FINITE ELEMENT METHOD

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7

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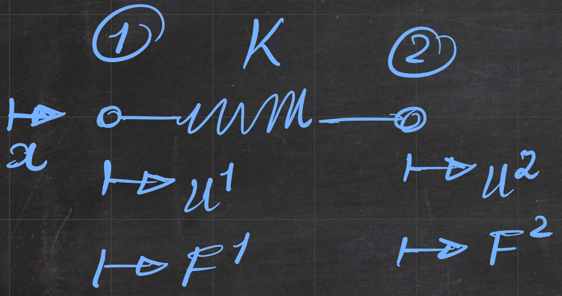
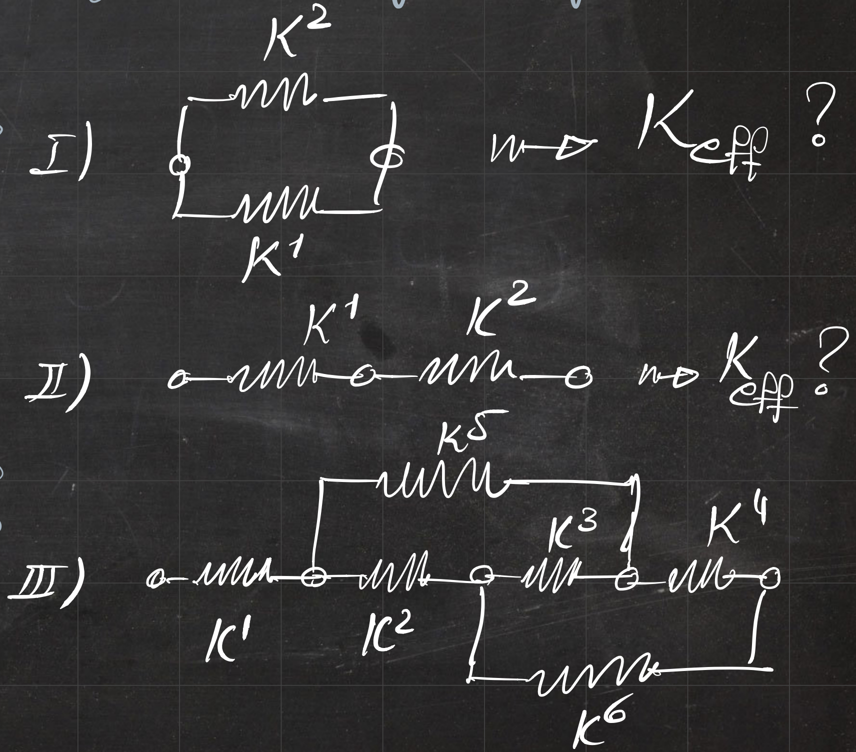
Understanding key ingredients of FEM using springs:

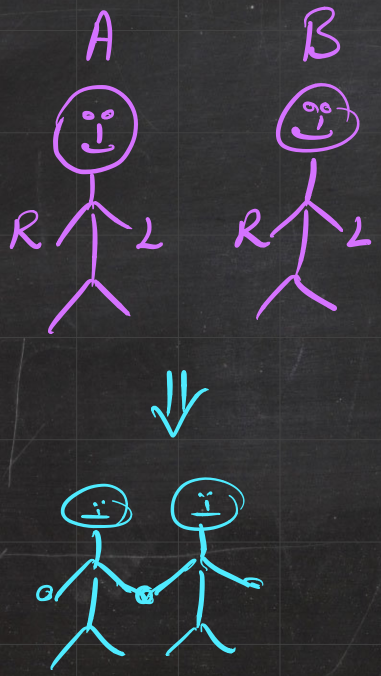


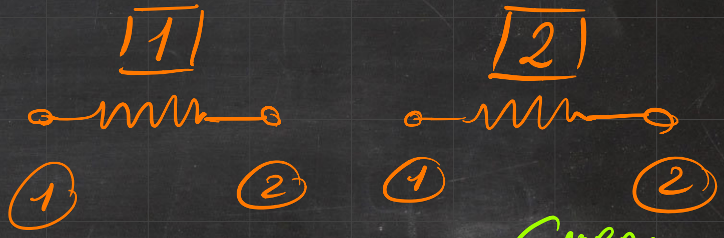
$$\begin{bmatrix}F^1\\F^2\end{bmatrix} = \begin{bmatrix}K & -K\\ -K & K\end{bmatrix}\begin{bmatrix}u^1\\u^2\end{bmatrix}$$

$$\mathbb{F} = \mathbb{K}\cdot\mathbb{U}$$



TOWARDS AN ALGORITHMIC APPROACH TO ASSEMBLY:





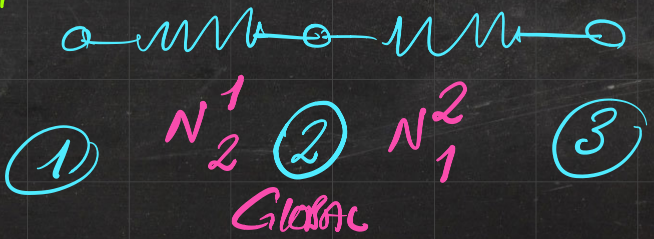
GLOBAL ELEMENT

Superscript: GLOBAL

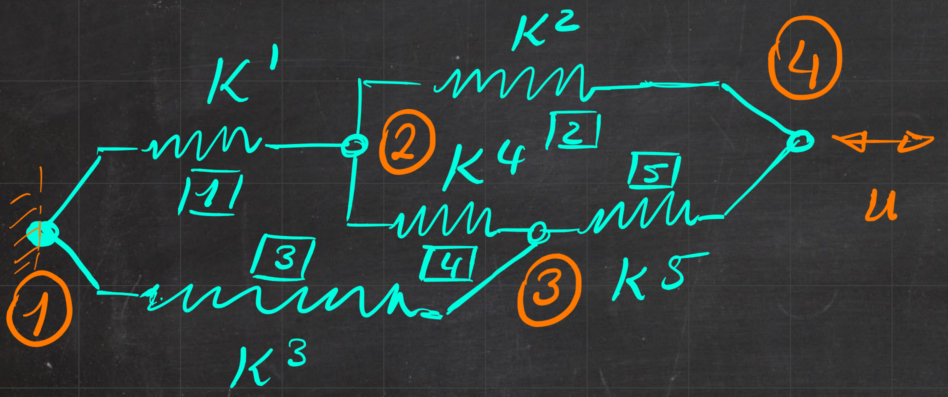
subscript: LOCAL

GLOBAL NODE

$$N^{2} = N_{2}^{1} = N_{1}^{2}$$



TOWARDS AN ALGORITHMIC APPROACH TO ASSEMBLY:



ELEMENT 5

$$\mathbb{K}^5 = \begin{bmatrix} K^5 & -K^5 \\ -K^5 & K^5 \end{bmatrix}$$

ELEMENT 1

$$\mathbb{K}^1 = \begin{bmatrix} K^1 & -K^1 \\ -K^1 & K^1 \end{bmatrix}$$

BOLD

ELEMENT 2

$$\mathbb{K}^2 = \begin{bmatrix} K^2 & -K^2 \\ -K^2 & K^2 \end{bmatrix}$$

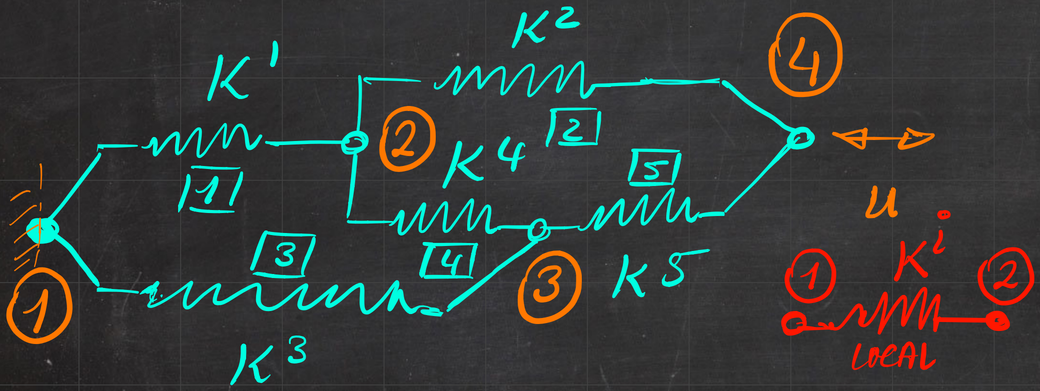
ELEMENT 3

$$\mathbb{K}^3 = \begin{bmatrix} K^3 & -K^3 \\ -K^3 & K^3 \end{bmatrix}$$

ELEMENT 4

$$\mathbb{K}^4 = \begin{bmatrix} K^4 & -K^4 \\ -K^4 & K^4 \end{bmatrix}$$

Towards an algorithmic approach to assembly:



$$\mathbb{K}^5 = \begin{array}{cc|c} ③ & ④ & \\ \hline \end{array}\quad
\begin{array}{c} \\ \end{array}$$

$$\mathbb{K}^5 = \begin{array}{c@{}c} & \begin{array}{cc} ③ & ④ \end{array} \\ & \left[\begin{array}{cc} K^5 & -K^5 \\ -K^5 & K^5 \end{array}\right] \begin{array}{c} ③ \\ ④ \end{array} \end{array}$$

$$\mathbb{K}^1 = \begin{array}{c} \begin{array}{cc} ① & ② \end{array} \\ \left[\begin{array}{cc} K^1 & -K^1 \\ -K^1 & K^1 \end{array}\right] \end{array} \begin{array}{c} ① \\ ② \end{array} \quad
\mathbb{K}^2 = \begin{array}{c} \begin{array}{cc} ② & ④ \end{array} \\ \left[\begin{array}{cc} K^2 & -K^2 \\ -K^2 & K^2 \end{array}\right] \end{array} \begin{array}{c} ② \\ ④ \end{array}$$

$$\mathbb{K}^3 = \begin{array}{c} \begin{array}{cc} ① & ③ \end{array} \\ \left[\begin{array}{cc} K^3 & -K^3 \\ -K^3 & K^3 \end{array}\right] \end{array} \begin{array}{c} ① \\ ③ \end{array} \quad
\mathbb{K}^4 = \begin{array}{c} \begin{array}{cc} ② & ③ \end{array} \\ \left[\begin{array}{cc} K^4 & -K^4 \\ -K^4 & K^4 \end{array}\right] \end{array} \begin{array}{c} ② \\ ③ \end{array}$$

$$\mathbb{K}^4 = \begin{array}{c} \\ \\ \end{array}\begin{array}{cc|c} ② & ③ & \\ \end{array}$$

$$\mathbb{K}^4 = \begin{array}{cc} ② & ③ \end{array} \qquad \begin{bmatrix} K^4 & -K^4 \\ -K^4 & K^4 \end{bmatrix} \begin{array}{c} ② \\ ③ \end{array}$$

$$\mathbb{K}^5 = \begin{array}{cc} ③ & ④ \end{array} \qquad \begin{bmatrix} K^5 & -K^5 \\ -K^5 & K^5 \end{bmatrix} \begin{array}{c} ③ \\ ④ \end{array}$$

GLOBAL

$$\mathbb{K} = \begin{array}{c|cccc} & ① & ② & ③ & ④ \\ \hline ① & K^1 + K^3 & -K^1 & -K^3 & 0 \\ ② & -K^1 & K^1 + K^2 + K^4 & -K^4 & -K^2 \\ ③ & -K^3 & -K^4 & K^3 + K^4 + K^5 & -K^5 \\ ④ & 0 & -K^2 & -K^5 & K^2 + K^5 \end{array}$$

$$\text{DET } \mathbb{K}^{\text{GLOBAL}} = 0 \qquad \mathbb{K}^{\text{GLOBAL}} : \text{SYM}$$

$$\mathbb{K}^1 = \begin{array}{cc} ① & ② \end{array} \qquad \begin{bmatrix} K^1 & -K^1 \\ -K^1 & K^1 \end{bmatrix} \begin{array}{c} ① \\ ② \end{array}$$

$$\mathbb{K}^2 = \begin{array}{cc} ② & ④ \end{array} \qquad \begin{bmatrix} K^2 & -K^2 \\ -K^2 & K^2 \end{bmatrix} \begin{array}{c} ② \\ ④ \end{array}$$

$$\mathbb{K}^3 = \begin{array}{cc} ① & ③ \end{array} \qquad \begin{bmatrix} K^3 & -K^3 \\ -K^3 & K^3 \end{bmatrix} \begin{array}{c} ① \\ ③ \end{array}$$

$$\begin{bmatrix} F^1 \\ F^2 \\ F^3 \\ F^4 \end{bmatrix} = \begin{bmatrix} K^{11} & K^{12} & K^{13} & K^{14} \\ K^{21} & K^{22} & K^{23} & K^{24} \\ K^{31} & K^{32} & K^{33} & K^{34} \\ K^{41} & K^{42} & K^{43} & K^{44} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \\ u^4 \end{bmatrix}$$

GLOBAL GLOBAL GLOBAL

$$\mathbb{F} = \mathbb{K} \cdot \mathbb{U}$$

NoN x 1 — NoN x NoN — NoN x 1

[PD x NoN] x 1

NoN ≡ NoN x PD

1D Problem Dimension

⇒ 4 EQ. & 4 Unknowns ← BCs ?

