrac{E}{2[1+\nu]} & E_{1222} = 0 \\
E_{2111} = 0 & E_{2112} = \dfrac{E}{2[1+\nu]} & E_{2121} = \dfrac{E}{2[1+\nu]} & E_{2122} = 0 \\
E_{2211} = \dfrac{E\,\nu}{1-\nu^2} & E_{2212} = 0 & E_{2221} = 0 & E_{2222} = \dfrac{E}{1-\nu^2}
\end{array}$$



natural space

$N^1 = \xi$ $\quad N^1_{,\xi} = 1 \quad N^1_{,\eta} = 0$

$N^2 = \eta$ $\quad N^2_{,\xi} = 0 \quad N^2_{,\eta} = 1$

$N^3 = (1 - \xi - \eta)$ $\quad N^3_{,\xi} = -1 \quad N^3_{,\eta}(\xi, \eta) = -1$

$$\boldsymbol{K} = \begin{bmatrix} \boldsymbol{K}^{11} & \boldsymbol{K}^{12} & \boldsymbol{K}^{13} \\ \boldsymbol{K}^{21} & \boldsymbol{K}^{22} & \boldsymbol{K}^{23} \\ \boldsymbol{K}^{31} & \boldsymbol{K}^{32} & \boldsymbol{K}^{33} \end{bmatrix}$$

Example (D2TR3N) … constant strain triangle … stiffness

in this particular case, the coordinates of the Gauss points do not enter the calculations, since the derivatives of the shape functions are constant over the element … however, this is usually not the case and particularly holds for constant strain triangle (CST) … also, in this case we have only one Gauss point that makes the calculations more straightforward … otherwise the summation over the Gauss points must be carefully taken into account …

$$\int_0^1 \int_0^{1-\eta} \{\bullet\} \; \mathrm{d}\xi \; \mathrm{d}\eta \approx \frac{1}{2} \sum_{i=1}^{N_{\text{Gauss Points}}} \alpha_i \; \{\bullet\}|_{\text{Gauss Point}^i}$$

Gauss Point Number	Coordinates		Weight Factor
	ξ	η	α
1	1/3	1/3	1

Gauss Point Number	Coordinates		Weight Factor
	ξ	η	α
1	1/6	1/6	1/3
2	4/6	1/6	1/3
3	1/6	4/6	1/3

natural space

$$N^1 = \xi \qquad N^1_{,\xi} = 1 \qquad N^1_{,\eta} = 0$$

$$N^2 = \eta \qquad N^2_{,\xi} = 0 \qquad N^2_{,\eta} = 1$$

$$N^3 = (1 - \xi - \eta) \qquad N^3_{,\xi} = -1 \qquad N^3_{,\eta}(\xi, \eta) = -1$$

$$\boldsymbol{K} = \begin{bmatrix} \boldsymbol{K}^{11} & \boldsymbol{K}^{12} & \boldsymbol{K}^{13} \\ \boldsymbol{K}^{21} & \boldsymbol{K}^{22} & \boldsymbol{K}^{23} \\ \boldsymbol{K}^{31} & \boldsymbol{K}^{32} & \boldsymbol{K}^{33} \end{bmatrix}$$

Example (D2TR3N) … constant strain triangle … stiffness

Triangular Elements

$$K^{ij}_{ac} = \sum_{\text{gp}=1}^{\text{GPE}} \left[\boldsymbol{J}^{-T}\cdot N^i_{,\boldsymbol{\xi}}\right]_b E_{abcd} \left[\boldsymbol{J}^{-T}\cdot N^j_{,\boldsymbol{\xi}}\right]_d \operatorname{Det}\boldsymbol{J}\,\alpha_{gp} \times \frac{1}{2}$$

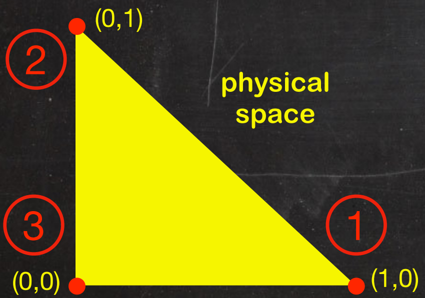
$$E_{abcd} = \frac{E}{2[1+\nu]}[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}] + \frac{E\,\nu}{1-\nu^2}\delta_{ab}\,\delta_{cd}$$

plane strain

$$\boldsymbol{J} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$J = \operatorname{Det}\boldsymbol{J} = \frac{\mathrm{d}A_{\boldsymbol{x}}}{\mathrm{d}A_{\boldsymbol{\xi}}} = 1$$

$$\begin{array}{llll}
E_{1111} = \dfrac{E}{1-\nu^2} & E_{1112} = 0 & E_{1121} = 0 & E_{1122} = \dfrac{E\,\nu}{1-\nu^2} \\
E_{1211} = 0 & E_{1212} = \dfrac{E}{2[1+\nu]} & E_{1221} = \dfrac{E}{2[1+\nu]} & E_{1222} = 0 \\
E_{2111} = 0 & E_{2112} = \dfrac{E}{2[1+\nu]} & E_{2121} = \dfrac{E}{2[1+\nu]} & E_{2122} = 0 \\
E_{2211} = \dfrac{E\,\nu}{1-\nu^2} & E_{2212} = 0 & E_{2221} = 0 & E_{2222} = \dfrac{E}{1-\nu^2}
\end{array}$$



natural space

$$N^1 = \xi \qquad N^1_{,\xi} = 1 \qquad N^1_{,\eta} = 0$$

$$N^2 = \eta \qquad N^2_{,\xi} = 0 \qquad N^2_{,\eta} = 1$$

$$N^3 = (1-\xi-\eta) \qquad N^3_{,\xi} = -1 \qquad N^3_{,\eta}(\xi,\eta) = -1$$

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Example (D2TR3N) … constant strain triangle … stiffness

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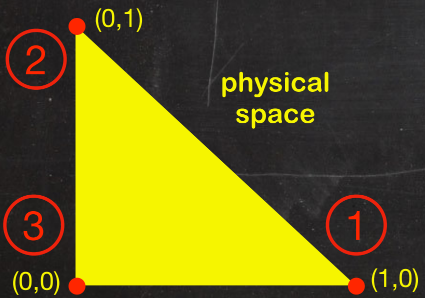
E_{2211} = \frac{E\,\nu}{1-\nu^2} & E_{2212} = 0 & E_{2221} = 0 & E_{2222} = \frac{E}{1-\nu^2}
\end{array}$$



natural space

$N^1 = \xi$

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Example (D2TR3N) … constant strain triangle … stiffness

Triangular Elements

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plane strain

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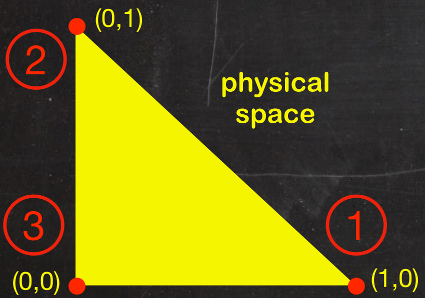
\end{array}$$



natural space

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plane strain

$$K^{ij}_{ac} = \left[N^{i}_{,\boldsymbol{\xi}}\right]_b E_{abcd} \left[N^{j}_{,\boldsymbol{\xi}}\right]_d \, \alpha_{gp} \times \frac{1}{2}$$

$$K^{ij}_{ac} = \frac{1}{2} \left[N^{i}_{,\boldsymbol{\xi}}\right]_b E_{abcd} \left[N^{j}_{,\boldsymbol{\xi}}\right]_d$$

$$\begin{array}{llll}
E_{1111} = \frac{E}{1-\nu^2} & E_{1112} = 0 & E_{1121} = 0 & E_{1122} = \frac{E\,\nu}{1-\nu^2} \\
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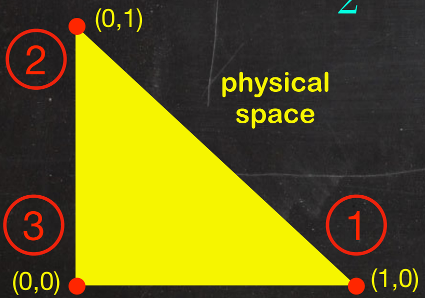
E_{2211} = \frac{E\,\nu}{1-\nu^2} & E_{2212} = 0 & E_{2221} = 0 & E_{2222} = \frac{E}{1-\nu^2}
\end{array}$$



natural space

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Example (D2TR3N) … constant strain triangle … stiffness

$$K^{ij}_{ac} = \frac{1}{2}\,\left[N^i_{,\boldsymbol{\xi}}\right]_b E_{abcd}\,\left[N^j_{,\boldsymbol{\xi}}\right]_d$$

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E_{1111} = \dfrac{E}{1-\nu^2} & E_{1112} = 0 & E_{1121} = 0 & E_{1122} = \dfrac{E\,\nu}{1-\nu^2} \\
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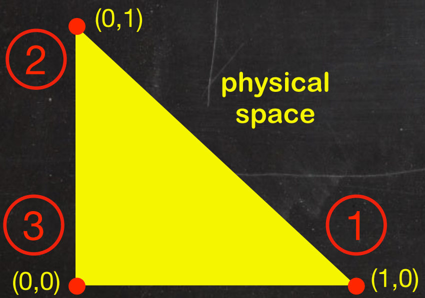
E_{2211} = \dfrac{E\,\nu}{1-\nu^2} & E_{2212} = 0 & E_{2221} = 0 & E_{2222} = \dfrac{E}{1-\nu^2}
\end{array}$$

$$\boldsymbol{K} = \begin{bmatrix}
K^{11}_{11} & K^{11}_{12} & K^{12}_{11} & K^{12}_{12} & K^{13}_{11} & K^{13}_{12} \\
K^{11}_{21} & K^{11}_{22} & K^{12}_{21} & K^{12}_{22} & K^{13}_{21} & K^{13}_{22} \\
K^{21}_{11} & K^{21}_{12} & K^{22}_{11} & K^{22}_{12} & K^{23}_{11} & K^{23}_{12} \\
K^{21}_{21} & K^{21}_{22} & K^{22}_{21} & K^{22}_{22} & K^{23}_{21} & K^{23}_{22} \\
K^{31}_{11} & K^{31}_{12} & K^{32}_{11} & K^{32}_{12} & K^{33}_{11} & K^{33}_{12} \\
K^{31}_{21} & K^{31}_{22} & K^{32}_{21} & K^{32}_{22} & K^{33}_{21} & K^{33}_{22}
\end{bmatrix}$$



natural space

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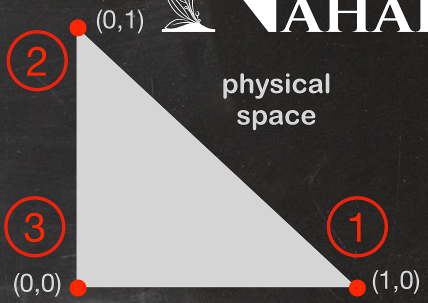
\boldsymbol{K}^{31} & \boldsymbol{K}^{32} & \boldsymbol{K}^{33}
\end{bmatrix}$$

Example (D2TR3N) … constant strain triangle … stiffness

$$K^{i\,j}_{a\,c} = \frac{1}{2}\,[N^i_{,\xi}]_b\,E_{abcd}\,[N^j_{,\xi}]_d$$

$$K^{\overset{i}{1}\overset{j}{1}}_{\underset{a}{1}\underset{c}{1}} = \frac{1}{2}\,[N^{\overset{i}{1}}_{,\xi}]_b\,E_{\underset{a}{1}b\underset{c}{1}d}\,[N^{\overset{j}{1}}_{,\xi}]_d$$

$$\overset{\text{PD}}{\sum_{b=1}^{2}}\;\overset{\text{PD}}{\sum_{d=1}^{2}}$$



$N^1 = \xi$	$N^1_{,\xi} = 1$	$N^1_{,\eta} = 0$
$N^2 = \eta$	$N^2_{,\xi} = 0$	$N^2_{,\eta} = 1$
$N^3 = (1-\xi-\eta)$	$N^3_{,\xi} = -1$	$N^3_{,\eta}(\xi,\eta) = -1$

$E_{1111} = \frac{E}{1-\nu^2}$	$E_{1112} = 0$	$E_{1121} = 0$	$E_{1122} = \frac{E\,\nu}{1-\nu^2}$
$E_{1211} = 0$	$E_{1212} = \frac{E}{2[1+\nu]}$	$E_{1221} = \frac{E}{2[1+\nu]}$	$E_{1222} = 0$
$E_{2111} = 0$	$E_{2112} = \frac{E}{2[1+\nu]}$	$E_{2121} = \frac{E}{2[1+\nu]}$	$E_{2122} = 0$
$E_{2211} = \frac{E\,\nu}{1-\nu^2}$	$E_{2212} = 0$	$E_{2221} = 0$	$E_{2222} = \frac{E}{1-\nu^2}$

Example (D2TR3N) … constant strain triangle … stiffness

$$K^{ij}_{ac} = \frac{1}{2}\,[N^i_{,\xi}]_b\, E_{abcd}\, [N^j_{,\xi}]_d$$

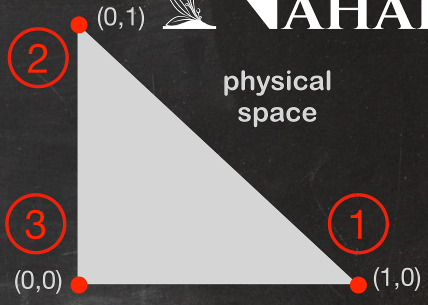
i j

$$K^{11}_{11} = \frac{1}{2}\,[N^1_{,\xi}]_b\, E_{1b1d}\, [N^1_{,\xi}]_d \qquad \sum_{b=1}^{2}\sum_{d=1}^{2}$$

a c

PD PD

$$= \frac{1}{2}\,[N^1_{,\xi}]_1\, E_{1111}\, [N^1_{,\xi}]_1 \qquad b=1 \quad d=1$$



$N^1 = \xi$	$N^1_{,\xi} = 1$	$N^1_{,\eta} = 0$
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$E_{1111} = \frac{E}{1-\nu^2}$	$E_{1112} = 0$	$E_{1121} = 0$	$E_{1122} = \frac{E\,\nu}{1-\nu^2}$
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Example (D2TR3N) … constant strain triangle … stiffness

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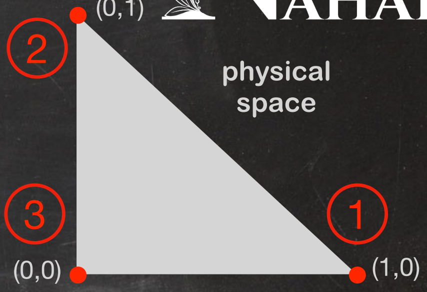
$$K^{ij}_{ac} = \frac{1}{2}\,[N^i_{,\xi}]_b\,E_{abcd}\,[N^j_{,\xi}]_d$$

$$\overset{\text{PD}}{\sum_{b=1}^{2}}\ \overset{\text{PD}}{\sum_{d=1}^{2}}$$

$$K^{\overset{i}{1}\overset{j}{1}}_{\underset{a}{1}\underset{c}{1}} = \frac{1}{2}\,[N^{\overset{i}{1}}_{,\xi}]_b\,E_{\underset{a}{1}b\underset{c}{1}d}\,[N^{\overset{j}{1}}_{,\xi}]_d$$

$$= \frac{1}{2}\,[N^1_{,\xi}]_1\,E_{1111}\,[N^1_{,\xi}]_1 \qquad \begin{matrix} b=1 \\ d=1 \end{matrix}$$

$$+ \frac{1}{2}\,[N^1_{,\xi}]_1\,E_{1112}\,[N^1_{,\xi}]_2 \qquad \begin{matrix} b=1 \\ d=2 \end{matrix}$$



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Example (D2TR3N) … constant strain triangle … stiffness

$$K^{ij}_{ac} = \frac{1}{2}\,[N^i_{,\xi}]_b\, E_{abcd}\,[N^j_{,\xi}]_d$$

i j i j

$$\sum_{b=1}^{2}\sum_{d=1}^{2}$$

PD PD

$$K^{11}_{11} = \frac{1}{2}\,[N^1_{,\xi}]_b\, E_{1b1d}\,[N^1_{,\xi}]_d$$

a c a c

$$= \frac{1}{2}\,[N^1_{,\xi}]_1\, E_{1111}\,[N^1_{,\xi}]_1$$

b=1 d=1

$$+ \frac{1}{2}\,[N^1_{,\xi}]_1\, E_{1112}\,[N^1_{,\xi}]_2$$

b=1 d=2

$$+ \frac{1}{2}\,[N^1_{,\xi}]_2\, E_{1211}\,[N^1_{,\xi}]_1$$

b=2 d=1

$N^1 = \xi$	$N^1_{,\xi} = 1$	$N^1_{,\eta} = 0$
$N^2 = \eta$	$N^2_{,\xi} = 0$	$N^2_{,\eta} = 1$
$N^3 = (1 - \xi - \eta)$	$N^3_{,\xi} = -1$	$N^3_{,\eta}(\xi,\eta) = -1$

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Example (D2TR3N) … constant strain triangle … stiffness

$$K^{ij}_{ac} = \frac{1}{2}\,[N^i_{,\xi}]_b\,E_{abcd}\,[N^j_{,\xi}]_d \quad \overset{\text{PD}}{\sum_{b=1}^{2}}\ \overset{\text{PD}}{\sum_{d=1}^{2}}$$

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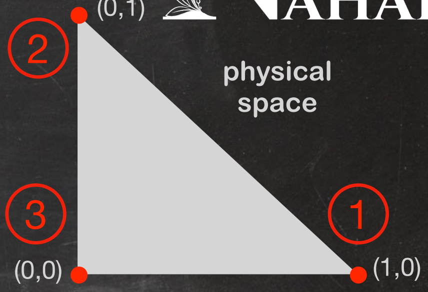

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Example (D2TR3N) … constant strain triangle … stiffness

$$K^{ij}_{ac} = \frac{1}{2}\,[N^i_{,\xi}]_b\,E_{abcd}\,[N^j_{,\xi}]_d$$

$$\overset{\text{PD}}{\sum_{b=1}^{2}}\;\overset{\text{PD}}{\sum_{d=1}^{2}}$$

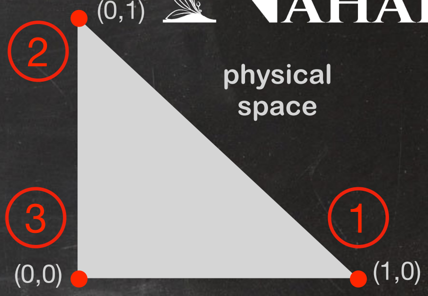
$$K^{\overset{i}{1}\overset{j}{1}}_{\underset{a}{1}\underset{c}{1}} = \frac{1}{2}\,[N^{\overset{i}{1}}_{,\xi}]_b\,E_{\underset{a}{1}b\underset{c}{1}d}\,[N^{\overset{j}{1}}_{,\xi}]_d$$

$$= \frac{1}{2}\,[N^1_{,\xi}]_1\,\underline{E_{1111}}\,[N^1_{,\xi}]_1$$

$$+ \frac{1}{2}\,[N^1_{,\xi}]_1\,\underline{E_{1112}}\,[N^1_{,\xi}]_2$$

$$+ \frac{1}{2}\,[N^1_{,\xi}]_2\,\underline{E_{1211}}\,[N^1_{,\xi}]_1$$

$$+ \frac{1}{2}\,[N^1_{,\xi}]_2\,\underline{E_{1212}}\,[N^1_{,\xi}]_2$$



$N^1 = \xi$	$N^1_{,\xi} = 1$	$N^1_{,\eta} = 0$
$N^2 = \eta$	$N^2_{,\xi} = 0$	$N^2_{,\eta} = 1$
$N^3 = (1-\xi-\eta)$	$N^3_{,\xi} = -1$	$N^3_{,\eta}(\xi,\eta) = -1$

$E_{1111} = \frac{E}{1-\nu^2}$	$E_{1112} = 0$	$E_{1121} = 0$	$E_{1122} = \frac{E\nu}{1-\nu^2}$
$E_{1211} = 0$	$E_{1212} = \frac{E}{2[1+\nu]}$	$E_{1221} = \frac{E}{2[1+\nu]}$	$E_{1222} = 0$
$E_{2111} = 0$	$E_{2112} = \frac{E}{2[1+\nu]}$	$E_{2121} = \frac{E}{2[1+\nu]}$	$E_{2122} = 0$
$E_{2211} = \frac{E\nu}{1-\nu^2}$	$E_{2212} = 0$	$E_{2221} = 0$	$E_{2222} = \frac{E}{1-\nu^2}$

Example (D2TR3N) … constant strain triangle … stiffness

$$K^{ij}_{ac} = \frac{1}{2}\,[N^i_{,\xi}]_b\,E_{abcd}\,[N^j_{,\xi}]_d \quad \overset{\text{PD}}{\sum_{b=1}^{2}}\ \overset{\text{PD}}{\sum_{d=1}^{2}}$$

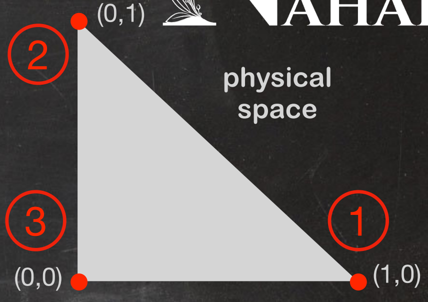
$$K^{\overset{i}{1}\overset{j}{1}}_{\underset{a}{1}\underset{c}{1}} = \frac{1}{2}\,[N^{\overset{i}{1}}_{,\xi}]_b\,E_{\underset{a}{1}b\underset{c}{1}d}\,[N^{\overset{j}{1}}_{,\xi}]_d$$

$$= \frac{1}{2}\,[N^1_{,\xi}]_1\,\underline{E_{1111}}\,[N^1_{,\xi}]_1$$

$$+ \frac{1}{2}\,[N^1_{,\xi}]_1\,\underline{\cancel{E_{1112}}}^{0}\,[N^1_{,\xi}]_2$$

$$+ \frac{1}{2}\,[N^1_{,\xi}]_2\,\underline{\cancel{E_{1211}}}^{0}\,[N^1_{,\xi}]_1$$

$$+ \frac{1}{2}\,[N^1_{,\xi}]_2\,\underline{E_{1212}}\,[N^1_{,\xi}]_2$$



$N^1 = \xi$	$N^1_{,\xi} = 1$	$N^1_{,\eta} = 0$
$N^2 = \eta$	$N^2_{,\xi} = 0$	$N^2_{,\eta} = 1$
$N^3 = (1-\xi-\eta)$	$N^3_{,\xi} = -1$	$N^3_{,\eta}(\xi,\eta) = -1$

$E_{1111} = \frac{E}{1-\nu^2}$	$E_{1112} = 0$	$E_{1121} = 0$	$E_{1122} = \frac{E\nu}{1-\nu^2}$
$E_{1211} = 0$	$E_{1212} = \frac{E}{2[1+\nu]}$	$E_{1221} = \frac{E}{2[1+\nu]}$	$E_{1222} = 0$
$E_{2111} = 0$	$E_{2112} = \frac{E}{2[1+\nu]}$	$E_{2121} = \frac{E}{2[1+\nu]}$	$E_{2122} = 0$
$E_{2211} = \frac{E\nu}{1-\nu^2}$	$E_{2212} = 0$	$E_{2221} = 0$	$E_{2222} = \frac{E}{1-\nu^2}$

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Example (D2TR3N) … constant strain triangle … stiffness

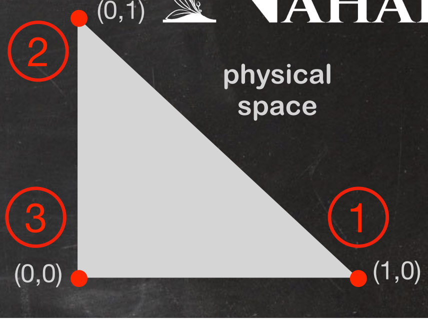
$$K^{ij}_{ac} = \frac{1}{2}\,[N^i_{,\xi}]_b\,E_{abcd}\,[N^j_{,\xi}]_d$$

$$E_{1111} = E/[1-\nu^2]$$

$$E_{1212} = E/2[1+\nu]$$

$$\overset{i\;j}{K}{}^{11}_{\underset{a\;c}{11}} = \frac{1}{2}\,[N^1_{,\xi}]_1\;E_{1111}\;[N^1_{,\xi}]_1$$

$$+\;\frac{1}{2}\,[N^1_{,\xi}]_2\;E_{1212}\;[N^1_{,\xi}]_2$$



$N^1 = \xi$	$N^1_{,\xi} = 1$	$N^1_{,\eta} = 0$
$N^2 = \eta$	$N^2_{,\xi} = 0$	$N^2_{,\eta} = 1$
$N^3 = (1-\xi-\eta)$	$N^3_{,\xi} = -1$	$N^3_{,\eta}(\xi,\eta) = -1$