NEUMANN BCs. ⇐ FORCE BASED ← BCs → DISPLACEMENT BASED ⇒ DIRICHLET BCs.

$$\begin{bmatrix} F^1 \\ F^2 \\ F^3 \\ F^4 \end{bmatrix} = \begin{bmatrix} K^{11} & K^{12} & K^{13} & K^{14} \\ K^{21} & K^{22} & K^{23} & K^{24} \\ K^{31} & K^{32} & K^{33} & K^{34} \\ K^{41} & K^{42} & K^{43} & K^{44} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \\ u^4 \end{bmatrix}$$

$\to u^1 = 0$

$\to u^4 = u$

	Homogeneous	Non Homogeneous
DIRICHLET $\hookrightarrow$ Displacement	$=0$	$\neq 0$
NEUMANN $\hookrightarrow$ Force	$=0$	$\neq 0$

$$= \begin{bmatrix} K^{11} \\ K^{21} \\ K^{31} \\ K^{41} \end{bmatrix} u^1 + \begin{bmatrix} K^{12} \\ K^{22} \\ K^{32} \\ K^{42} \end{bmatrix} u^2 + \begin{bmatrix} K^{13} \\ K^{23} \\ K^{33} \\ K^{43} \end{bmatrix} u^3 + \begin{bmatrix} K^{14} \\ K^{24} \\ K^{34} \\ K^{44} \end{bmatrix} u^4$$

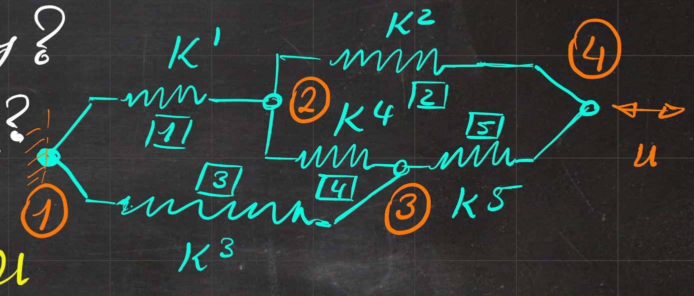
4 EQN. & 4 Unknowns

NEUMANN

NEUMANN

$$\begin{bmatrix} \cancel{F^1} \\ \cancel{F^2} \\ \cancel{F^3} \\ \cancel{F^4} \end{bmatrix} \begin{matrix} ? \\ 0 \\ 0 \\ ? \end{matrix} \quad \begin{bmatrix} K^{11} & K^{12} & K^{13} & K^{14} \\ K^{21} & K^{22} & K^{23} & K^{24} \\ K^{31} & K^{32} & K^{33} & K^{34} \\ K^{41} & K^{42} & K^{43} & K^{44} \end{bmatrix} \begin{bmatrix} \cancel{u^1} \\ \cancel{u^2} \\ \cancel{u^3} \\ \cancel{u^4} \end{bmatrix} \begin{matrix} 0 \\ ? \\ ? \\ u \end{matrix}$$

DIRICHLET (Prescribed Displacement)



DIRICHLET (Prescribed Displ.)

$$\overset{\checkmark}{[\mathbb{b}]} = [\mathbb{A}] \overset{?}{[\mathbb{X}]}$$

$$\mathbb{A} \cdot \mathbb{X} = \mathbb{b} \Rightarrow \mathbb{X} = \mathbb{A}^{-1} \cdot \mathbb{b}$$

4 EQN. & 4 Unknowns

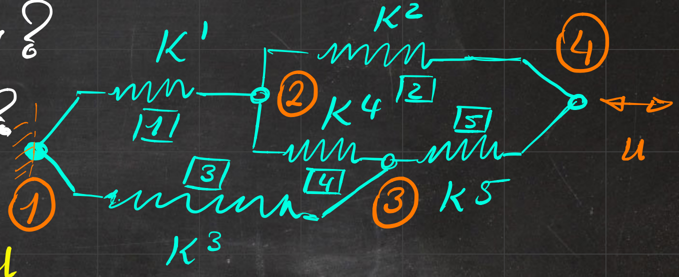
$$\begin{bmatrix} \cancel{F^1} \\ \cancel{F^2} \\ \cancel{F^3} \\ \cancel{F^4} \end{bmatrix} = \begin{bmatrix} K^{11} & K^{12} & K^{13} & K^{14} \\ K^{21} & K^{22} & K^{23} & K^{24} \\ K^{31} & K^{32} & K^{33} & K^{34} \\ K^{41} & K^{42} & K^{43} & K^{44} \end{bmatrix} \begin{bmatrix} \cancel{u^1} \\ \cancel{u^2} \\ \cancel{u^3} \\ \cancel{u^4} \end{bmatrix}$$

$F^1 \to ?$ $\qquad$ $u^1 \to 0$ — DIRICHLET (Prescribed Displacement)

$F^2 \to 0$ — NEUMANN $\qquad$ $u^2 \to ?$

$F^3 \to 0$ — NEUMANN $\qquad$ $u^3 \to ?$

$F^4 \to ?$ $\qquad$ $u^4 \to u$ — DIRICHLET (Prescribed Displ.)



$$\begin{bmatrix} F^P \\ \hline F^u \end{bmatrix} = \left[\begin{array}{c|c} K^{Pu} & K^{PP} \\ \hline K^{uu} & K^{uP} \end{array}\right] \begin{bmatrix} u^u \\ \hline u^P \end{bmatrix}$$

NEUMANN

FREE NODES $\Rightarrow$ DoF — DEGREES OF FREEDOM

CONSTRAINED NODES $\Rightarrow$ DoC — DEGREES OF CONSTRAINT

DIRICHLET

4 EQN. & 4 Unknowns

$$\begin{bmatrix} \cancel{F^1} \\ \cancel{F^2} \\ \cancel{F^3} \\ \cancel{F^4} \end{bmatrix} \begin{matrix} ? \\ 0 \\ 0 \\ ? \end{matrix} = \begin{bmatrix} K^{11} & K^{12} & K^{13} & K^{14} \\ K^{21} & K^{22} & K^{23} & K^{24} \\ K^{31} & K^{32} & K^{33} & K^{34} \\ K^{41} & K^{42} & K^{43} & K^{44} \end{bmatrix} \begin{bmatrix} \cancel{u^1} \\ \cancel{u^2} \\ \cancel{u^3} \\ \cancel{u^4} \end{bmatrix} \begin{matrix} 0 \\ ? \\ ? \\ u \end{matrix}$$

NEUMANN

NEUMANN

$$[F^P] = [K^{Pu}][u^u] + [K^{PP}][u^P]$$

$$\Downarrow$$

$$\underbrace{[K^{Pu}]}_{A}\underbrace{[u^u]}_{x} = \underbrace{[F^P] - [K^{PP}][u^P]}_{b}$$

$$[x] = [A]^{-1}[b] \Leftarrow A \cdot x = b$$

$$\begin{bmatrix} F^P \\ \hline F^u \end{bmatrix} = \begin{bmatrix} K^{Pu} & K^{PP} \\ \hline K^{uu} & K^{uP} \end{bmatrix} \begin{bmatrix} u^u \\ \hline u^P \end{bmatrix}$$

$$\begin{bmatrix} DoF \\ \hline DoC \end{bmatrix} = \begin{bmatrix} DoF \times DoF & DoF \times DoC \\ \hline DoC \times DoF & DoC \times DoC \end{bmatrix} \begin{bmatrix} DoF \\ \hline DoC \end{bmatrix}$$

FORCE — STIFFNESS — DISPLACEMENT

4 EQN. & 4 Unknowns

$$\begin{bmatrix} \cancel{F^1} \; ? \\ \cancel{F^2} \; 0 \\ \cancel{F^3} \; 0 \\ \cancel{F^4} \; ? \end{bmatrix} = \begin{bmatrix} K^{11} & K^{12} & K^{13} & K^{14} \\ K^{21} & K^{22} & K^{23} & K^{24} \\ K^{31} & K^{32} & K^{33} & K^{34} \\ K^{41} & K^{42} & K^{43} & K^{44} \end{bmatrix} \begin{bmatrix} \cancel{u^1} \; 0 \\ \cancel{u^2} \; ? \\ \cancel{u^3} \; ? \\ \cancel{u^4} \; u \end{bmatrix}$$

NEUMANN

NEUMANN

$$[F^P] = [K^{Pu}][u^u] + [K^{PP}][u^P]$$

$$\Downarrow$$

$$\underbrace{[K^{Pu}]}_{\text{REDUCED STIFFNESS}}[u^u] = [F^P] - [K^{PP}][u^P]$$

$$\Rightarrow [u^u] = [K^{Pu}]^{-1} \cdot \underbrace{\{[F^P] - [K^{PP}][u^P]\}}$$

$$\begin{bmatrix} F^P \\ \hline F^u \end{bmatrix} = \begin{bmatrix} K^{Pu} & K^{PP} \\ \hline K^{uu} & K^{uP} \end{bmatrix} \begin{bmatrix} u^u \\ \hline u^P \end{bmatrix}$$

REDUCED SYSTEM

$$\underset{\text{DOF}\times\text{DOF}}{A} \cdot x = b$$

4 EQN. & 4 Unknowns

$$\begin{bmatrix} \cancel{F^1} \\ \cancel{F^2} \\ \cancel{F^3} \\ \cancel{F^4} \end{bmatrix} = \begin{bmatrix} K^{11} & K^{12} & K^{13} & K^{14} \\ K^{21} & K^{22} & K^{23} & K^{24} \\ K^{31} & K^{32} & K^{33} & K^{34} \\ K^{41} & K^{42} & K^{43} & K^{44} \end{bmatrix} \begin{bmatrix} \cancel{u^1} \\ \cancel{u^2} \\ \cancel{u^3} \\ \cancel{u^4} \end{bmatrix}$$

F^1: ?, F^2: 0, F^3: 0 (NEUMANN), F^4: ? (NEUMANN)

u^1: 0, u^2: ?, u^3: ?, u^4: u

$$\begin{bmatrix} F^P \\ \hline F^u \end{bmatrix} = \begin{bmatrix} K^{Pu} & K^{PP} \\ \hline K^{uu} & K^{uP} \end{bmatrix} \begin{bmatrix} u^u \\ \hline u^P \end{bmatrix}$$

$$[F^P] = [K^{Pu}][u^u] + [K^{PP}][u^P]$$

$$\Downarrow$$

$$\underbrace{[K^{Pu}][u^u] = [F^P] - [K^{PP}][u^P]}$$

REDUCED SYSTEM

$$\Rightarrow [u^u] = [K^{Pu}]^{-1} \cdot \{[F^P] - [K^{PP}][u^P]\}$$

$$\Rightarrow [F^u] = [K^{uu}][u^u] + [K^{uP}][u^P]$$

STATIC CONDENSATION ✓

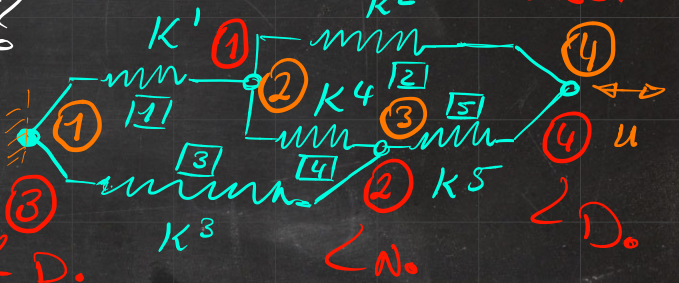
4 EQN. & 4 Unknowns

$$\begin{bmatrix} F^1 \to 0 \\ F^2 \to 0 \\ F^3 \to ? \\ F^4 \to ? \end{bmatrix} = \left[\begin{array}{cc|cc} K^{11} & K^{12} & K^{13} & K^{14} \\ K^{21} & K^{22} & K^{23} & K^{24} \\ \hline K^{31} & K^{32} & K^{33} & K^{34} \\ K^{41} & K^{42} & K^{43} & K^{44} \end{array}\right] \begin{bmatrix} u^1 \to ? \\ u^2 \to ? \\ u^3 \to 0 \\ u^4 \to ? \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} F^{P} \\ \hline F^{u} \end{bmatrix} = \left[\begin{array}{c|c} K^{Pu} & K^{PP} \\ \hline K^{uu} & K^{uP} \end{array}\right] \begin{bmatrix} u^{u} \\ \hline u^{P} \end{bmatrix}$$

NUMBERING CAREFULLY FROM THE ONSET

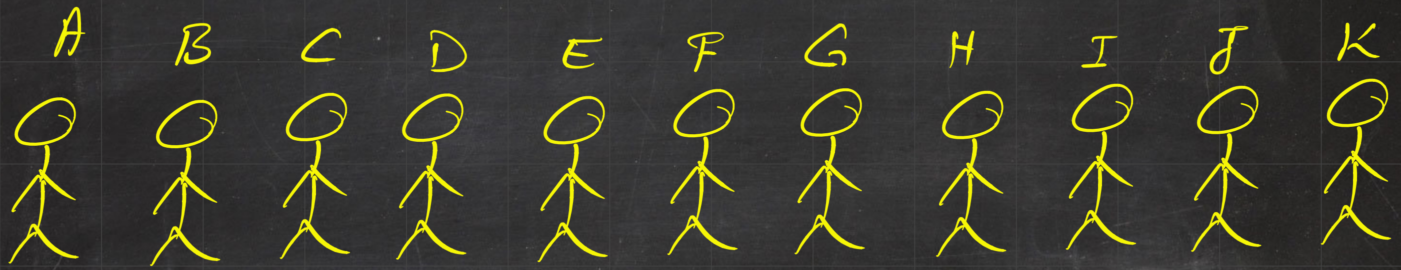


	NL	DEGREE
N.	(X,Y) 1	→ 3
	(X,Y) 2	→ 1
D.	(X,Y) 3	→ 2
	(X,Y) 4	→ 4

ENL

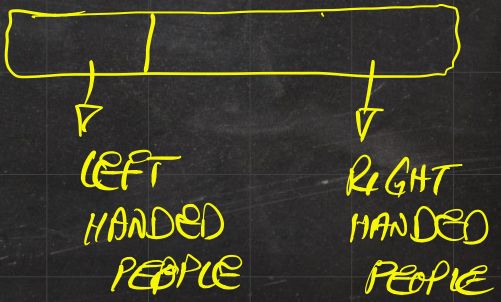
Loop over nodes
ASSIGN DEGREES TO NODES
end

EXTENDED NODE LIST → THE NAMING (NUMBERING) IS ARBITRARY!



→ Every Person Can Say One Word ← Programming: One Loop!

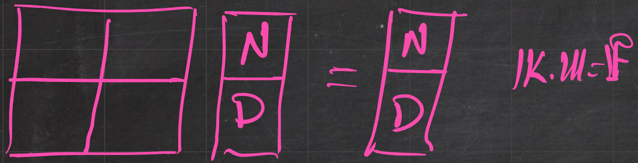
How many people?

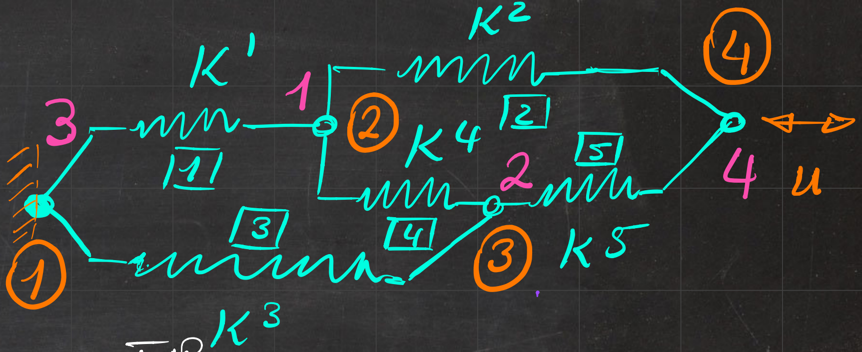


How many right-handed? → Assign Degrees

How many left-handed? → Assign Degrees

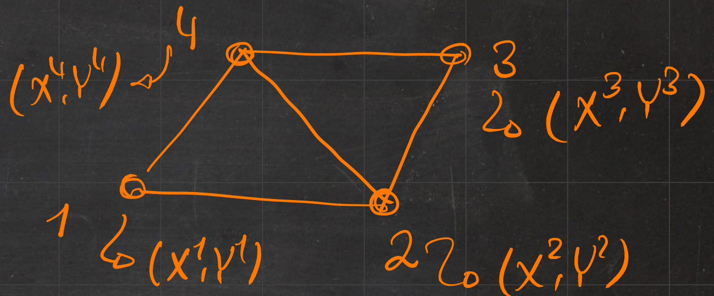
EXTENDED NODE LIST





NL	COOR.	BC INFO	TMP DEGREE	DEGREE	DISP	FORCE
1	0 0 0	D	-1	3	0	?
2	0 0 0	N	1	1	?	0
3	0 0 0	N	2	2	?	0
4	0 0 0	D	-2	4	u (↳ Pres.)	?

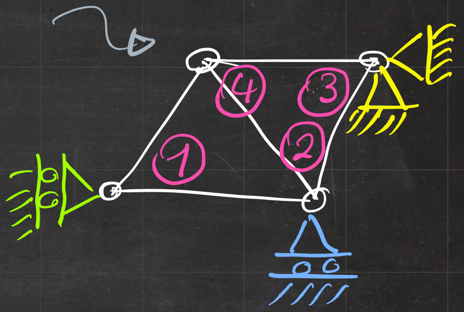
Truss Structures in 2D:



NL	COOR.	BC INFO	TMP DEGREE	DEGREE	DISP	FORCE
1	X^1, Y^1	D, N	-1, 1	5, 1		
2	⋮	N, D	2, -2	2, 6		
3		D, D	-3, -4	7, 8		
4		N, N	3, 4	3, 4		

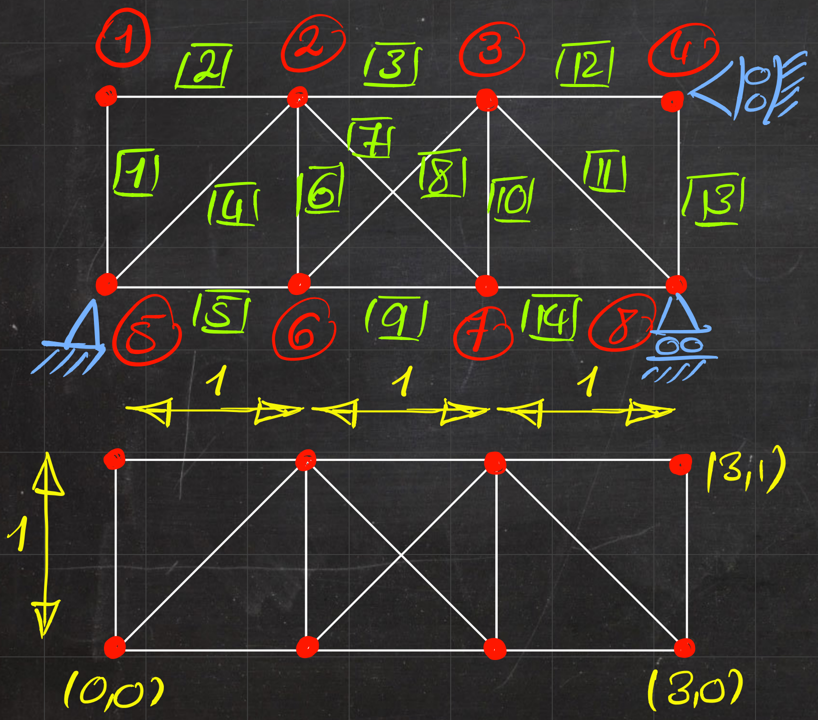
Truss Structures in 2D:

System with 4 DoFs



NL	Coor.	BC info	TMP Degree	Degree	Disp	Force
1	X^1, Y^1	D, N	-1, 1	5, 1		
2	⋮	N, D	2, -2	2, 6		
3	⋮	D, D	-3, -4	7, 8		
4	⋮	N, N	3, 4	3, 4		

NODE LIST ⟶ EXTENDED NODE LIST   CONNECTIVITY

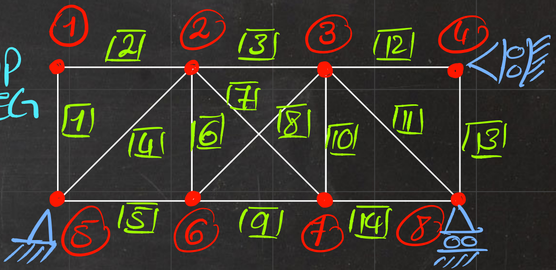


NL	COOR {x, y}
1	0, 1
2	1, 1
3	2, 1
4	3, 1
5	0, 0
6	1, 0
7	2, 0
8	3, 0

EL	
1	5, 1
2	1, 2
3	2, 3
4	5, 2
5	5, 6
6	6, 2
7	2, 7
8	⋮
9	⋮
10	⋮
11	⋮
⋮	

EXTENDED NODE LIST

NODE NUMBER	COOR X	COOR Y	BC INFO X	BC INFO Y	IMP DEG X	IMP DEG Y
1	0	1	N	N	1	2
2	1	1	N	N	3	4
3	2	1	N	N	5	6
4	3	1	D	N	-1	7
5	0	0	D	D	-2	-3
6	1	0	N	N	8	9
7	2	0	N	N	10	11
8	3	0	N	D	12	-4



EXTENDED NODE LIST

NODE NUMBER	COOR X	COOR Y	BC INFO X	BC INFO Y	TMP DEG X	TMP DEG Y	(GLOBAL) DEGREE X	(GLOBAL) DEGREE Y	DISP X	DISP Y	FORCE X	FORCE Y
1	0	1	N	N	1	2	1	2				
2	1	1	N	N	3	4	3	4				
3	2	1	N	N	5	6	5	6				
4	3	1	D	N	-1	7	13	7				
5	0	0	D	D	-2	-3	14	15				
6	1	0	N	N	8	9	8	9				
7	2	0	N	N	10	11	10	11				
8	3	0	N	D	12	-4	12	16				

EXTENDED NODE LIST

$$\underbrace{\mathbb{F}}_{\text{Force}} = \underbrace{\mathbb{K}}_{\text{Global Stiffness Matrix}} \; \underbrace{\mathbb{u}}_{\text{Disp}}$$

$$F^p \quad \begin{bmatrix} 12\times 1 \;\checkmark \\ \hdashline 4\times1 \;? \end{bmatrix} = \left[\begin{array}{c:c} 12\times12 & 12\times4 \\ \hdashline 4\times12 & 4\times4 \end{array}\right] \begin{bmatrix} 12\times1 \;? \\ \hdashline 4\times1 \;\checkmark \end{bmatrix} \quad \begin{matrix} u^u \\ u^p \end{matrix}$$