$E_{1111} = \frac{E}{1-\nu^2}$	$E_{1112} = 0$	$E_{1121} = 0$	$E_{1122} = \frac{E\,\nu}{1-\nu^2}$
$E_{1211} = 0$	$E_{1212} = \frac{E}{2[1+\nu]}$	$E_{1221} = \frac{E}{2[1+\nu]}$	$E_{1222} = 0$
$E_{2111} = 0$	$E_{2112} = \frac{E}{2[1+\nu]}$	$E_{2121} = \frac{E}{2[1+\nu]}$	$E_{2122} = 0$
$E_{2211} = \frac{E\,\nu}{1-\nu^2}$	$E_{2212} = 0$	$E_{2221} = 0$	$E_{2222} = \frac{E}{1-\nu^2}$

Example (D2TR3N) … constant strain triangle … stiffness

$$K^{ij}_{ac} = \frac{1}{2}\,[N^i_{,\xi}]_b\, E_{abcd}\, [N^j_{,\xi}]_d$$

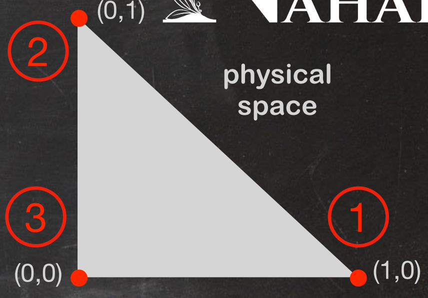
$$\overset{i\ j}{K^{11}_{11}} = \frac{1}{2}\,[N^1_{,\xi}]_1\, E_{1111}\, [N^1_{,\xi}]_1$$

$$\underset{a\ c}{} + \frac{1}{2}\,[N^1_{,\xi}]_2\, E_{1212}\, [N^1_{,\xi}]_2$$

$$E_{1111} = E/[1-\nu^2]$$

$$E_{1212} = E/2[1+\nu]$$

$$[N^1_{,\xi}] = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} [N^1_{,\xi}]_1 = 1 \\ [N^1_{,\xi}]_2 = 0 \end{array}$$



$N^1 = \xi$	$N^1_{,\xi} = 1$	$N^1_{,\eta} = 0$
$N^2 = \eta$	$N^2_{,\xi} = 0$	$N^2_{,\eta} = 1$
$N^3 = (1-\xi-\eta)$	$N^3_{,\xi} = -1$	$N^3_{,\eta}(\xi,\eta) = -1$

$E_{1111} = \frac{E}{1-\nu^2}$	$E_{1112} = 0$	$E_{1121} = 0$	$E_{1122} = \frac{E\,\nu}{1-\nu^2}$
$E_{1211} = 0$	$E_{1212} = \frac{E}{2[1+\nu]}$	$E_{1221} = \frac{E}{2[1+\nu]}$	$E_{1222} = 0$
$E_{2111} = 0$	$E_{2112} = \frac{E}{2[1+\nu]}$	$E_{2121} = \frac{E}{2[1+\nu]}$	$E_{2122} = 0$
$E_{2211} = \frac{E\,\nu}{1-\nu^2}$	$E_{2212} = 0$	$E_{2221} = 0$	$E_{2222} = \frac{E}{1-\nu^2}$

Example (D2TR3N) … constant strain triangle … stiffness

$$K^{ij}_{ac} = \frac{1}{2}\,[N^i_{,\xi}]_b\, E_{abcd}\, [N^j_{,\xi}]_d$$

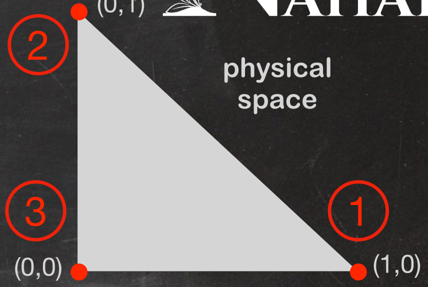
$$K^{\overset{i\,j}{11}}_{\underset{a\,c}{11}} = \frac{1}{2}\,\big[\overset{1}{\cancel{N^1_{,\xi}}}\big]_1\, E_{1111}\, \big[\overset{1}{\cancel{N^1_{,\xi}}}\big]_1 + \frac{1}{2}\,\big[\underset{0}{\cancel{N^1_{,\xi}}}\big]_2\, E_{1212}\, \big[\underset{0}{\cancel{N^1_{,\xi}}}\big]_2$$

$$E_{1111} = E/[1-\nu^2]$$

$$E_{1212} = E/2[1+\nu]$$

$$[N^1_{,\xi}]_1 = 1$$

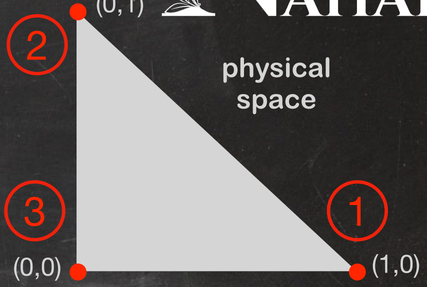
$$[N^1_{,\xi}]_2 = 0$$



$N^1 = \xi$	$N^1_{,\xi} = 1$	$N^1_{,\eta} = 0$
$N^2 = \eta$	$N^2_{,\xi} = 0$	$N^2_{,\eta} = 1$
$N^3 = (1-\xi-\eta)$	$N^3_{,\xi} = -1$	$N^3_{,\eta}(\xi,\eta) = -1$

$E_{1111} = \frac{E}{1-\nu^2}$	$E_{1112} = 0$	$E_{1121} = 0$	$E_{1122} = \frac{E\,\nu}{1-\nu^2}$
$E_{1211} = 0$	$E_{1212} = \frac{E}{2[1+\nu]}$	$E_{1221} = \frac{E}{2[1+\nu]}$	$E_{1222} = 0$
$E_{2111} = 0$	$E_{2112} = \frac{E}{2[1+\nu]}$	$E_{2121} = \frac{E}{2[1+\nu]}$	$E_{2122} = 0$
$E_{2211} = \frac{E\,\nu}{1-\nu^2}$	$E_{2212} = 0$	$E_{2221} = 0$	$E_{2222} = \frac{E}{1-\nu^2}$

Example (D2TR3N) … constant strain triangle … stiffness



$$K^{ij}_{ac} = \frac{1}{2}\,[N^i_{,\xi}]_b\,E_{abcd}\,[N^j_{,\xi}]_d$$

$$E_{1111} = E/[1-\nu^2]$$

$$E_{1212} = E/2[1+\nu]$$

$$\underset{a\;c}{K}{}^{\overset{i\;j}{11}}_{11} = \frac{1}{2}\,\overset{1}{\cancel{[N^1_{,\xi}]_1}}\,E_{1111}\,\overset{1}{\cancel{[N^1_{,\xi}]_1}} + \frac{1}{2}\,\underset{0}{\cancel{[N^1_{,\xi}]_2}}\,E_{1212}\,\underset{0}{\cancel{[N^1_{,\xi}]_2}}$$

$$= \frac{1}{2}E_{1111} = \frac{E}{2[1-\nu^2]}$$

$N^1 = \xi$	$N^1_{,\xi} = 1$	$N^1_{,\eta} = 0$
$N^2 = \eta$	$N^2_{,\xi} = 0$	$N^2_{,\eta} = 1$
$N^3 = (1-\xi-\eta)$	$N^3_{,\xi} = -1$	$N^3_{,\eta}(\xi,\eta) = -1$

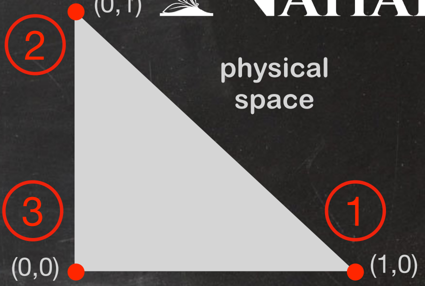
$E_{1111} = \frac{E}{1-\nu^2}$	$E_{1112} = 0$	$E_{1121} = 0$	$E_{1122} = \frac{E\nu}{1-\nu^2}$
$E_{1211} = 0$	$E_{1212} = \frac{E}{2[1+\nu]}$	$E_{1221} = \frac{E}{2[1+\nu]}$	$E_{1222} = 0$
$E_{2111} = 0$	$E_{2112} = \frac{E}{2[1+\nu]}$	$E_{2121} = \frac{E}{2[1+\nu]}$	$E_{2122} = 0$
$E_{2211} = \frac{E\nu}{1-\nu^2}$	$E_{2212} = 0$	$E_{2221} = 0$	$E_{2222} = \frac{E}{1-\nu^2}$

Example (D2TR3N) … constant strain triangle … stiffness

$$K^{ij}_{ac} = \frac{1}{2}\,\left[N^i_{,\xi}\right]_b E_{abcd} \left[N^j_{,\xi}\right]_d$$

$$K^{\overset{i}{1}\overset{j}{1}}_{\underset{a}{1}\underset{c}{1}} = \frac{1}{2}E_{1111}$$

$$K^{11}_{11} = \frac{E}{2[1-\nu^2]}$$



$N^1 = \xi$	$N^1_{,\xi} = 1$	$N^1_{,\eta} = 0$
$N^2 = \eta$	$N^2_{,\xi} = 0$	$N^2_{,\eta} = 1$
$N^3 = (1-\xi-\eta)$	$N^3_{,\xi} = -1$	$N^3_{,\eta}(\xi,\eta) = -1$

$E_{1111} = \frac{E}{1-\nu^2}$	$E_{1112} = 0$	$E_{1121} = 0$	$E_{1122} = \frac{E\,\nu}{1-\nu^2}$
$E_{1211} = 0$	$E_{1212} = \frac{E}{2[1+\nu]}$	$E_{1221} = \frac{E}{2[1+\nu]}$	$E_{1222} = 0$
$E_{2111} = 0$	$E_{2112} = \frac{E}{2[1+\nu]}$	$E_{2121} = \frac{E}{2[1+\nu]}$	$E_{2122} = 0$
$E_{2211} = \frac{E\,\nu}{1-\nu^2}$	$E_{2212} = 0$	$E_{2221} = 0$	$E_{2222} = \frac{E}{1-\nu^2}$

Example (D2TR3N) … constant strain triangle … stiffness

Triangular Elements

$$K^{ij}_{ac} = \sum_{\mathrm{gp}=1}^{\mathrm{GPE}} \left[\boldsymbol{J}^{-T}\cdot N^{i}_{,\boldsymbol{\xi}}\right]_b E_{abcd} \left[\boldsymbol{J}^{-T}\cdot N^{j}_{,\boldsymbol{\xi}}\right]_d \,\mathrm{Det}\boldsymbol{J}\,\alpha_{gp} \times \frac{1}{2}$$

$$E_{abcd} = \frac{E}{2[1+\nu]}\left[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}\right] + \frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

plane strain

$$\frac{E}{2[1-\nu^2]}$$

$$\boldsymbol{K} = \begin{bmatrix} \boldsymbol{K}^{11} & \boldsymbol{K}^{12} & \boldsymbol{K}^{13} \\ \boldsymbol{K}^{21} & \boldsymbol{K}^{22} & \boldsymbol{K}^{23} \\ \boldsymbol{K}^{31} & \boldsymbol{K}^{32} & \boldsymbol{K}^{33} \end{bmatrix} = \begin{bmatrix} K^{11}_{11} & K^{11}_{12} & K^{12}_{11} & K^{12}_{12} & K^{13}_{11} & K^{13}_{12} \\ K^{11}_{21} & K^{11}_{22} & K^{12}_{21} & K^{12}_{22} & K^{13}_{21} & K^{13}_{22} \\ K^{21}_{11} & K^{21}_{12} & K^{22}_{11} & K^{22}_{12} & K^{23}_{11} & K^{23}_{12} \\ K^{21}_{21} & K^{21}_{22} & K^{22}_{21} & K^{22}_{22} & K^{23}_{21} & K^{23}_{22} \\ K^{31}_{11} & K^{31}_{12} & K^{32}_{11} & K^{32}_{12} & K^{33}_{11} & K^{33}_{12} \\ K^{31}_{21} & K^{31}_{22} & K^{32}_{21} & K^{32}_{22} & K^{33}_{21} & K^{33}_{22} \end{bmatrix}$$

$$\boldsymbol{J} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\mathrm{NPE}} \\ y^1 & y^2 & \dots & y^{\mathrm{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\mathrm{NPE}}_\xi & N^{\mathrm{NPE}}_\eta \end{bmatrix}$$