↳ Force

↳ Global Stiffness Matrix

↳ Disp

(Global) Degree

X	Y
1	2
3	4
5	6
13	7
14	15
8	9
10	11
12	16

12 DoF

EXTENDED NODE LIST

$$
\begin{array}{cc}
F^P & \mathbb{F} \\
\checkmark & \begin{bmatrix} 12 \times 1 \\ \hline 4 \times 1 \end{bmatrix} \\
? &
\end{array}
=
\begin{array}{c}
\mathbb{K} \\
\begin{bmatrix} 12 \times 12 & 12 \times 4 \\ 4 \times 12 & 4 \times 4 \end{bmatrix}
\end{array}
\begin{array}{cc}
\mathbb{u} & \\
\begin{bmatrix} 12 \times 1 \\ \hline 4 \times 1 \end{bmatrix} & \begin{array}{c} u^u \; ? \\ u^p \; \checkmark \end{array}
\end{array}
$$

↳ FORCE

↳ GLOBAL STIFFNESS MATRIX

↳ DISP

→ STATIC CONDENSATION

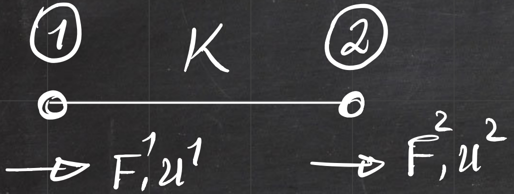
↓

REDUCED SYSTEM

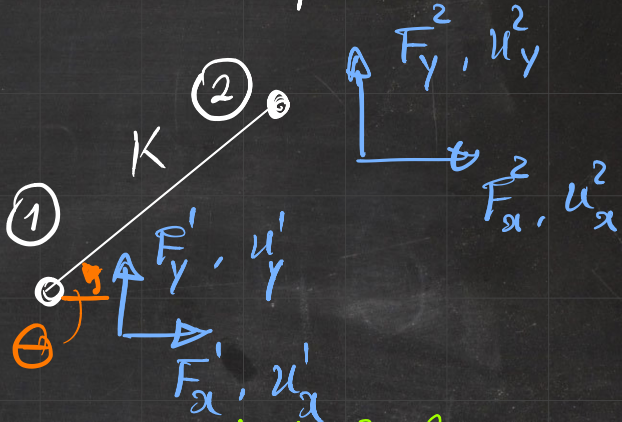
↳ $\mathbb{A} \cdot x = \mathbb{b}$

↳ $\square$ 12x12

To compute stiffness of 1D element in 2D space



$$\begin{bmatrix} F^1 \\ F^2 \end{bmatrix} = \begin{bmatrix} K & -K \\ -K & K \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix}$$



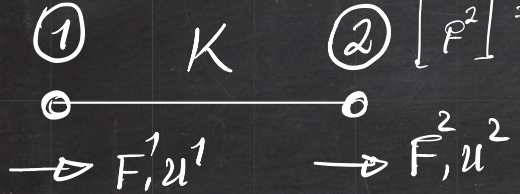
$$\mathbb{K} = \begin{array}{c} \begin{array}{cccc} 1_x & 1_y & 2_x & 2_y \end{array} \\ \begin{bmatrix} \; & \; & \; & \; \\ \; & \; & \; & \; \\ \; & \; & \; & \; \\ \; & \; & \; & \; \end{bmatrix} \end{array}$$

?

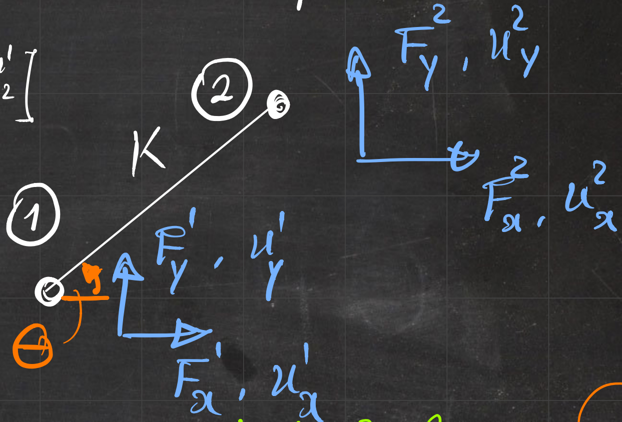
$f(\theta)$

4x4

To compute stiffness of 1D element in 2D space



$$\begin{bmatrix} F^1 \\ F^2 \end{bmatrix} = \begin{bmatrix} K & -K \\ -K & K \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix}$$



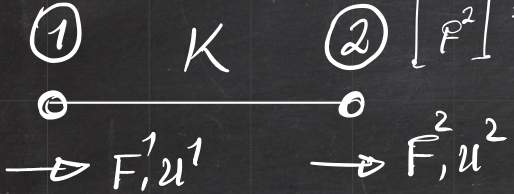
$$\begin{bmatrix} F_x^1 \\ F_y^1 \\ F_x^2 \\ F_y^2 \end{bmatrix} = \begin{array}{c} \begin{matrix} 1_x & 1_y & 2_x & 2_y \end{matrix} \\ \begin{bmatrix} \; & \; & \; & \; \\ \; & \; & \; & \; \\ \; & \; & \; & \; \\ \; & \; & \; & \; \end{bmatrix} \end{array} \begin{bmatrix} u_x^1 \\ u_y^1 \\ u_x^2 \\ u_y^2 \end{bmatrix}$$

$$\mathbb{K} = \begin{array}{c} \begin{matrix} 1_x & 1_y & 2_x & 2_y \end{matrix} \\ \begin{bmatrix} \; & \; & \; & \; \\ \; & \; & \; & \; \\ \; & \; & \; & \; \\ \; & \; & \; & \; \end{bmatrix} \end{array} \quad 4 \times 4$$

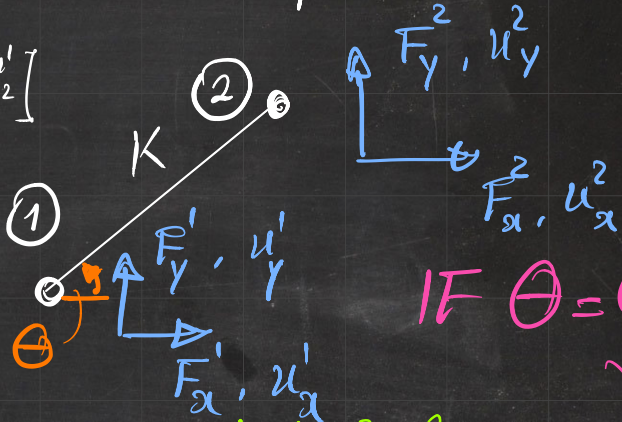
?

$f(\theta)$

To compute stiffness of 1D element in 2D space



$$\begin{bmatrix} F^1 \\ F^2 \end{bmatrix} = \begin{bmatrix} K & -K \\ -K & K \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix}$$



IF $\theta = 0$

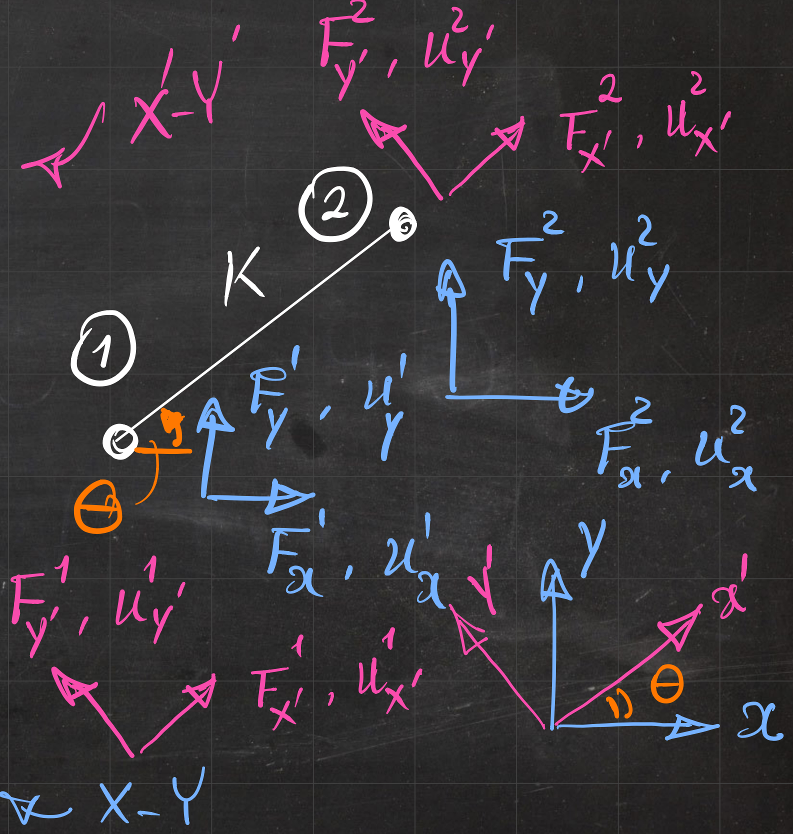
$$\begin{bmatrix} F_x^1 \\ F_y^1 \\ F_x^2 \\ F_y^2 \end{bmatrix} = \begin{array}{c} \begin{array}{cccc} 1_x & 1_y & 2_x & 2_y \end{array} \\ \begin{bmatrix} \; & \; & \; & \; \\ \; & \; & \; & \; \\ \; & \; & \; & \; \\ \; & \; & \; & \; \end{bmatrix} \end{array} \begin{bmatrix} u_x^1 \\ u_y^1 \\ u_x^2 \\ u_y^2 \end{bmatrix}$$

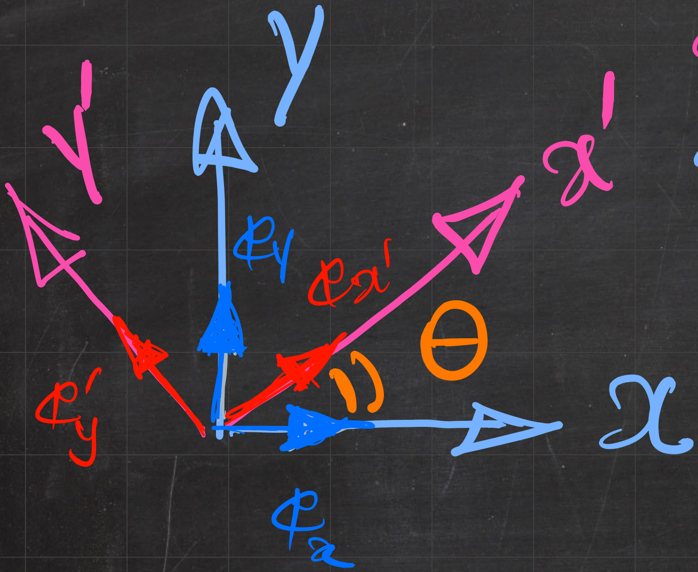
$$\mathbb{K} = \begin{array}{c} \begin{array}{cccc} 1_x & 1_y & 2_x & 2_y \end{array} \\ \begin{bmatrix} K & 0 & -K & 0 \\ 0 & 0 & 0 & 0 \\ -K & 0 & K & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

4x4

$$
\begin{bmatrix} F^1_{x'} \\ F^1_{y'} \\ F^2_{x'} \\ F^2_{y'} \end{bmatrix}
=
\begin{array}{c}
\begin{array}{cccc} 1_{x'} & 1_{y'} & 2_{x'} & 2_{y'} \end{array} \\
\begin{bmatrix} K & 0 & -K & 0 \\ 0 & 0 & 0 & 0 \\ -K & 0 & K & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{array}
\begin{bmatrix} u^1_{x'} \\ u^1_{y'} \\ u^2_{x'} \\ u^2_{y'} \end{bmatrix}
$$