! ATTENTION !

element stiffness matrix is
symmetric and
its determinant is zero

Example (D2TR3N) … constant strain triangle … stiffness

Triangular Elements

$$K^{i\,j}_{a\,c} = \sum_{\text{gp}\,=\,1}^{\text{GPE}} \left[\boldsymbol{J}^{-T}\cdot N^i_{,\boldsymbol{\xi}}\right]_b E_{a\,b\,c\,d} \left[\boldsymbol{J}^{-T}\cdot N^j_{,\boldsymbol{\xi}}\right]_d \,\text{Det}\boldsymbol{J}\,\alpha_{gp} \times \frac{1}{2}$$

$$E_{a\,b\,c\,d} = \frac{E}{2[1+\nu]}[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}] + \frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

plane strain

$$\boldsymbol{K} = \begin{bmatrix} \boldsymbol{K}^{11} & \boldsymbol{K}^{12} & \boldsymbol{K}^{13} \\ \boldsymbol{K}^{21} & \boldsymbol{K}^{22} & \boldsymbol{K}^{23} \\ \boldsymbol{K}^{31} & \boldsymbol{K}^{32} & \boldsymbol{K}^{33} \end{bmatrix} = \begin{bmatrix} K^{11}_{11} & K^{11}_{12} & K^{12}_{11} & K^{12}_{12} & K^{13}_{11} & K^{13}_{12} \\ K^{11}_{21} & K^{11}_{22} & K^{12}_{21} & K^{12}_{22} & K^{13}_{21} & K^{13}_{22} \\ K^{21}_{11} & K^{21}_{12} & K^{22}_{11} & K^{22}_{12} & K^{23}_{11} & K^{23}_{12} \\ K^{21}_{21} & K^{21}_{22} & K^{22}_{21} & K^{22}_{22} & K^{23}_{21} & K^{23}_{22} \\ K^{31}_{11} & K^{31}_{12} & K^{32}_{11} & K^{32}_{12} & K^{33}_{11} & K^{33}_{12} \\ K^{31}_{21} & K^{31}_{22} & K^{32}_{21} & K^{32}_{22} & K^{33}_{21} & K^{33}_{22} \end{bmatrix}$$

$$\boldsymbol{J} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\text{NPE}} \\ y^1 & y^2 & \dots & y^{\text{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\text{NPE}}_\xi & N^{\text{NPE}}_\eta \end{bmatrix}$$

! ATTENTION !

element stiffness matrix is symmetric and its determinant is zero

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Example (D2TR3N) … constant strain triangle … stiffness

Triangular Elements

$$K^{ij}_{ac} = \sum_{\text{gp}=1}^{\text{GPE}} \left[\boldsymbol{J}^{-T} \cdot N^{i}_{,\boldsymbol{\xi}}\right]_b E_{abcd} \left[\boldsymbol{J}^{-T} \cdot N^{j}_{,\boldsymbol{\xi}}\right]_d \operatorname{Det}\boldsymbol{J}\, \alpha_{gp} \times \frac{1}{2}$$

$$E_{abcd} = \frac{E}{2[1+\nu]}[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}]$$

plane strain

$$+\frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

$$\boldsymbol{K} = \begin{bmatrix} \boldsymbol{K}^{11} & \boldsymbol{K}^{12} & \boldsymbol{K}^{13} \\ \boldsymbol{K}^{21} & \boldsymbol{K}^{22} & \boldsymbol{K}^{23} \\ \boldsymbol{K}^{31} & \boldsymbol{K}^{32} & \boldsymbol{K}^{33} \end{bmatrix} = \left[\begin{array}{|cc|cccc}
\cline{1-2}
K^{11}_{11} & K^{11}_{12} & K^{12}_{11} & K^{12}_{12} & K^{13}_{11} & K^{13}_{12} \\
K^{11}_{21} & K^{11}_{22} & K^{12}_{21} & K^{12}_{22} & K^{13}_{21} & K^{13}_{22} \\
\cline{1-2}
K^{21}_{11} & K^{21}_{12} & K^{22}_{11} & K^{22}_{12} & K^{23}_{11} & K^{23}_{12} \\
K^{21}_{21} & K^{21}_{22} & K^{22}_{21} & K^{22}_{22} & K^{23}_{21} & K^{23}_{22} \\
\cline{1-2}
K^{31}_{11} & K^{31}_{12} & K^{32}_{11} & K^{32}_{12} & K^{33}_{11} & K^{33}_{12} \\
K^{31}_{21} & K^{31}_{22} & K^{32}_{21} & K^{32}_{22} & K^{33}_{21} & K^{33}_{22} \\
\cline{1-2}
\end{array}\right]$$

$$\boldsymbol{J} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\text{NPE}} \\ y^1 & y^2 & \dots & y^{\text{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\text{NPE}}_\xi & N^{\text{NPE}}_\eta \end{bmatrix}$$

! ATTENTION !

element stiffness matrix is symmetric and its determinant is zero

Example (D2TR3N) … linear triangular element

Triangular Elements

$$K^{\,i\,j}_{\,a\,c} = \sum_{\text{gp}\,=\,1}^{\text{GPE}} \left[\boldsymbol{J}^{-T} \cdot N^{i}_{,\boldsymbol{\xi}}\right]_b E_{a\,b\,c\,d} \left[\boldsymbol{J}^{-T} \cdot N^{j}_{,\boldsymbol{\xi}}\right]_d \text{Det}\boldsymbol{J}\, \alpha_{gp} \times \frac{1}{2}$$

$$E_{a\,b\,c\,d} = \frac{E}{2[1+\nu]}[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}] + \frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

plane strain

$$\boldsymbol{J} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\text{NPE}} \\ y^1 & y^2 & \dots & y^{\text{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\text{NPE}}_\xi & N^{\text{NPE}}_\eta \end{bmatrix}$$

$$K^{11}_{11} = \frac{E}{2\,[1-\nu^2]} \qquad K^{11}_{12} = 0$$

$$K^{11}_{21} = 0 \qquad K^{11}_{22} = \frac{E}{4\,[1+\nu]}$$

$$x^1 = [1,0]$$

$$x^2 = [0,1]$$

$$x^3 = [0,0]$$

Example (D2TR3N) … linear triangular element

Triangular Elements

$$K^{ij}_{ac} = \sum_{\text{gp}=1}^{\text{GPE}} \left[\boldsymbol{J}^{-T} \cdot N^{i}_{,\boldsymbol{\xi}}\right]_b E_{abcd} \left[\boldsymbol{J}^{-T} \cdot N^{j}_{,\boldsymbol{\xi}}\right]_d \text{Det}\boldsymbol{J}\,\alpha_{gp} \times \frac{1}{2}$$

$$E_{abcd} = \frac{E}{2[1+\nu]}[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}] + \frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

plane strain

$$\boldsymbol{J} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\text{NPE}} \\ y^1 & y^2 & \dots & y^{\text{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\text{NPE}}_\xi & N^{\text{NPE}}_\eta \end{bmatrix}$$

$$K^{21}_{11} = 0 \qquad K^{21}_{12} = \frac{E}{4[1+\nu]}$$

$$K^{21}_{21} = \frac{E\nu}{2[1-\nu^2]} \qquad K^{21}_{22} = 0$$

$$x^1 = [1,0]$$

$$x^2 = [0,1]$$

$$x^3 = [0,0]$$

Example (D2TR3N) … linear triangular element

Triangular Elements

$$K^{ij}_{ac} = \sum_{\text{gp}=1}^{\text{GPE}} \left[\boldsymbol{J}^{-T} \cdot N^i_{,\boldsymbol{\xi}}\right]_b E_{abcd} \left[\boldsymbol{J}^{-T} \cdot N^j_{,\boldsymbol{\xi}}\right]_d \text{Det}\boldsymbol{J}\, \alpha_{gp} \times \frac{1}{2}$$

$$E_{abcd} = \frac{E}{2[1+\nu]}[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}] + \frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

plane strain

$$\boldsymbol{J} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\text{NPE}} \\ y^1 & y^2 & \dots & y^{\text{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\text{NPE}}_\xi & N^{\text{NPE}}_\eta \end{bmatrix}$$

$$K^{31}_{11} = -\frac{E}{2[1-\nu^2]} \qquad K^{31}_{12} = -\frac{E}{4[1+\nu]}$$

$$K^{31}_{21} = -\frac{E\nu}{2[1-\nu^2]} \qquad K^{31}_{22} = -\frac{E}{4[1+\nu]}$$

$x^1 = [1, 0]$

$x^2 = [0, 1]$

$x^3 = [0, 0]$

Example (D2TR3N) … linear triangular element

Triangular Elements

$$K^{ij}_{ac} = \sum_{\text{gp}=1}^{\text{GPE}} \left[\boldsymbol{J}^{-T}\cdot N^{i}_{,\boldsymbol{\xi}}\right]_b E_{abcd} \left[\boldsymbol{J}^{-T}\cdot N^{j}_{,\boldsymbol{\xi}}\right]_d \operatorname{Det}\boldsymbol{J}\,\alpha_{gp} \times \frac{1}{2}$$

$$E_{abcd} = \frac{E}{2[1+\nu]}[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}] + \frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

plane strain

$$\boldsymbol{J} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\text{NPE}} \\ y^1 & y^2 & \dots & y^{\text{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\text{NPE}}_\xi & N^{\text{NPE}}_\eta \end{bmatrix}$$

$$\boldsymbol{K} = \begin{bmatrix}
\frac{E}{2\,[1-\nu^2]} & 0 & 0 & \frac{E\nu}{2\,[1-\nu^2]} & -\frac{E}{2\,[1-\nu^2]} & -\frac{E\nu}{2\,[1-\nu^2]} \\
0 & \frac{E}{4\,[\nu+1]} & \frac{E}{4\,[\nu+1]} & 0 & -\frac{E}{4\,[\nu+1]} & -\frac{E}{4\,[\nu+1]} \\
0 & \frac{E}{4\,[\nu+1]} & \frac{E}{4\,[\nu+1]} & 0 & -\frac{E}{4\,[\nu+1]} & -\frac{E}{4\,[\nu+1]} \\
\frac{E\nu}{2\,[1-\nu^2]} & 0 & 0 & \frac{E}{2\,[1-\nu^2]} & -\frac{E\nu}{2\,[1-\nu^2]} & -\frac{E}{2\,[1-\nu^2]} \\
-\frac{E}{2\,[1-\nu^2]} & -\frac{E}{4\,[\nu+1]} & -\frac{E}{4\,[\nu+1]} & -\frac{E\nu}{2\,[1-\nu^2]} & \frac{E\,[3-\nu]}{4\,[1-\nu^2]} & \frac{E}{4\,[1-\nu]} \\
-\frac{E\nu}{2\,[1-\nu^2]} & -\frac{E}{4\,[\nu+1]} & -\frac{E}{4\,[\nu+1]} & -\frac{E}{2\,[1-\nu^2]} & \frac{E}{4\,[1-\nu]} & \frac{E\,[3-\nu]}{4\,[1-\nu^2]}
\end{bmatrix}$$

$$\boldsymbol{x}^1 = [\,1\,,\,0\,]$$

$$\boldsymbol{x}^2 = [\,0\,,\,1\,]$$

$$\boldsymbol{x}^3 = [\,0\,,\,0\,]$$

Example (D2TR3N) … constant strain triangle … stiffness

Triangular Elements

$$K^{i\,j}_{a\,c} = \sum_{\mathrm{gp}=1}^{\mathrm{GPE}} \left[\boldsymbol{J}^{-T}\cdot N^{i}_{,\boldsymbol{\xi}}\right]_b E_{a\,b\,c\,d} \left[\boldsymbol{J}^{-T}\cdot N^{j}_{,\boldsymbol{\xi}}\right]_d \,\mathrm{Det}\boldsymbol{J}\,\alpha_{gp} \times \frac{1}{2}$$