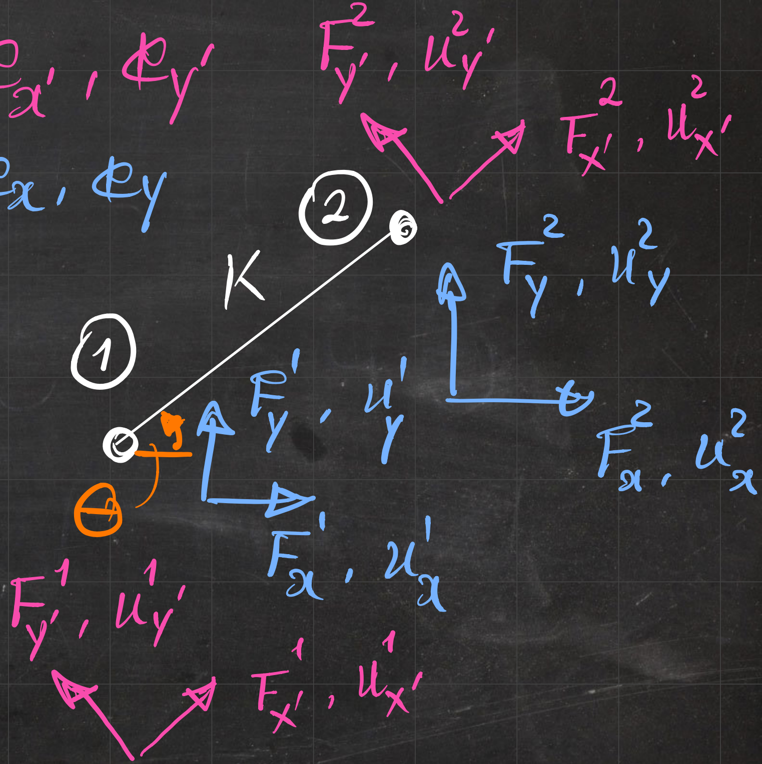
$$
\begin{bmatrix} F^1_{x} \\ F^1_{y} \\ F^2_{x} \\ F^2_{y} \end{bmatrix}
=
\begin{array}{c}
\begin{array}{cccc} 1_{x} & 1_{y} & 2_{x} & 2_{y} \end{array} \\
\begin{bmatrix} \; & \; & \; & \; \\ \; & \; & \; & \; \\ \; & \; & \; & \; \\ \; & \; & \; & \; \end{bmatrix}
\end{array}
\begin{bmatrix} u^1_{x} \\ u^1_{y} \\ u^2_{x} \\ u^2_{y} \end{bmatrix}
$$





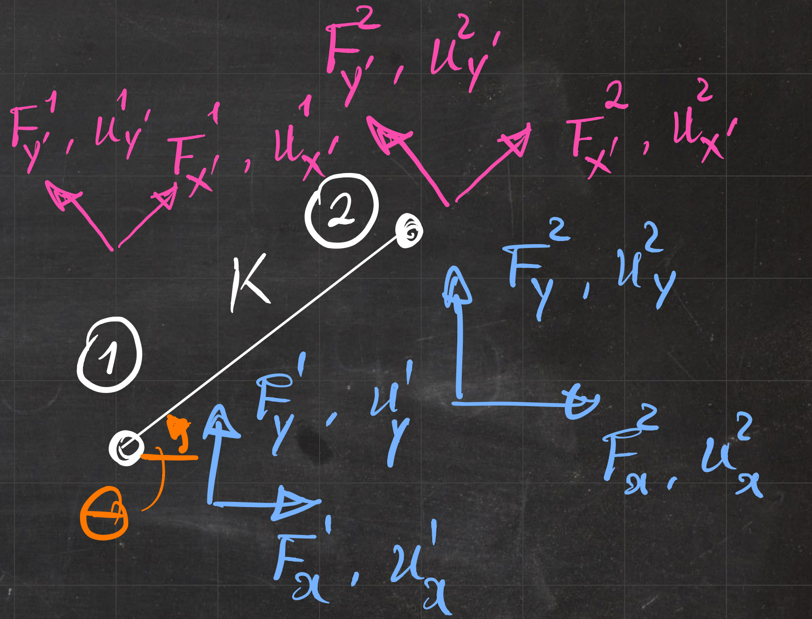
$e_{x'}, e_{y'}$

e_x, e_y



$$
\begin{cases}
e_{x'} = \cos\theta \, e_x + \sin\theta \, e_y \\
e_{y'} = -\sin\theta \, e_x + \cos\theta \, e_y
\end{cases}
$$

$$
\begin{cases}
\boldsymbol{e}_{x'} = \cos\theta \, \boldsymbol{e}_x + \sin\theta \, \boldsymbol{e}_y \\
\boldsymbol{e}_{y'} = -\sin\theta \, \boldsymbol{e}_x + \cos\theta \, \boldsymbol{e}_y
\end{cases}
$$

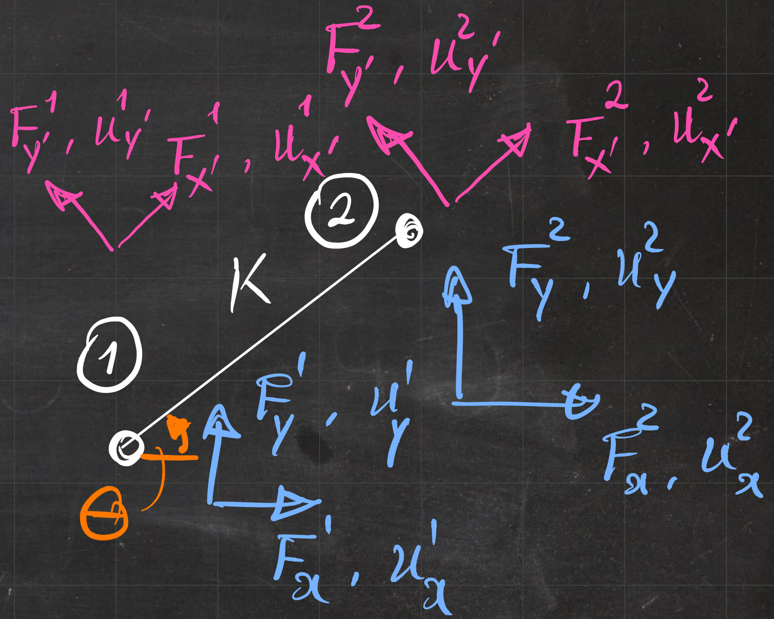


$$
\boldsymbol{F} = F_{x'} \boldsymbol{e}_{x'} + F_{y'} \boldsymbol{e}_{y'}
$$

$$
= F_x \boldsymbol{e}_x + F_y \boldsymbol{e}_y \quad \ldots
$$

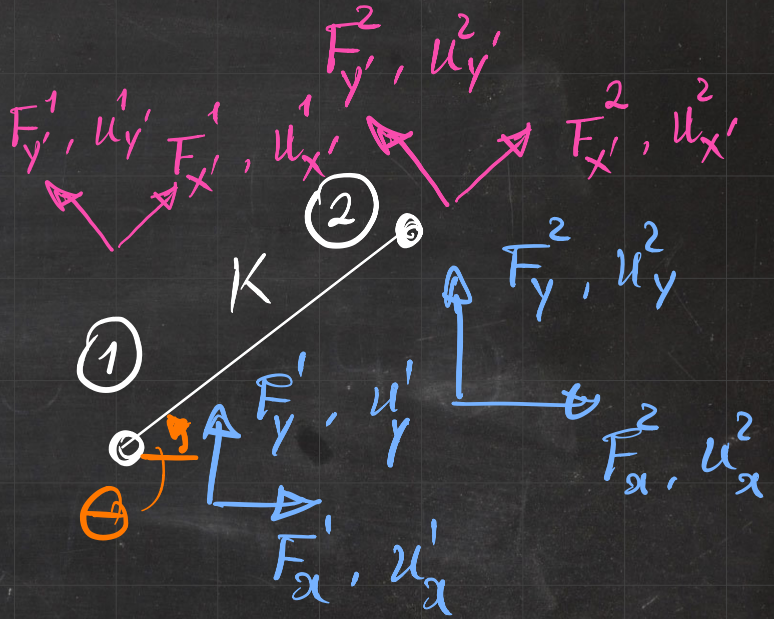
$$
\begin{bmatrix} F_x \\ F_y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ +\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} F_{x'} \\ F_{y'} \end{bmatrix} \iff \begin{bmatrix} F_{x'} \\ F_{y'} \end{bmatrix} = \begin{bmatrix} \cos\theta & +\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix}
$$

$$
\begin{bmatrix} F^1_{x'} \\ F^1_{y'} \\ F^2_{x'} \\ F^2_{y'} \end{bmatrix}
=
\underbrace{\begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix}}_{\mathbb{R}}
\begin{bmatrix} F^1_{x} \\ F^1_{y} \\ F^2_{x} \\ F^2_{y} \end{bmatrix}
$$



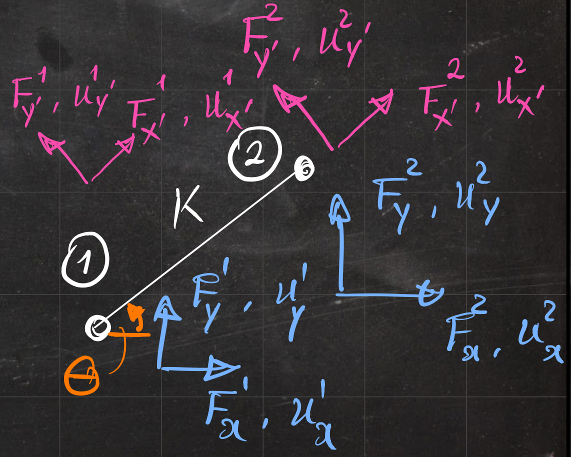
$$
\begin{bmatrix} F_x \\ F_y \end{bmatrix}
=
\begin{bmatrix} \cos\theta & -\sin\theta \\ +\sin\theta & \cos\theta \end{bmatrix}
\begin{bmatrix} F_{x'} \\ F_{y'} \end{bmatrix}
\iff
\begin{bmatrix} F_{x'} \\ F_{y'} \end{bmatrix}
=
\begin{bmatrix} \cos\theta & +\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}
\begin{bmatrix} F_x \\ F_y \end{bmatrix}
$$

$$\begin{bmatrix} F_{x'}^1 \\ F_{y'}^1 \\ F_{x'}^2 \\ F_{y'}^2 \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix}}_{\mathbb{R}(\theta)} \begin{bmatrix} F_x^1 \\ F_y^1 \\ F_x^2 \\ F_y^2 \end{bmatrix}$$



$$\begin{bmatrix} F^1_{x'} \\ F^1_{y'} \\ F^2_{x'} \\ F^2_{y'} \end{bmatrix} = \mathbb{R}_\theta \begin{bmatrix} F^1_x \\ F^1_y \\ F^2_x \\ F^2_y \end{bmatrix}, \quad \begin{bmatrix} u^1_{x'} \\ u^1_{y'} \\ u^2_{x'} \\ u^2_{y'} \end{bmatrix} = \mathbb{R}_\theta \begin{bmatrix} u^1_x \\ u^1_y \\ u^2_x \\ u^2_y \end{bmatrix}$$

$$\mathbb{R}_\theta^{-1} = \mathbb{R}_{-\theta}$$



$$
\begin{bmatrix} F_x^{1'} \\ F_y^{1'} \\ F_x^{2'} \\ F_y^{2'} \end{bmatrix}
=
\begin{array}{c}
\begin{array}{cccc} 1_x' & 1_y' & 2_x' & 2_y' \end{array} \\
\underbrace{\begin{bmatrix} K & 0 & -K & 0 \\ 0 & 0 & 0 & 0 \\ -K & 0 & K & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathbb{K}_0}
\end{array}
\begin{bmatrix} u_x^{1'} \\ u_y^{1'} \\ u_x^{2'} \\ u_y^{2'} \end{bmatrix}
$$

$$
\begin{bmatrix} F_x^{1'} \\ F_y^{1'} \\ F_x^{2'} \\ F_y^{2'} \end{bmatrix}
= \mathbb{R}_\theta
\begin{bmatrix} F_x^{1} \\ F_y^{1} \\ F_x^{2} \\ F_y^{2} \end{bmatrix}
,\quad
\begin{bmatrix} u_x^{1'} \\ u_y^{1'} \\ u_x^{2'} \\ u_y^{2'} \end{bmatrix}
= \mathbb{R}_\theta
\begin{bmatrix} u_x^{1} \\ u_y^{1} \\ u_x^{2} \\ u_y^{2} \end{bmatrix}
$$

$$\mathbb{F}' = \mathbb{K}_0\, \mathbb{u}'$$

$$
\begin{bmatrix} F_x^{1} \\ F_y^{1} \\ F_x^{2} \\ F_y^{2} \end{bmatrix}
=
\begin{array}{c}
\begin{array}{cccc} 1_x & 1_y & 2_x & 2_y \end{array} \\
\begin{bmatrix} \; & \; & \; & \; \\ \; & \; & \; & \; \\ \; & \; & \; & \; \\ \; & \; & \; & \; \end{bmatrix}
\end{array}
\begin{bmatrix} u_x^{1} \\ u_y^{1} \\ u_x^{2} \\ u_y^{2} \end{bmatrix}
$$