$$E_{a\,b\,c\,d} = \frac{E}{2[1+\nu]}\left[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}\right] + \frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

plane strain

$$\boldsymbol{J} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\mathrm{NPE}} \\ y^1 & y^2 & \dots & y^{\mathrm{NPE}} \end{bmatrix} \begin{bmatrix} N^1_{\xi} & N^1_{\eta} \\ N^2_{\xi} & N^2_{\eta} \\ \vdots & \vdots \\ N^{\mathrm{NPE}}_{\xi} & N^{\mathrm{NPE}}_{\eta} \end{bmatrix}$$

$$\boldsymbol{K} = \begin{bmatrix} \boldsymbol{K}^{11} & \boldsymbol{K}^{12} & \boldsymbol{K}^{13} \\ \boldsymbol{K}^{21} & \boldsymbol{K}^{22} & \boldsymbol{K}^{23} \\ \boldsymbol{K}^{31} & \boldsymbol{K}^{32} & \boldsymbol{K}^{33} \end{bmatrix} = \begin{bmatrix} K^{11}_{11} & K^{11}_{12} & K^{12}_{11} & K^{12}_{12} & K^{13}_{11} & K^{13}_{12} \\ K^{11}_{21} & K^{11}_{22} & K^{12}_{21} & K^{12}_{22} & K^{13}_{21} & K^{13}_{22} \\ K^{21}_{11} & K^{21}_{12} & K^{22}_{11} & K^{22}_{12} & K^{23}_{11} & K^{23}_{12} \\ K^{21}_{21} & K^{21}_{22} & K^{22}_{21} & K^{22}_{22} & K^{23}_{21} & K^{23}_{22} \\ K^{31}_{11} & K^{31}_{12} & K^{32}_{11} & K^{32}_{12} & K^{33}_{11} & K^{33}_{12} \\ K^{31}_{21} & K^{31}_{22} & K^{32}_{21} & K^{32}_{22} & K^{33}_{21} & K^{33}_{22} \end{bmatrix}$$

! ATTENTION !

**element stiffness matrix is
symmetric and
its determinant is zero**

Example (D2TR3N) … linear triangular element

Triangular Elements

$$K^{\,i\,j}_{\,a\,c} = \sum_{\mathrm{gp}\,=\,1}^{\mathrm{GPE}} \left[\boldsymbol{J}^{-T}\cdot N^{i}_{,\boldsymbol{\xi}}\right]_b E_{a\,b\,c\,d} \left[\boldsymbol{J}^{-T}\cdot N^{j}_{,\boldsymbol{\xi}}\right]_d \,\mathrm{Det}\boldsymbol{J}\,\alpha_{gp} \times \frac{1}{2}$$

$$E_{a\,b\,c\,d} = \frac{E}{2[1+\nu]}\left[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}\right] + \frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

plane strain

$$\boldsymbol{J} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\mathrm{NPE}} \\ y^1 & y^2 & \dots & y^{\mathrm{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\mathrm{NPE}}_\xi & N^{\mathrm{NPE}}_\eta \end{bmatrix}$$

$$\boldsymbol{K} = \begin{bmatrix}
\frac{E}{2\,[1-\nu^2]} & 0 & 0 & \frac{E\nu}{2\,[1-\nu^2]} & -\frac{E}{2\,[1-\nu^2]} & -\frac{E\nu}{2\,[1-\nu^2]} \\
0 & \frac{E}{4\,[\nu+1]} & \frac{E}{4\,[\nu+1]} & 0 & -\frac{E}{4\,[\nu+1]} & -\frac{E}{4\,[\nu+1]} \\
0 & \frac{E}{4\,[\nu+1]} & \frac{E}{4\,[\nu+1]} & 0 & -\frac{E}{4\,[\nu+1]} & -\frac{E}{4\,[\nu+1]} \\
\frac{E\nu}{2\,[1-\nu^2]} & 0 & 0 & \frac{E}{2\,[1-\nu^2]} & -\frac{E\nu}{2\,[1-\nu^2]} & -\frac{E}{2\,[1-\nu^2]} \\
-\frac{E}{2\,[1-\nu^2]} & -\frac{E}{4\,[\nu+1]} & -\frac{E}{4\,[\nu+1]} & -\frac{E\nu}{2\,[1-\nu^2]} & \frac{E\,[3-\nu]}{4\,[1-\nu^2]} & \frac{E}{4\,[1-\nu]} \\
-\frac{E\nu}{2\,[1-\nu^2]} & -\frac{E}{4\,[\nu+1]} & -\frac{E}{4\,[\nu+1]} & -\frac{E}{2\,[1-\nu^2]} & \frac{E}{4\,[1-\nu]} & \frac{E\,[3-\nu]}{4\,[1-\nu^2]}
\end{bmatrix}$$

$$\boldsymbol{x}^1 = [\,1\,,\,0\,]$$

$$\boldsymbol{x}^2 = [\,0\,,\,1\,]$$

$$\boldsymbol{x}^3 = [\,0\,,\,0\,]$$

Example (D2TR3N) … linear triangular element

… using one Gauss point …

Triangular Elements

$$K^{ij}_{ac} = \sum_{\text{gp}=1}^{\text{GPE}} \left[\boldsymbol{J}^{-T} \cdot N^{i}_{,\boldsymbol{\xi}}\right]_b E_{abcd} \left[\boldsymbol{J}^{-T} \cdot N^{j}_{,\boldsymbol{\xi}}\right]_d \operatorname{Det}\boldsymbol{J}\, \alpha_{gp} \times \frac{1}{2}$$

$$E_{abcd} = \frac{E}{2[1+\nu]}[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}] + \frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

plane strain

$$\boldsymbol{J} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\text{NPE}} \\ y^1 & y^2 & \dots & y^{\text{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\text{NPE}}_\xi & N^{\text{NPE}}_\eta \end{bmatrix}$$

$$\boldsymbol{K} = \begin{bmatrix} 5.000 & 0.000 & 0.000 & 0.000 & -5.000 & 0.000 \\ 0.000 & 2.500 & 2.500 & 0.000 & -2.500 & -2.500 \\ 0.000 & 2.500 & 2.500 & 0.000 & -2.500 & -2.500 \\ 0.000 & 0.000 & 0.000 & 5.000 & 0.000 & -5.000 \\ -5.000 & -2.500 & -2.500 & 0.000 & 7.500 & 2.500 \\ 0.000 & -2.500 & -2.500 & -5.000 & 2.500 & 7.500 \end{bmatrix}$$

$$\boldsymbol{x}^1 = [1\,,\,0]$$

$$\boldsymbol{x}^2 = [0\,,\,1]$$

$$\boldsymbol{x}^3 = [0\,,\,0]$$

$$E = 10\,,\ \nu = 0$$

Example (D2TR3N) … linear triangular element

… using one Gauss point …

Triangular Elements

$$K^{i\,j}_{a\,c} = \sum_{\mathrm{gp}\,=\,1}^{\mathrm{GPE}} \left[\boldsymbol{J}^{-T}\cdot N^{i}_{,\boldsymbol{\xi}}\right]_b E_{a\,b\,c\,d} \left[\boldsymbol{J}^{-T}\cdot N^{j}_{,\boldsymbol{\xi}}\right]_d \mathrm{Det}\boldsymbol{J}\,\alpha_{gp} \times \frac{1}{2}$$

$$E_{a\,b\,c\,d} = \frac{E}{2[1+\nu]}[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}] + \frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

plane strain

$$\boldsymbol{J} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\mathrm{NPE}} \\ y^1 & y^2 & \dots & y^{\mathrm{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\mathrm{NPE}}_\xi & N^{\mathrm{NPE}}_\eta \end{bmatrix}$$

$$\boldsymbol{K} = \begin{bmatrix} 5.333 & 0.000 & 0.000 & 1.333 & -5.333 & -1.333 \\ 0.000 & 2.000 & 2.000 & 0.000 & -2.000 & -2.000 \\ 0.000 & 2.000 & 2.000 & 0.000 & -2.000 & -2.000 \\ 1.333 & 0.000 & 0.000 & 5.333 & -1.333 & -5.333 \\ -5.333 & -2.000 & -2.000 & -1.333 & 7.333 & 3.333 \\ -1.333 & -2.000 & -2.000 & -5.333 & 3.333 & 7.333 \end{bmatrix}$$

$$\boldsymbol{x}^1 = [1\,,\,0]$$

$$\boldsymbol{x}^2 = [0\,,\,1]$$

$$\boldsymbol{x}^3 = [0\,,\,0]$$

$$E = 10\,,\ \nu = 0.25$$

Example (D2TR3N) … linear triangular element

… using one Gauss point …

Triangular Elements

$$K^{\,i\,j}_{\,a\,c} = \sum_{\mathrm{gp}\,=\,1}^{\mathrm{GPE}} \left[\boldsymbol{J}^{-T}\cdot N^{i}_{,\boldsymbol{\xi}}\right]_b E_{a\,b\,c\,d} \left[\boldsymbol{J}^{-T}\cdot N^{j}_{,\boldsymbol{\xi}}\right]_d \mathrm{Det}\boldsymbol{J}\,\alpha_{gp} \times \frac{1}{2}$$

$$E_{a\,b\,c\,d} = \frac{E}{2[1+\nu]}[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}] + \frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

plane strain

$$\boldsymbol{K} = \begin{bmatrix} 6.667 & 0.000 & 0.000 & 3.333 & -6.667 & -3.333 \\ 0.000 & 1.667 & 1.667 & 0.000 & -1.667 & -1.667 \\ 0.000 & 1.667 & 1.667 & 0.000 & -1.667 & -1.667 \\ 3.333 & 0.000 & 0.000 & 6.667 & -3.333 & -6.667 \\ -6.667 & -1.667 & -1.667 & -3.333 & 8.333 & 5.000 \\ -3.333 & -1.667 & -1.667 & -6.667 & 5.000 & 8.333 \end{bmatrix}$$

$$\boldsymbol{J} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\mathrm{NPE}} \\ y^1 & y^2 & \dots & y^{\mathrm{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\mathrm{NPE}}_\xi & N^{\mathrm{NPE}}_\eta \end{bmatrix}$$

$$\boldsymbol{x}^1 = [\,1\,,\,0\,]$$

$$\boldsymbol{x}^2 = [\,0\,,\,1\,]$$

$$\boldsymbol{x}^3 = [\,0\,,\,0\,]$$

$$E = 10\,,\ \nu = 0.5$$

Example (D2TR3N) … linear triangular element

… using one Gauss point …

Triangular Elements

$$K^{\,i\,j}_{\,a\,c} = \sum_{\mathrm{gp}\,=\,1}^{\mathrm{GPE}} \left[\boldsymbol{J}^{-T} \cdot N^{i}_{,\boldsymbol{\xi}}\right]_b E_{a\,b\,c\,d} \left[\boldsymbol{J}^{-T} \cdot N^{j}_{,\boldsymbol{\xi}}\right]_d \mathrm{Det}\boldsymbol{J}\,\alpha_{gp} \times \frac{1}{2}$$

$$E_{a\,b\,c\,d} = \frac{E}{2[1+\nu]}[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}] + \frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

plane strain

$$\boldsymbol{K} = \begin{bmatrix} 11.429 & 0.000 & 0.000 & 8.571 & -11.429 & -8.571 \\ 0.000 & 1.429 & 1.429 & 0.000 & -1.429 & -1.429 \\ 0.000 & 1.429 & 1.429 & 0.000 & -1.429 & -1.429 \\ 8.571 & 0.000 & 0.000 & 11.429 & -8.571 & -11.429 \\ -11.429 & -1.429 & -1.429 & -8.571 & 12.857 & 10.000 \\ -8.571 & -1.429 & -1.429 & -11.429 & 10.000 & 12.857 \end{bmatrix}$$

$$\boldsymbol{J} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\mathrm{NPE}} \\ y^1 & y^2 & \dots & y^{\mathrm{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\mathrm{NPE}}_\xi & N^{\mathrm{NPE}}_\eta \end{bmatrix}$$