$$\mathbb{R}_\theta\, \mathbb{F} = \mathbb{K}_0\, \mathbb{R}_\theta\, \mathbb{u}$$

$$\mathbb{F} = \mathbb{R}_\theta^{T}\, \mathbb{K}_0\, \mathbb{R}_\theta\, \mathbb{u}$$

$$
\begin{array}{c}
\\
\begin{bmatrix} F_{x'}^{1} \\ F_{y'}^{1} \\ F_{x'}^{2} \\ F_{y'}^{2} \end{bmatrix}
\end{array}
=
\begin{array}{c}
\begin{array}{cccc} 1_{x'} & 1_{y'} & 2_{x'} & 2_{y'} \end{array} \\
\underbrace{\begin{bmatrix} K & 0 & -K & 0 \\ 0 & 0 & 0 & 0 \\ -K & 0 & K & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathbb{K}_0}
\end{array}
\begin{array}{c}
\\
\begin{bmatrix} u_{x'}^{1} \\ u_{y'}^{1} \\ u_{x'}^{2} \\ u_{y'}^{2} \end{bmatrix}
\end{array}
$$

$$
\begin{bmatrix} F_{x'}^{1} \\ F_{y'}^{1} \\ F_{x'}^{2} \\ F_{y'}^{2} \end{bmatrix} = \mathbb{R}_\theta \begin{bmatrix} F_{x}^{1} \\ F_{y}^{1} \\ F_{x}^{2} \\ F_{y}^{2} \end{bmatrix}, \quad \begin{bmatrix} u_{x'}^{1} \\ u_{y'}^{1} \\ u_{x'}^{2} \\ u_{y'}^{2} \end{bmatrix} = \mathbb{R}_\theta \begin{bmatrix} u_{x}^{1} \\ u_{y}^{1} \\ u_{x}^{2} \\ u_{y}^{2} \end{bmatrix}
$$

$$\mathbb{F}' = \mathbb{K}_0 \, \mathbb{u}'$$

$$\mathbb{F} = \underbrace{\mathbb{R}_\theta^T \, \mathbb{K}_0 \, \mathbb{R}_\theta}_{\mathbb{K}} \, \mathbb{u}$$

$$\mathbb{F} = \mathbb{K} \, \mathbb{u}$$

$$
\begin{array}{c}
\\
\begin{bmatrix} F_{x}^{1} \\ F_{y}^{1} \\ F_{x}^{2} \\ F_{y}^{2} \end{bmatrix}
\end{array}
=
\begin{array}{c}
\begin{array}{cccc} 1_{x} & 1_{y} & 2_{x} & 2_{y} \end{array} \\
\begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \\ & & & \end{bmatrix}
\end{array}
\begin{array}{c}
\\
\begin{bmatrix} u_{x}^{1} \\ u_{y}^{1} \\ u_{x}^{2} \\ u_{y}^{2} \end{bmatrix}
\end{array}
$$

$$\begin{bmatrix} F_x^{1'} \\ F_y^{1'} \\ F_x^{2'} \\ F_y^{2'} \end{bmatrix} = \begin{array}{cccc} 1_x' & 1_y' & 2_x' & 2_y' \end{array} \underbrace{\begin{bmatrix} K & 0 & -K & 0 \\ 0 & 0 & 0 & 0 \\ -K & 0 & K & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathbb{K}_0} \begin{bmatrix} u_x^{1'} \\ u_y^{1'} \\ u_x^{2'} \\ u_y^{2'} \end{bmatrix}$$

$$\begin{bmatrix} F_{x'}^{1} \\ F_{y'}^{1} \\ F_{x'}^{2} \\ F_{y'}^{2} \end{bmatrix} = \mathbb{R}_\theta \begin{bmatrix} F_x^{1} \\ F_y^{1} \\ F_x^{2} \\ F_y^{2} \end{bmatrix}, \quad \begin{bmatrix} u_{x'}^{1} \\ u_{y'}^{1} \\ u_{x'}^{2} \\ u_{y'}^{2} \end{bmatrix} = \mathbb{R}_\theta \begin{bmatrix} u_x^{1} \\ u_y^{1} \\ u_x^{2} \\ u_y^{2} \end{bmatrix}$$

$$\mathbb{F}' = \mathbb{K}_0 \, \mathbb{U}'$$

$$\begin{bmatrix} F_x^{1} \\ F_y^{1} \\ F_x^{2} \\ F_y^{2} \end{bmatrix} = \begin{array}{cccc} 1_x & 1_y & 2_x & 2_y \end{array} \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} u_x^{1} \\ u_y^{1} \\ u_x^{2} \\ u_y^{2} \end{bmatrix}$$

$$\mathbb{F} = \mathbb{K} \, \mathbb{U}$$

$$\mathbb{K} = \mathbb{R}_\theta^T \, \mathbb{K}_0 \, \mathbb{R}_\theta$$

$$
\begin{bmatrix} F^1_x \\ F^1_y \\ F^2_x \\ F^2_y \end{bmatrix}
=
\begin{array}{c}
\begin{array}{cccc} 1_x & 1_y & 2_x & 2_y \end{array} \\
\begin{bmatrix} \; & \; & \; & \; \\ \; & \; & \; & \; \\ \; & \; & \; & \; \\ \; & \; & \; & \; \end{bmatrix}
\end{array}
\begin{bmatrix} u^1_x \\ u^1_y \\ u^2_x \\ u^2_y \end{bmatrix}
$$

$$\mathbb{F} = \mathbb{K}\,\mathbb{u}$$

$$\mathbb{K} = \mathbb{R}_\theta^T\, \mathbb{K}_0\, \mathbb{R}_\theta$$

$$
\begin{bmatrix} F^1_x \\ F^1_y \\ F^2_x \\ F^2_y \end{bmatrix}
= \underset{\frac{EA}{L}}{K}
\begin{bmatrix}
\cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\
\sin\theta\cos\theta & \sin^2\theta & -\sin\theta\cos\theta & -\sin^2\theta \\
-\cos^2\theta & -\cos\theta\sin\theta & \cos^2\theta & \cos\theta\sin\theta \\
-\sin\theta\cos\theta & -\sin^2\theta & \sin\theta\cos\theta & \sin^2\theta
\end{bmatrix}
\begin{bmatrix} u^1_x \\ u^1_y \\ u^2_x \\ u^2_y \end{bmatrix}
$$

$$
\underbrace{\begin{bmatrix}
\cos\theta & \sin\theta & 0 & 0 \\
-\sin\theta & \cos\theta & 0 & 0 \\
0 & 0 & \cos\theta & \sin\theta \\
0 & 0 & -\sin\theta & \cos\theta
\end{bmatrix}}_{\mathbb{R}(\theta)}
$$

$$\begin{bmatrix} F^1_x \\ F^1_y \\ F^2_x \\ F^2_y \end{bmatrix} = \begin{array}{c} \begin{array}{cccc} 1_x & 1_y & 2_x & 2_y \end{array} \\ \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \end{array} \begin{bmatrix} u^1_x \\ u^1_y \\ u^2_x \\ u^2_y \end{bmatrix}$$

$$\mathbb{F} = \mathbb{K}\,\mathbb{u}$$

$$\theta = 0$$

$$\mathbb{K} = \mathbb{R}_\theta^T \, \mathbb{K}_0 \, \mathbb{R}_\theta$$

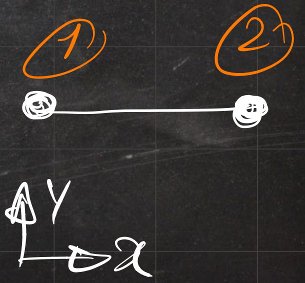
$$\boxed{\begin{matrix} F^1_x \\ F^1_y \\ F^2_x \\ F^2_y \end{matrix}} = K \, \frac{EA}{L} \boxed{\begin{matrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}} \boxed{\begin{matrix} u^1_x \\ u^1_y \\ u^2_x \\ u^2_y \end{matrix}}$$

$$\underbrace{\begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix}}_{\mathbb{R}(\theta)}$$

$$\begin{bmatrix} F^1_x \\ F^1_y \\ F^2_x \\ F^2_y \end{bmatrix} = K \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ \sin\theta\cos\theta & \sin^2\theta & -\sin\theta\cos\theta & -\sin^2\theta \\ -\cos^2\theta & -\cos\theta\sin\theta & \cos^2\theta & \cos\theta\sin\theta \\ -\sin\theta\cos\theta & -\sin^2\theta & \sin\theta\cos\theta & \sin^2\theta \end{bmatrix} \begin{bmatrix} u^1_x \\ u^1_y \\ u^2_x \\ u^2_y \end{bmatrix}, \qquad K = \frac{EA}{L}$$

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

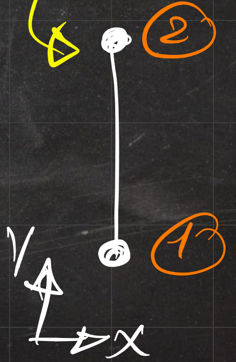
$$\theta = 0 \Rightarrow \mathbb{K} = \frac{EA}{L} \begin{array}{c} \begin{array}{cccc} 1x & 1y & 2x & 2y \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$



$$
\begin{bmatrix} F^1_x \\ F^1_y \\ F^2_x \\ F^2_y \end{bmatrix} = \underbrace{K}_{\frac{EA}{L}} \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ \sin\theta\cos\theta & \sin^2\theta & -\sin\theta\cos\theta & -\sin^2\theta \\ -\cos^2\theta & -\cos\theta\sin\theta & \cos^2\theta & \cos\theta\sin\theta \\ -\sin\theta\cos\theta & -\sin^2\theta & \sin\theta\cos\theta & \sin^2\theta \end{bmatrix} \begin{bmatrix} u^1_x \\ u^1_y \\ u^2_x \\ u^2_y \end{bmatrix}
$$

$$
\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}
$$

$$
\theta = 90^\circ \Rightarrow \mathbb{K} = \frac{EA}{L}
\begin{array}{c}
\begin{array}{cccc} 1x & 1y & 2x & 2y \end{array} \\
\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}
\end{array}
$$



$$
\begin{bmatrix} F_x^1 \\ F_y^1 \\ F_x^2 \\ F_y^2 \end{bmatrix} = K \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ \sin\theta\cos\theta & \sin^2\theta & -\sin\theta\cos\theta & -\sin^2\theta \\ -\cos^2\theta & -\cos\theta\sin\theta & \cos^2\theta & \cos\theta\sin\theta \\ -\sin\theta\cos\theta & -\sin^2\theta & \sin\theta\cos\theta & \sin^2\theta \end{bmatrix} \begin{bmatrix} u_x^1 \\ u_y^1 \\ u_x^2 \\ u_y^2 \end{bmatrix}
$$

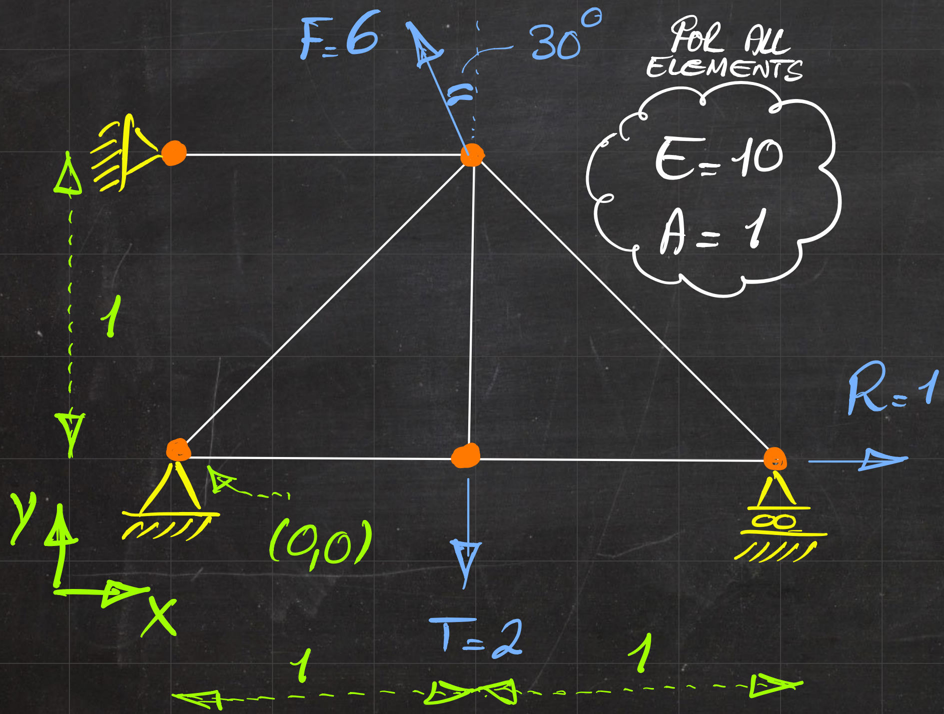
$$
K = \frac{EA}{L}
$$

$$
\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}
$$

$$
\theta = 45^\circ \Rightarrow \mathbb{K} = \frac{1}{2}\frac{EA}{L}
\begin{array}{c}
\begin{array}{cccc} 1x & 1y & 2x & 2y \end{array} \\
\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}
\end{array}
$$

$$
\cos\theta = \sin\theta = \frac{1}{\sqrt{2}}
$$

EXAMPLE:

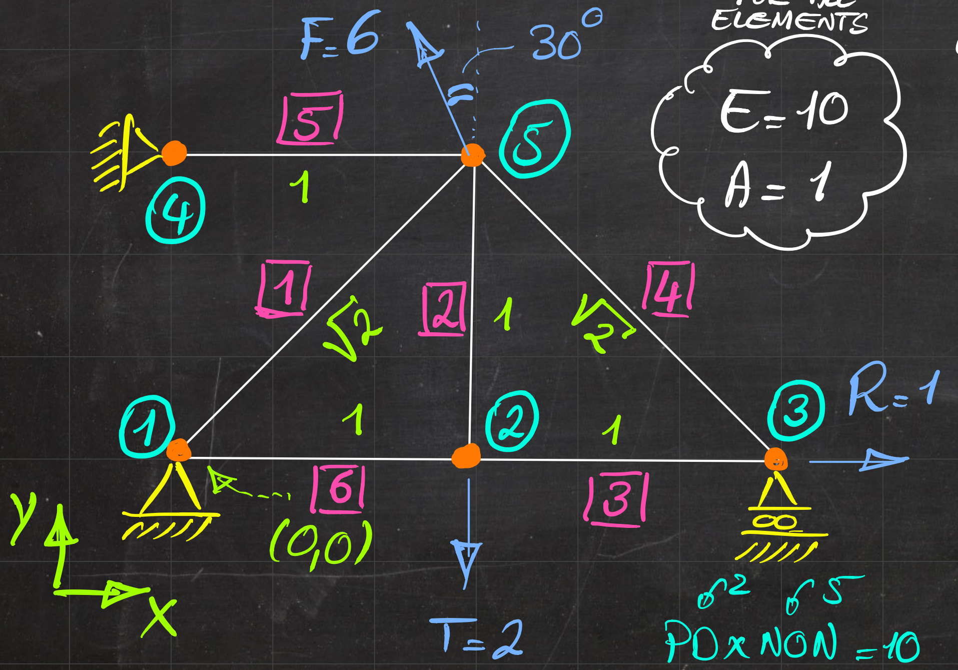


CALCULATE THE DISPLACEMENTS OF ALL THE NODES.

* NUMBER NODES & ELEMENTS
  ↳ Create NL & EL
* CREATE ENL
* COMPUTE ELEMENTS STIFFNESSES
* ASSEMBLE STIFFNESS
* SOLVE

EXAMPLE:

CALCULATE THE DISPLACEMENTS OF ALL THE NODES.



FOR ALL ELEMENTS

$E = 10$

$A = 1$

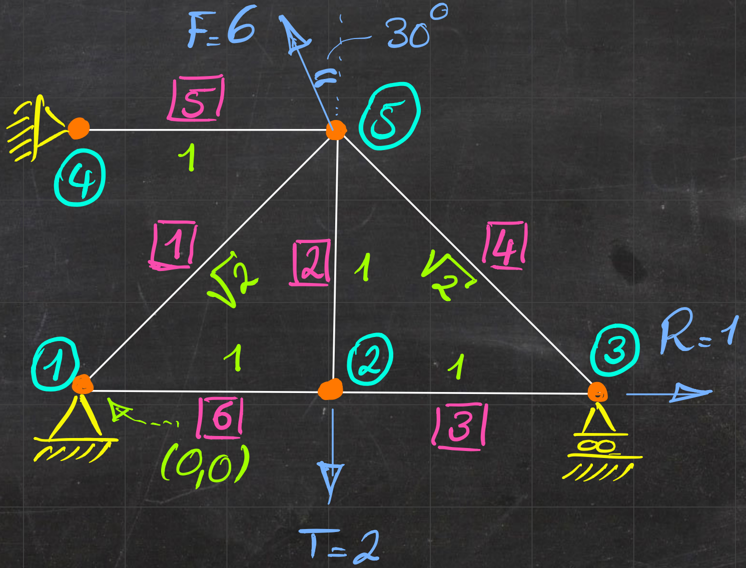
* NUMBER NODES & ELEMENTS
  ↳ Create NL & EL
* CREATE ENL
* COMPUTE ELEMENTS STIFFNESSES
* ASSEMBLE STIFFNESS
* SOLVE

$\overset{2}{PD} \times \overset{5}{NON} = 10$

EXAMPLE:

COOR

NL	X	Y
1	0	0
2	1	0
3	2	0
4	0	1
5	1	1

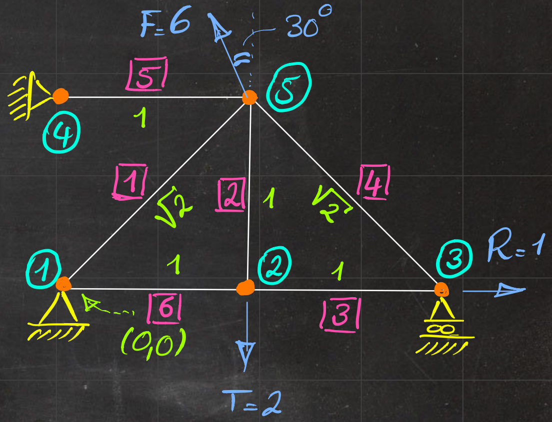


CONNECTIVITY

EL	CORNERS N1	N2
1	1	5
2	2	5
3	2	3
4	3	5
5	4	5
6	1	2

EXAMPLE:

NL	COOR X	COOR Y	BC INFO X	BC INFO Y	TMP DEG. X	TMP DEG. Y
1	0	0	D	D	-1	-2
2	1	0	N	N	1	2
3	2	0	N	D	3	-3
4	0	1	D	D	-4	-5
5	1	1	N	N	4	5



DoF = 5

DoC = 5 ← |-5|

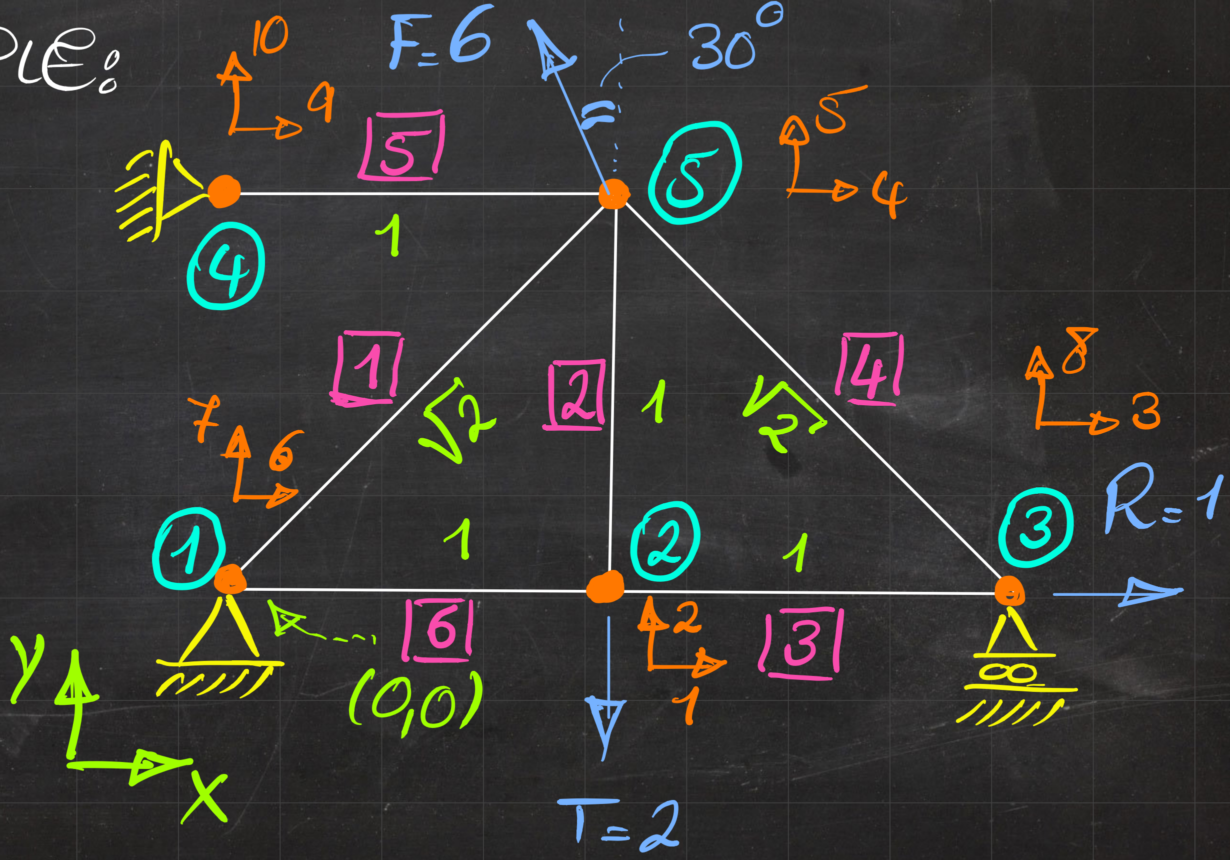
+ 10 = 2×5 ← NoN

EXAMPLE:

EXTENDED NODE LIST

	COOR		BC INFO		TMP DEG.		GLOBAL DEG		DISP		EXTERNALLY PRESCRIBED FORCE	
NL	X	Y	X	Y	X	Y	X	Y	X	Y	X	Y
1	0	0	D	D	-1	-2	6	7	0	0	?	?
2	1	0	N	N	1	2	1	2	?	?	0	-2
3	2	0	N	D	3	-3	3	8	?	0	1	?
4	0	1	D	D	-4	-5	9	10	0	0	?	?
5	1	1	N	N	4	5	4	5	?	?	$-F\sin 30^\circ$	$F\cos 30^\circ$

EXAMPLE:



EXAMPLE:

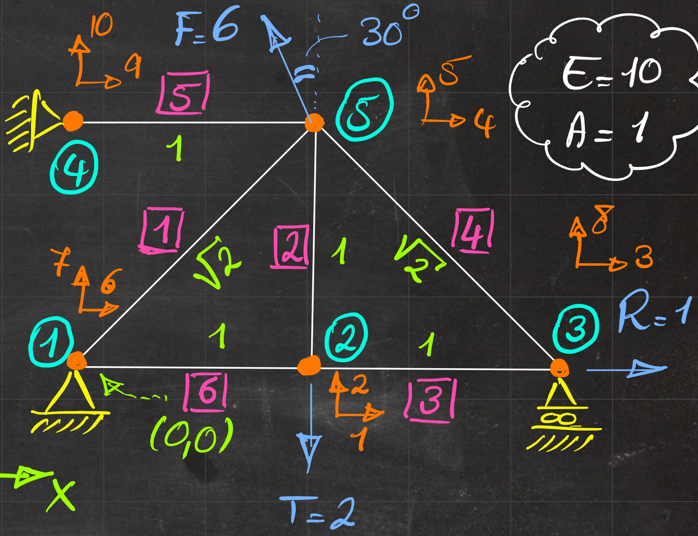
BETWEEN 1 - 5

$$\mathbb{K}^{1} = \frac{EA}{2\sqrt{2}}\;
\begin{array}{c|cccc}
 & \overset{6}{1_x} & \overset{7}{1_y} & \overset{4}{5_x} & \overset{5}{5_y} \\
\hline
\overset{6}{1_x} & 1 & 1 & -1 & -1 \\
\overset{7}{1_y} & 1 & 1 & -1 & -1 \\
\overset{4}{5_x} & -1 & -1 & 1 & 1 \\
\overset{5}{5_y} & -1 & -1 & 1 & 1
\end{array}$$