$$\boldsymbol{x}^1 = [\,1\,,\,0\,]$$

$$\boldsymbol{x}^2 = [\,0\,,\,1\,]$$

$$\boldsymbol{x}^3 = [\,0\,,\,0\,]$$

$$E = 10\,,\ \nu = 0.75$$

Example (D2QU4N) … bilinear quadrilateral element

Quadrilateral Elements

$$K^{ij}_{ac} = \sum_{\text{gp}\,=\,1}^{\text{GPE}} \left[\boldsymbol{J}^{-T} \cdot N^{i}_{,\boldsymbol{\xi}}\right]_b E_{abcd} \left[\boldsymbol{J}^{-T} \cdot N^{j}_{,\boldsymbol{\xi}}\right]_d \operatorname{Det}\boldsymbol{J}\, \alpha_{gp}$$

$$E_{abcd} = \frac{E}{2[1+\nu]}[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}] + \frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

plane strain

$$\boldsymbol{K} = \begin{bmatrix} \boldsymbol{K}^{11} & \boldsymbol{K}^{12} & \boldsymbol{K}^{13} & \boldsymbol{K}^{14} \\ \boldsymbol{K}^{21} & \boldsymbol{K}^{22} & \boldsymbol{K}^{23} & \boldsymbol{K}^{24} \\ \boldsymbol{K}^{31} & \boldsymbol{K}^{32} & \boldsymbol{K}^{33} & \boldsymbol{K}^{34} \\ \boldsymbol{K}^{41} & \boldsymbol{K}^{42} & \boldsymbol{K}^{43} & \boldsymbol{K}^{44} \end{bmatrix} = \begin{bmatrix} K^{11}_{11} & K^{11}_{12} & K^{12}_{11} & K^{12}_{12} & K^{13}_{11} & K^{13}_{12} & K^{14}_{11} & K^{14}_{12} \\ K^{11}_{21} & K^{11}_{22} & K^{12}_{21} & K^{12}_{22} & K^{13}_{21} & K^{13}_{22} & K^{14}_{21} & K^{14}_{22} \\ K^{21}_{11} & K^{21}_{12} & K^{22}_{11} & K^{22}_{12} & K^{23}_{11} & K^{23}_{12} & K^{24}_{11} & K^{24}_{12} \\ K^{21}_{21} & K^{21}_{22} & K^{22}_{21} & K^{22}_{22} & K^{23}_{21} & K^{23}_{22} & K^{24}_{21} & K^{24}_{22} \\ K^{31}_{11} & K^{31}_{12} & K^{32}_{11} & K^{32}_{12} & K^{33}_{11} & K^{33}_{12} & K^{34}_{11} & K^{34}_{12} \\ K^{31}_{21} & K^{31}_{22} & K^{32}_{21} & K^{32}_{22} & K^{33}_{21} & K^{33}_{22} & K^{34}_{21} & K^{34}_{22} \\ K^{41}_{11} & K^{41}_{12} & K^{42}_{11} & K^{42}_{12} & K^{43}_{11} & K^{43}_{12} & K^{44}_{11} & K^{44}_{12} \\ K^{41}_{21} & K^{41}_{22} & K^{42}_{21} & K^{42}_{22} & K^{43}_{21} & K^{43}_{22} & K^{44}_{21} & K^{44}_{22} \end{bmatrix}$$

$$\boldsymbol{J} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\text{NPE}} \\ y^1 & y^2 & \dots & y^{\text{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\text{NPE}}_\xi & N^{\text{NPE}}_\eta \end{bmatrix}$$

! ATTENTION !

element stiffness matrix is symmetric and its determinant is zero

Example (D2QU4N) … bilinear quadrilateral element

… using one Gauss point …

Quadrilateral Elements

$$K^{ij}_{ac} = \sum_{\mathrm{gp}\,=\,1}^{\mathrm{GPE}} \left[\boldsymbol{J}^{-T}\cdot N^{i}_{,\boldsymbol{\xi}}\right]_b E_{abcd} \left[\boldsymbol{J}^{-T}\cdot N^{j}_{,\boldsymbol{\xi}}\right]_d \mathrm{Det}\boldsymbol{J}\,\alpha_{gp}$$

$$E_{abcd} = \frac{E}{2[1+\nu]}\left[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}\right] + \frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

plane strain

$$\boldsymbol{J} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\mathrm{NPE}} \\ y^1 & y^2 & \dots & y^{\mathrm{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\mathrm{NPE}}_\xi & N^{\mathrm{NPE}}_\eta \end{bmatrix}$$

$$\boldsymbol{x}^1 = [-1\,,\,-1]$$

$$\boldsymbol{x}^2 = [1\,,\,-1]$$

$$\boldsymbol{x}^3 = [1\,,\,1]$$

$$\boldsymbol{x}^4 = [-1\,,\,1]$$

$$\boldsymbol{K} = \begin{bmatrix}
\frac{E\,[\nu-3]}{8\,[\nu^2-1]} & -\frac{E}{8\,[\nu-1]} & \frac{E}{8\,[\nu-1]} & \frac{E\,[1-3\nu]}{8\,[\nu^2-1]} & -\frac{E\,[\nu-3]}{8\,[\nu^2-1]} & \frac{E}{8\,[\nu-1]} & -\frac{E}{8\,[\nu-1]} & -\frac{E\,[1-3\nu]}{8\,[\nu^2-1]} \\
-\frac{E}{8\,[\nu-1]} & \frac{E\,[\nu-3]}{8\,[\nu^2-1]} & -\frac{E\,[1-3\nu]}{8\,[\nu^2-1]} & -\frac{E}{8\,[\nu-1]} & \frac{E}{8\,[\nu-1]} & -\frac{E\,[\nu-3]}{8\,[\nu^2-1]} & \frac{E\,[1-3\nu]}{8\,[\nu^2-1]} & \frac{E}{8\,[\nu-1]} \\
\frac{E}{8\,[\nu-1]} & -\frac{E\,[1-3\nu]}{8\,[\nu^2-1]} & \frac{E\,[\nu-3]}{8\,[\nu^2-1]} & \frac{E}{8\,[\nu-1]} & -\frac{E}{8\,[\nu-1]} & \frac{E\,[1-3\nu]}{8\,[\nu^2-1]} & -\frac{E\,[\nu-3]}{8\,[\nu^2-1]} & -\frac{E}{8\,[\nu-1]} \\
\frac{E\,[1-3\nu]}{8\,[\nu^2-1]} & -\frac{E}{8\,[\nu-1]} & \frac{E}{8\,[\nu-1]} & \frac{E\,[\nu-3]}{8\,[\nu^2-1]} & -\frac{E\,[1-3\nu]}{8\,[\nu^2-1]} & \frac{E}{8\,[\nu-1]} & -\frac{E}{8\,[\nu-1]} & -\frac{E\,[\nu-3]}{8\,[\nu^2-1]} \\
-\frac{E\,[\nu-3]}{8\,[\nu^2-1]} & \frac{E}{8\,[\nu-1]} & -\frac{E}{8\,[\nu-1]} & -\frac{E\,[1-3\nu]}{8\,[\nu^2-1]} & \frac{E\,[\nu-3]}{8\,[\nu^2-1]} & -\frac{E}{8\,[\nu-1]} & \frac{E}{8\,[\nu-1]} & \frac{E\,[1-3\nu]}{8\,[\nu^2-1]} \\
\frac{E}{8\,[\nu-1]} & -\frac{E\,[\nu-3]}{8\,[\nu^2-1]} & \frac{E\,[1-3\nu]}{8\,[\nu^2-1]} & \frac{E}{8\,[\nu-1]} & -\frac{E}{8\,[\nu-1]} & \frac{E\,[\nu-3]}{8\,[\nu^2-1]} & -\frac{E\,[1-3\nu]}{8\,[\nu^2-1]} & -\frac{E}{8\,[\nu-1]} \\
-\frac{E}{8\,[\nu-1]} & \frac{E\,[1-3\nu]}{8\,[\nu^2-1]} & -\frac{E\,[\nu-3]}{8\,[\nu^2-1]} & -\frac{E}{8\,[\nu-1]} & \frac{E}{8\,[\nu-1]} & -\frac{E\,[1-3\nu]}{8\,[\nu^2-1]} & \frac{E\,[\nu-3]}{8\,[\nu^2-1]} & \frac{E}{8\,[\nu-1]} \\
-\frac{E\,[1-3\nu]}{8\,[\nu^2-1]} & \frac{E}{8\,[\nu-1]} & -\frac{E}{8\,[\nu-1]} & -\frac{E\,[\nu-3]}{8\,[\nu^2-1]} & \frac{E\,[1-3\nu]}{8\,[\nu^2-1]} & -\frac{E}{8\,[\nu-1]} & \frac{E}{8\,[\nu-1]} & \frac{E\,[\nu-3]}{8\,[\nu^2-1]}
\end{bmatrix}$$

Example (D2QU4N) … bilinear quadrilateral element

… using one Gauss point …

Quadrilateral Elements

$$K^{\,i\,j}_{\,a\,c} = \sum_{\text{gp}\,=\,1}^{\text{GPE}} \left[\boldsymbol{J}^{-T} \cdot N^{i}_{,\boldsymbol{\xi}}\right]_b E_{a\,b\,c\,d} \left[\boldsymbol{J}^{-T} \cdot N^{j}_{,\boldsymbol{\xi}}\right]_d \, \text{Det}\boldsymbol{J}\, \alpha_{gp}$$

$$E_{a\,b\,c\,d} = \frac{E}{2[1+\nu]}[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}] + \frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

plane strain

$$\boldsymbol{J} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\text{NPE}} \\ y^1 & y^2 & \dots & y^{\text{NPE}} \end{bmatrix} \begin{bmatrix} N^1_{\xi} & N^1_{\eta} \\ N^2_{\xi} & N^2_{\eta} \\ \vdots & \vdots \\ N^{\text{NPE}}_{\xi} & N^{\text{NPE}}_{\eta} \end{bmatrix}$$

$$\boldsymbol{K} = \begin{bmatrix} 3.750 & 1.250 & -1.250 & -1.250 & -3.750 & -1.250 & 1.250 & 1.250 \\ 1.250 & 3.750 & 1.250 & 1.250 & -1.250 & -3.750 & -1.250 & -1.250 \\ -1.250 & 1.250 & 3.750 & -1.250 & 1.250 & -1.250 & -3.750 & 1.250 \\ -1.250 & 1.250 & -1.250 & 3.750 & 1.250 & -1.250 & 1.250 & -3.750 \\ -3.750 & -1.250 & 1.250 & 1.250 & 3.750 & 1.250 & -1.250 & -1.250 \\ -1.250 & -3.750 & -1.250 & -1.250 & 1.250 & 3.750 & 1.250 & 1.250 \\ 1.250 & -1.250 & -3.750 & 1.250 & -1.250 & 1.250 & 3.750 & -1.250 \\ 1.250 & -1.250 & 1.250 & -3.750 & -1.250 & 1.250 & -1.250 & 3.750 \end{bmatrix}$$

$$\boldsymbol{x}^1 = [-1\,,\,-1]$$

$$\boldsymbol{x}^2 = [1\,,\,-1]$$

$$\boldsymbol{x}^3 = [1\,,\,1]$$

$$\boldsymbol{x}^4 = [-1\,,\,1]$$

$$E = 10\,,\ \nu = 0$$

Example (D2QU4N) … bilinear quadrilateral element

… using four Gauss points …

Quadrilateral Elements

$$K^{ij}_{ac} = \sum_{\text{gp}\,=\,1}^{\text{GPE}} \left[\boldsymbol{J}^{-T}\cdot N^{i}_{,\boldsymbol{\xi}}\right]_b E_{abcd} \left[\boldsymbol{J}^{-T}\cdot N^{j}_{,\boldsymbol{\xi}}\right]_d \operatorname{Det}\boldsymbol{J}\,\alpha_{gp}$$

$$E_{abcd} = \frac{E}{2[1+\nu]}[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}] + \frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

plane strain

$$\boldsymbol{K} = \begin{bmatrix} 5.000 & 1.250 & -2.500 & -1.250 & -2.500 & -1.250 & 0.000 & 1.250 \\ 1.250 & 5.000 & 1.250 & -0.000 & -1.250 & -2.500 & -1.250 & -2.500 \\ -2.500 & 1.250 & 5.000 & -1.250 & 0.000 & -1.250 & -2.500 & 1.250 \\ -1.250 & -0.000 & -1.250 & 5.000 & 1.250 & -2.500 & 1.250 & -2.500 \\ -2.500 & -1.250 & 0.000 & 1.250 & 5.000 & 1.250 & -2.500 & -1.250 \\ -1.250 & -2.500 & -1.250 & -2.500 & 1.250 & 5.000 & 1.250 & -0.000 \\ 0.000 & -1.250 & -2.500 & 1.250 & -2.500 & 1.250 & 5.000 & -1.250 \\ 1.250 & -2.500 & 1.250 & -2.500 & -1.250 & -0.000 & -1.250 & 5.000 \end{bmatrix}$$

$$\boldsymbol{J} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\text{NPE}} \\ y^1 & y^2 & \dots & y^{\text{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\text{NPE}}_\xi & N^{\text{NPE}}_\eta \end{bmatrix}$$

$$\boldsymbol{x}^1 = [\,-1\,,\,-1\,]$$
$$\boldsymbol{x}^2 = [\,1\,,\,-1\,]$$
$$\boldsymbol{x}^3 = [\,1\,,\,1\,]$$
$$\boldsymbol{x}^4 = [\,-1\,,\,1\,]$$

$$E = 10\,,\ \nu = 0$$

Example (D2QU4N) … bilinear quadrilateral element

… using nine Gauss points …

Quadrilateral Elements

$$K^{\,i\,j}_{\,a\,c} = \sum_{\mathrm{gp}\,=\,1}^{\mathrm{GPE}} \left[\boldsymbol{J}^{-T}\cdot N^{i}_{,\boldsymbol{\xi}}\right]_b E_{a\,b\,c\,d} \left[\boldsymbol{J}^{-T}\cdot N^{j}_{,\boldsymbol{\xi}}\right]_d \,\mathrm{Det}\boldsymbol{J}\,\alpha_{gp}$$

$$E_{a\,b\,c\,d} = \frac{E}{2[1+\nu]}[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}] + \frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

plane strain

$$\boldsymbol{J} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\mathrm{NPE}} \\ y^1 & y^2 & \dots & y^{\mathrm{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\mathrm{NPE}}_\xi & N^{\mathrm{NPE}}_\eta \end{bmatrix}$$

$$\boldsymbol{K} = \begin{bmatrix}
5.000 & 1.250 & -2.500 & -1.250 & -2.500 & -1.250 & -0.000 & 1.250 \\
1.250 & 5.000 & 1.250 & -0.000 & -1.250 & -2.500 & -1.250 & -2.500 \\
-2.500 & 1.250 & 5.000 & -1.250 & -0.000 & -1.250 & -2.500 & 1.250 \\
-1.250 & -0.000 & -1.250 & 5.000 & 1.250 & -2.500 & 1.250 & -2.500 \\
-2.500 & -1.250 & -0.000 & 1.250 & 5.000 & 1.250 & -2.500 & -1.250 \\
-1.250 & -2.500 & -1.250 & -2.500 & 1.250 & 5.000 & 1.250 & -0.000 \\
-0.000 & -1.250 & -2.500 & 1.250 & -2.500 & 1.250 & 5.000 & -1.250 \\
1.250 & -2.500 & 1.250 & -2.500 & -1.250 & -0.000 & -1.250 & 5.000
\end{bmatrix}$$

$$\boldsymbol{x}^1 = [-1\,,\,-1]$$
$$\boldsymbol{x}^2 = [1\,,\,-1]$$
$$\boldsymbol{x}^3 = [1\,,\,1]$$
$$\boldsymbol{x}^4 = [-1\,,\,1]$$

$$E = 10\,,\ \nu = 0$$

Example (D2QU4N) … bilinear quadrilateral element

… using one Gauss point …

Quadrilateral Elements

$$K^{ij}_{ac} = \sum_{\mathrm{gp}\,=\,1}^{\mathrm{GPE}} \left[\boldsymbol{J}^{-T}\cdot N^{i}_{,\boldsymbol{\xi}}\right]_b E_{abcd} \left[\boldsymbol{J}^{-T}\cdot N^{j}_{,\boldsymbol{\xi}}\right]_d \, \mathrm{Det}\boldsymbol{J}\, \alpha_{gp}$$

$$E_{abcd} = \frac{E}{2[1+\nu]}\left[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}\right] + \frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

plane strain

$$\boldsymbol{J} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\mathrm{NPE}} \\ y^1 & y^2 & \dots & y^{\mathrm{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\mathrm{NPE}}_\xi & N^{\mathrm{NPE}}_\eta \end{bmatrix}$$

$$\boldsymbol{K} = \begin{bmatrix} 3.667 & 1.667 & -1.667 & -0.333 & -3.667 & -1.667 & 1.667 & 0.333 \\ 1.667 & 3.667 & 0.333 & 1.667 & -1.667 & -3.667 & -0.333 & -1.667 \\ -1.667 & 0.333 & 3.667 & -1.667 & 1.667 & -0.333 & -3.667 & 1.667 \\ -0.333 & 1.667 & -1.667 & 3.667 & 0.333 & -1.667 & 1.667 & -3.667 \\ -3.667 & -1.667 & 1.667 & 0.333 & 3.667 & 1.667 & -1.667 & -0.333 \\ -1.667 & -3.667 & -0.333 & -1.667 & 1.667 & 3.667 & 0.333 & 1.667 \\ 1.667 & -0.333 & -3.667 & 1.667 & -1.667 & 0.333 & 3.667 & -1.667 \\ 0.333 & -1.667 & 1.667 & -3.667 & -0.333 & 1.667 & -1.667 & 3.667 \end{bmatrix}$$

$$\boldsymbol{x}^1 = [-1\,,\,-1]$$

$$\boldsymbol{x}^2 = [1\,,\,-1]$$

$$\boldsymbol{x}^3 = [1\,,\,1]$$

$$\boldsymbol{x}^4 = [-1\,,\,1]$$

$$E = 10\,,\ \nu = 0.25$$

Example (D2QU4N) … bilinear quadrilateral element

… using four Gauss points …

Quadrilateral Elements

$$K^{ij}_{ac} = \sum_{\text{gp}=1}^{\text{GPE}} \left[\boldsymbol{J}^{-T}\cdot N^i_{,\boldsymbol{\xi}}\right]_b E_{abcd} \left[\boldsymbol{J}^{-T}\cdot N^j_{,\boldsymbol{\xi}}\right]_d \,\text{Det}\boldsymbol{J}\,\alpha_{gp}$$

$$E_{abcd} = \frac{E}{2[1+\nu]}\left[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}\right] + \frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

plane strain

$$\boldsymbol{J} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\text{NPE}} \\ y^1 & y^2 & \dots & y^{\text{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\text{NPE}}_\xi & N^{\text{NPE}}_\eta \end{bmatrix}$$

$$\boldsymbol{K} = \begin{bmatrix} 4.889 & 1.667 & -2.889 & -0.333 & -2.444 & -1.667 & 0.444 & 0.333 \\ 1.667 & 4.889 & 0.333 & 0.444 & -1.667 & -2.444 & -0.333 & -2.889 \\ -2.889 & 0.333 & 4.889 & -1.667 & 0.444 & -0.333 & -2.444 & 1.667 \\ -0.333 & 0.444 & -1.667 & 4.889 & 0.333 & -2.889 & 1.667 & -2.444 \\ -2.444 & -1.667 & 0.444 & 0.333 & 4.889 & 1.667 & -2.889 & -0.333 \\ -1.667 & -2.444 & -0.333 & -2.889 & 1.667 & 4.889 & 0.333 & 0.444 \\ 0.444 & -0.333 & -2.444 & 1.667 & -2.889 & 0.333 & 4.889 & -1.667 \\ 0.333 & -2.889 & 1.667 & -2.444 & -0.333 & 0.444 & -1.667 & 4.889 \end{bmatrix}$$

$$\boldsymbol{x}^1 = [-1\,,\,-1]$$
$$\boldsymbol{x}^2 = [1\,,\,-1]$$
$$\boldsymbol{x}^3 = [1\,,\,1]$$
$$\boldsymbol{x}^4 = [-1\,,\,1]$$

$$E = 10\,,\ \nu = 0.25$$

Example (D2QU4N) … bilinear quadrilateral element

… using nine Gauss points …

Quadrilateral Elements

$$K^{ij}_{ac} = \sum_{\mathrm{gp}=1}^{\mathrm{GPE}} \left[\boldsymbol{J}^{-T}\cdot N^{i}_{,\boldsymbol{\xi}}\right]_b E_{abcd} \left[\boldsymbol{J}^{-T}\cdot N^{j}_{,\boldsymbol{\xi}}\right]_d \,\mathrm{Det}\boldsymbol{J}\,\alpha_{gp}$$

$$E_{abcd} = \frac{E}{2[1+\nu]}[\delta_{ad}\,\delta_{bc}+\delta_{ac}\,\delta_{bd}] + \frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

plane strain

$$\boldsymbol{K} = \begin{bmatrix} 4.889 & 1.667 & -2.889 & -0.333 & -2.444 & -1.667 & 0.444 & 0.333 \\ 1.667 & 4.889 & 0.333 & 0.444 & -1.667 & -2.444 & -0.333 & -2.889 \\ -2.889 & 0.333 & 4.889 & -1.667 & 0.444 & -0.333 & -2.444 & 1.667 \\ -0.333 & 0.444 & -1.667 & 4.889 & 0.333 & -2.889 & 1.667 & -2.444 \\ -2.444 & -1.667 & 0.444 & 0.333 & 4.889 & 1.667 & -2.889 & -0.333 \\ -1.667 & -2.444 & -0.333 & -2.889 & 1.667 & 4.889 & 0.333 & 0.444 \\ 0.444 & -0.333 & -2.444 & 1.667 & -2.889 & 0.333 & 4.889 & -1.667 \\ 0.333 & -2.889 & 1.667 & -2.444 & -0.333 & 0.444 & -1.667 & 4.889 \end{bmatrix}$$

$$\boldsymbol{J} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\mathrm{NPE}} \\ y^1 & y^2 & \dots & y^{\mathrm{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\mathrm{NPE}}_\xi & N^{\mathrm{NPE}}_\eta \end{bmatrix}$$

$$\boldsymbol{x}^1 = [-1\,,\,-1]$$

$$\boldsymbol{x}^2 = [1\,,\,-1]$$

$$\boldsymbol{x}^3 = [1\,,\,1]$$

$$\boldsymbol{x}^4 = [-1\,,\,1]$$

$$E = 10\,,\ \nu = 0.25$$

Example (D2QU4N) … bilinear quadrilateral element

… using one Gauss point …

Quadrilateral Elements

$$K^{ij}_{ac} = \sum_{\text{gp}=1}^{\text{GPE}} \left[\boldsymbol{J}^{-T} \cdot N^{i}_{,\boldsymbol{\xi}}\right]_b E_{abcd} \left[\boldsymbol{J}^{-T} \cdot N^{j}_{,\boldsymbol{\xi}}\right]_d \operatorname{Det}\boldsymbol{J}\, \alpha_{gp}$$

$$E_{abcd} = \frac{E}{2[1+\nu]}[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}] + \frac{E\,\nu}{1-\nu^2}\delta_{ab}\,\delta_{cd}$$

plane strain

$$\boldsymbol{J} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\text{NPE}} \\ y^1 & y^2 & \dots & y^{\text{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\text{NPE}}_\xi & N^{\text{NPE}}_\eta \end{bmatrix}$$

$$\boldsymbol{K} = \begin{bmatrix} 4.167 & 2.500 & -2.500 & 0.833 & -4.167 & -2.500 & 2.500 & -0.833 \\ 2.500 & 4.167 & -0.833 & 2.500 & -2.500 & -4.167 & 0.833 & -2.500 \\ -2.500 & -0.833 & 4.167 & -2.500 & 2.500 & 0.833 & -4.167 & 2.500 \\ 0.833 & 2.500 & -2.500 & 4.167 & -0.833 & -2.500 & 2.500 & -4.167 \\ -4.167 & -2.500 & 2.500 & -0.833 & 4.167 & 2.500 & -2.500 & 0.833 \\ -2.500 & -4.167 & 0.833 & -2.500 & 2.500 & 4.167 & -0.833 & 2.500 \\ 2.500 & 0.833 & -4.167 & 2.500 & -2.500 & -0.833 & 4.167 & -2.500 \\ -0.833 & -2.500 & 2.500 & -4.167 & 0.833 & 2.500 & -2.500 & 4.167 \end{bmatrix}$$

$$\boldsymbol{x}^1 = [-1, -1]$$
$$\boldsymbol{x}^2 = [1, -1]$$
$$\boldsymbol{x}^3 = [1, 1]$$
$$\boldsymbol{x}^4 = [-1, 1]$$

$$E = 10\,,\ \nu = 0.5$$

Example (D2QU4N) … bilinear quadrilateral element

… using four Gauss points …

Quadrilateral Elements

$$K^{\,i\,j}_{\,a\,c} = \sum_{\text{gp}\,=\,1}^{\text{GPE}} \left[\boldsymbol{J}^{-T}\cdot N^{i}_{,\boldsymbol{\xi}}\right]_b E_{a\,b\,c\,d} \left[\boldsymbol{J}^{-T}\cdot N^{j}_{,\boldsymbol{\xi}}\right]_d \,\text{Det}\boldsymbol{J}\,\alpha_{gp}$$

$$E_{a\,b\,c\,d} = \frac{E}{2[1+\nu]}[\delta_{ad}\,\delta_{bc}+\delta_{ac}\,\delta_{bd}] + \frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

plane strain

$$\boldsymbol{K} = \begin{bmatrix} 5.556 & 2.500 & -3.889 & 0.833 & -2.778 & -2.500 & 1.111 & -0.833 \\ 2.500 & 5.556 & -0.833 & 1.111 & -2.500 & -2.778 & 0.833 & -3.889 \\ -3.889 & -0.833 & 5.556 & -2.500 & 1.111 & 0.833 & -2.778 & 2.500 \\ 0.833 & 1.111 & -2.500 & 5.556 & -0.833 & -3.889 & 2.500 & -2.778 \\ -2.778 & -2.500 & 1.111 & -0.833 & 5.556 & 2.500 & -3.889 & 0.833 \\ -2.500 & -2.778 & 0.833 & -3.889 & 2.500 & 5.556 & -0.833 & 1.111 \\ 1.111 & 0.833 & -2.778 & 2.500 & -3.889 & -0.833 & 5.556 & -2.500 \\ -0.833 & -3.889 & 2.500 & -2.778 & 0.833 & 1.111 & -2.500 & 5.556 \end{bmatrix}$$

$$\boldsymbol{J} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\text{NPE}} \\ y^1 & y^2 & \dots & y^{\text{NPE}} \end{bmatrix} \begin{bmatrix} N^1_{\xi} & N^1_{\eta} \\ N^2_{\xi} & N^2_{\eta} \\ \vdots & \vdots \\ N^{\text{NPE}}_{\xi} & N^{\text{NPE}}_{\eta} \end{bmatrix}$$

$$\boldsymbol{x}^1 = [-1\,,\,-1]$$
$$\boldsymbol{x}^2 = [1\,,\,-1]$$
$$\boldsymbol{x}^3 = [1\,,\,1]$$
$$\boldsymbol{x}^4 = [-1\,,\,1]$$

$$E = 10\,,\ \nu = 0.5$$

Example (D2QU4N) … bilinear quadrilateral element

… using nine Gauss points …

Quadrilateral Elements

$$K^{\,i\,j}_{\,a\,c} = \sum_{\mathrm{gp}\,=\,1}^{\mathrm{GPE}} \left[\boldsymbol{J}^{-T}\cdot N^{i}_{,\boldsymbol{\xi}}\right]_b E_{a\,b\,c\,d}\left[\boldsymbol{J}^{-T}\cdot N^{j}_{,\boldsymbol{\xi}}\right]_d \,\mathrm{Det}\boldsymbol{J}\,\alpha_{gp}$$

$$E_{a\,b\,c\,d} = \frac{E}{2[1+\nu]}[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}] + \frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

plane strain

$$\boldsymbol{K} = \begin{bmatrix} 5.556 & 2.500 & -3.889 & 0.833 & -2.778 & -2.500 & 1.111 & -0.833 \\ 2.500 & 5.556 & -0.833 & 1.111 & -2.500 & -2.778 & 0.833 & -3.889 \\ -3.889 & -0.833 & 5.556 & -2.500 & 1.111 & 0.833 & -2.778 & 2.500 \\ 0.833 & 1.111 & -2.500 & 5.556 & -0.833 & -3.889 & 2.500 & -2.778 \\ -2.778 & -2.500 & 1.111 & -0.833 & 5.556 & 2.500 & -3.889 & 0.833 \\ -2.500 & -2.778 & 0.833 & -3.889 & 2.500 & 5.556 & -0.833 & 1.111 \\ 1.111 & 0.833 & -2.778 & 2.500 & -3.889 & -0.833 & 5.556 & -2.500 \\ -0.833 & -3.889 & 2.500 & -2.778 & 0.833 & 1.111 & -2.500 & 5.556 \end{bmatrix}$$

$$\boldsymbol{J} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\mathrm{NPE}} \\ y^1 & y^2 & \dots & y^{\mathrm{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\mathrm{NPE}}_\xi & N^{\mathrm{NPE}}_\eta \end{bmatrix}$$

$$\boldsymbol{x}^1 = [-1\,,\,-1]$$
$$\boldsymbol{x}^2 = [1\,,\,-1]$$
$$\boldsymbol{x}^3 = [1\,,\,1]$$
$$\boldsymbol{x}^4 = [-1\,,\,1]$$

$$E = 10\,,\ \nu = 0.5$$

Example (D2QU4N) … bilinear quadrilateral element

… using one Gauss point …

Quadrilateral Elements

$$K^{ij}_{ac} = \sum_{\mathrm{gp}=1}^{\mathrm{GPE}} \left[\boldsymbol{J}^{-T} \cdot N^{i}_{,\boldsymbol{\xi}}\right]_b E_{abcd} \left[\boldsymbol{J}^{-T} \cdot N^{j}_{,\boldsymbol{\xi}}\right]_d \mathrm{Det}\boldsymbol{J}\, \alpha_{gp}$$

$$E_{abcd} = \frac{E}{2[1+\nu]}[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}]$$

plane strain $$+\frac{E\,\nu}{1-\nu^2}\delta_{ab}\,\delta_{cd}$$

$$\boldsymbol{J} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\mathrm{NPE}} \\ y^1 & y^2 & \dots & y^{\mathrm{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\mathrm{NPE}}_\xi & N^{\mathrm{NPE}}_\eta \end{bmatrix}$$

$$\boldsymbol{K} = \begin{bmatrix} 6.429 & 5.000 & -5.000 & 3.571 & -6.429 & -5.000 & 5.000 & -3.571 \\ 5.000 & 6.429 & -3.571 & 5.000 & -5.000 & -6.429 & 3.571 & -5.000 \\ -5.000 & -3.571 & 6.429 & -5.000 & 5.000 & 3.571 & -6.429 & 5.000 \\ 3.571 & 5.000 & -5.000 & 6.429 & -3.571 & -5.000 & 5.000 & -6.429 \\ -6.429 & -5.000 & 5.000 & -3.571 & 6.429 & 5.000 & -5.000 & 3.571 \\ -5.000 & -6.429 & 3.571 & -5.000 & 5.000 & 6.429 & -3.571 & 5.000 \\ 5.000 & 3.571 & -6.429 & 5.000 & -5.000 & -3.571 & 6.429 & -5.000 \\ -3.571 & -5.000 & 5.000 & -6.429 & 3.571 & 5.000 & -5.000 & 6.429 \end{bmatrix}$$

$$\boldsymbol{x}^1 = [-1\,,\,-1]$$
$$\boldsymbol{x}^2 = [1\,,\,-1]$$
$$\boldsymbol{x}^3 = [1\,,\,1]$$
$$\boldsymbol{x}^4 = [-1\,,\,1]$$

$$E = 10\,,\ \nu = 0.75$$

Example (D2QU4N) … bilinear quadrilateral element

… using four Gauss points …

Quadrilateral Elements

$$K^{ij}_{ac} = \sum_{\text{gp}=1}^{\text{GPE}} \left[\boldsymbol{J}^{-T} \cdot N^i_{,\boldsymbol{\xi}}\right]_b E_{abcd} \left[\boldsymbol{J}^{-T} \cdot N^j_{,\boldsymbol{\xi}}\right]_d \operatorname{Det}\boldsymbol{J}\, \alpha_{gp}$$

$$E_{abcd} = \frac{E}{2[1+\nu]}[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}]$$

plane strain

$$+\frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

$$\boldsymbol{J} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\text{NPE}} \\ y^1 & y^2 & \dots & y^{\text{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\text{NPE}}_\xi & N^{\text{NPE}}_\eta \end{bmatrix}$$

$$\boldsymbol{K} = \begin{bmatrix} 8.571 & 5.000 & -7.143 & 3.571 & -4.286 & -5.000 & 2.857 & -3.571 \\ 5.000 & 8.571 & -3.571 & 2.857 & -5.000 & -4.286 & 3.571 & -7.143 \\ -7.143 & -3.571 & 8.571 & -5.000 & 2.857 & 3.571 & -4.286 & 5.000 \\ 3.571 & 2.857 & -5.000 & 8.571 & -3.571 & -7.143 & 5.000 & -4.286 \\ -4.286 & -5.000 & 2.857 & -3.571 & 8.571 & 5.000 & -7.143 & 3.571 \\ -5.000 & -4.286 & 3.571 & -7.143 & 5.000 & 8.571 & -3.571 & 2.857 \\ 2.857 & 3.571 & -4.286 & 5.000 & -7.143 & -3.571 & 8.571 & -5.000 \\ -3.571 & -7.143 & 5.000 & -4.286 & 3.571 & 2.857 & -5.000 & 8.571 \end{bmatrix}$$

$$\boldsymbol{x}^1 = [-1\,,\,-1]$$

$$\boldsymbol{x}^2 = [1\,,\,-1]$$

$$\boldsymbol{x}^3 = [1\,,\,1]$$

$$\boldsymbol{x}^4 = [-1\,,\,1]$$

$$E = 10\,,\ \nu = 0.75$$

Example (D2QU4N) … bilinear quadrilateral element

… using nine Gauss points …

Quadrilateral Elements

$$K^{ij}_{ac} = \sum_{\mathrm{gp}=1}^{\mathrm{GPE}} \left[\boldsymbol{J}^{-T}\cdot N^{i}_{,\boldsymbol{\xi}}\right]_b E_{abcd} \left[\boldsymbol{J}^{-T}\cdot N^{j}_{,\boldsymbol{\xi}}\right]_d \mathrm{Det}\boldsymbol{J}\,\alpha_{gp}$$

$$E_{abcd} = \frac{E}{2[1+\nu]}[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}] + \frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

plane strain

$$\boldsymbol{K} = \begin{bmatrix} 8.571 & 5.000 & -7.143 & 3.571 & -4.286 & -5.000 & 2.857 & -3.571 \\ 5.000 & 8.571 & -3.571 & 2.857 & -5.000 & -4.286 & 3.571 & -7.143 \\ -7.143 & -3.571 & 8.571 & -5.000 & 2.857 & 3.571 & -4.286 & 5.000 \\ 3.571 & 2.857 & -5.000 & 8.571 & -3.571 & -7.143 & 5.000 & -4.286 \\ -4.286 & -5.000 & 2.857 & -3.571 & 8.571 & 5.000 & -7.143 & 3.571 \\ -5.000 & -4.286 & 3.571 & -7.143 & 5.000 & 8.571 & -3.571 & 2.857 \\ 2.857 & 3.571 & -4.286 & 5.000 & -7.143 & -3.571 & 8.571 & -5.000 \\ -3.571 & -7.143 & 5.000 & -4.286 & 3.571 & 2.857 & -5.000 & 8.571 \end{bmatrix}$$

$$\boldsymbol{J} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\mathrm{NPE}} \\ y^1 & y^2 & \dots & y^{\mathrm{NPE}} \end{bmatrix} \begin{bmatrix} N^1_{\xi} & N^1_{\eta} \\ N^2_{\xi} & N^2_{\eta} \\ \vdots & \vdots \\ N^{\mathrm{NPE}}_{\xi} & N^{\mathrm{NPE}}_{\eta} \end{bmatrix}$$

$$\boldsymbol{x}^1 = [-1\,,\,-1]$$
$$\boldsymbol{x}^2 = [1\,,\,-1]$$
$$\boldsymbol{x}^3 = [1\,,\,1]$$
$$\boldsymbol{x}^4 = [-1\,,\,1]$$

$$E = 10\,,\;\nu = 0.75$$