$E = 10$

$A = 1$

$L = \sqrt{2}$



E=10
A=1

$$\mathbb{K}_\theta = \frac{EA}{L}\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ \sin\theta\cos\theta & \sin^2\theta & -\sin\theta\cos\theta & -\sin^2\theta \\ -\cos^2\theta & -\cos\theta\sin\theta & \cos^2\theta & \cos\theta\sin\theta \\ -\sin\theta\cos\theta & -\sin^2\theta & \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

EXAMPLE:

BETWEEN ① - ⑤

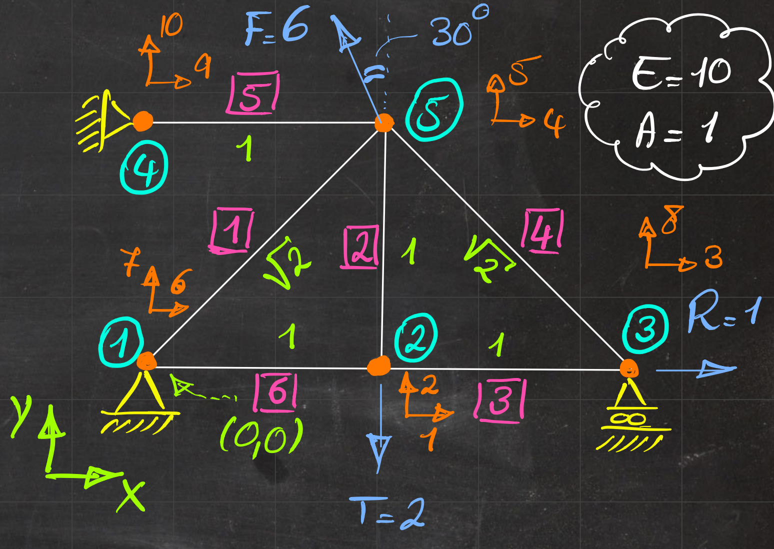
$$\mathbb{K}^{1} = \frac{EA}{2\sqrt{2}} \begin{array}{ccccc} 6 & 7 & 4 & 5 & \\ \end{array}$$

$$\begin{array}{c|cccc|c} & 6 & 7 & 4 & 5 & \\ \hline & 1 & 1 & -1 & -1 & 6 \\ & 1 & 1 & -1 & -1 & 7 \\ & -1 & -1 & 1 & 1 & 4 \\ & -1 & -1 & 1 & 1 & 5 \end{array}$$

$E = 10$

$A = 1$

$L = \sqrt{2}$



E=10
A=1

$$\mathbb{K}_{\theta} = \frac{EA}{L} \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ \sin\theta\cos\theta & \sin^2\theta & -\sin\theta\cos\theta & -\sin^2\theta \\ -\cos^2\theta & -\cos\theta\sin\theta & \cos^2\theta & \cos\theta\sin\theta \\ -\sin\theta\cos\theta & -\sin^2\theta & \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

EXAMPLE:

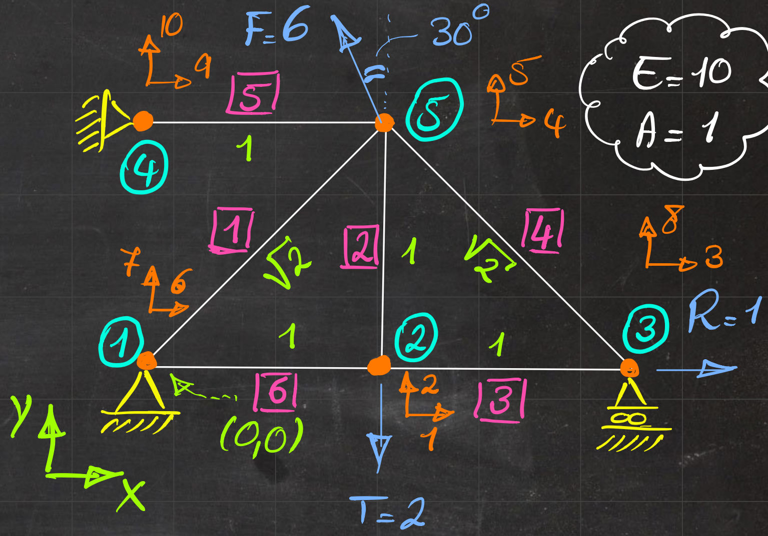
BETWEEN (2) - (5)

$$
\overset{2}{\mathbb{K}} = \frac{EA}{1}
\begin{array}{c}
\begin{array}{cccc} 1 & 2 & 4 & 5 \end{array} \\
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1
\end{bmatrix}
\end{array}
\begin{array}{c} 1 \\ 2 \\ 4 \\ 5 \end{array}
$$

$E = 10$

$A = 1$

$L = 1$



$E = 10$

$A = 1$

$$
\mathbb{K}_\theta = \frac{EA}{L}
\begin{bmatrix}
\cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\
\sin\theta\cos\theta & \sin^2\theta & -\sin\theta\cos\theta & -\sin^2\theta \\
-\cos^2\theta & -\cos\theta\sin\theta & \cos^2\theta & \cos\theta\sin\theta \\
-\sin\theta\cos\theta & -\sin^2\theta & \sin\theta\cos\theta & \sin^2\theta
\end{bmatrix}
$$

EXAMPLE:

② → ③ ⇒ $\theta = 0°$

③ → ② ⇒ $\theta = 180°$

BETWEEN ② - ③

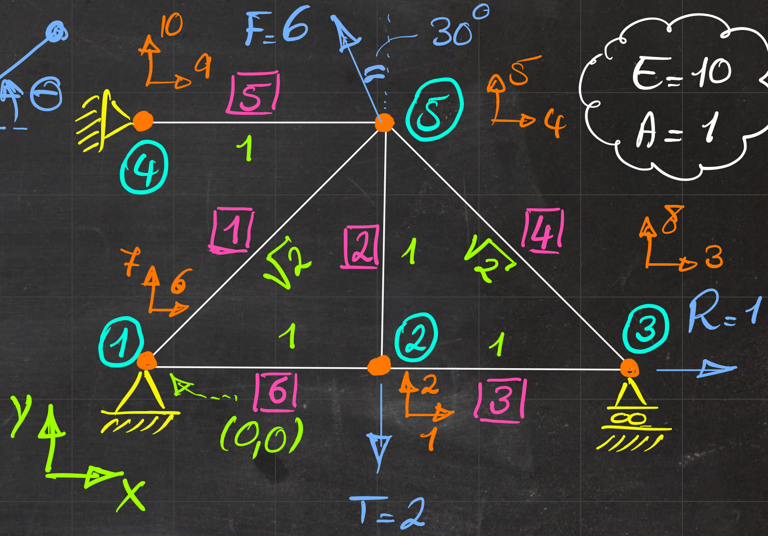
$$\mathbb{K}^{3} = \frac{EA}{1}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 8 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}\begin{array}{c} 1 \\ 2 \\ 3 \\ 8 \end{array}$$

$E = 10$

$A = 1$

$L = 1$

$E = 10$
$A = 1$



$$\mathbb{K}_{\theta} = \frac{EA}{L}\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ \sin\theta\cos\theta & \sin^2\theta & -\sin\theta\cos\theta & -\sin^2\theta \\ -\cos^2\theta & -\cos\theta\sin\theta & \cos^2\theta & \cos\theta\sin\theta \\ -\sin\theta\cos\theta & -\sin^2\theta & \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

EXAMPLE:

$\theta = 135^\circ$

BETWEEN 3 - 5

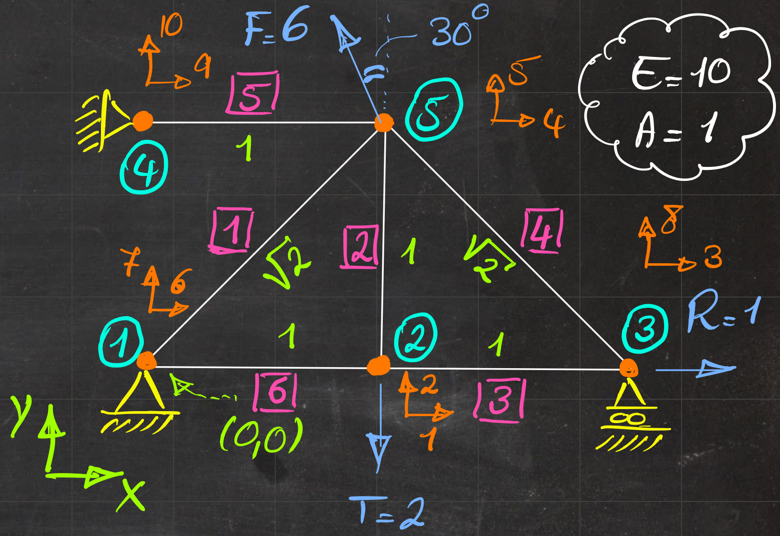
$$\mathbb{K}^{4} = \frac{EA}{2\sqrt{2}} \begin{array}{c} \\ \end{array}\begin{array}{cccc|c} 3 & 8 & 4 & 5 & \\ \end{array}$$

$$\mathbb{K}^{4} = \frac{EA}{2\sqrt{2}}\;\begin{array}{c} \begin{array}{cccc} 3 & 8 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ 3 \\ 8 \\ 4 \\ 5 \end{array}$$

$E = 10$

$A = 1$

$L = \sqrt{2}$



$$\mathbb{K}_{\theta} = \frac{EA}{L} \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ \sin\theta\cos\theta & \sin^2\theta & -\sin\theta\cos\theta & -\sin^2\theta \\ -\cos^2\theta & -\cos\theta\sin\theta & \cos^2\theta & \cos\theta\sin\theta \\ -\sin\theta\cos\theta & -\sin^2\theta & \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

EXAMPLE:

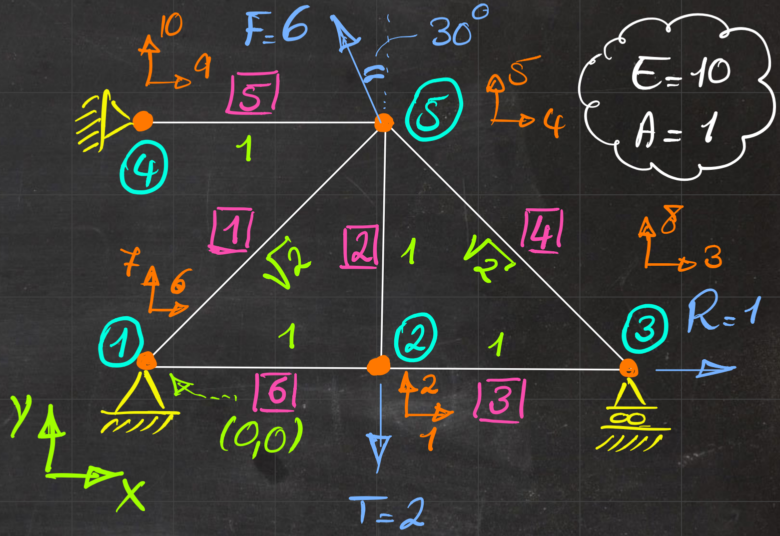
BETWEEN 4 - 5

$$\mathbb{K} = \frac{EA}{1}\begin{array}{c} \begin{array}{cccc} 9 & 10 & 4 & 5 \end{array} \\ \left[\begin{array}{cccc} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ 9 \\ 10 \\ 4 \\ 5 \end{array}$$

$E = 10$

$A = 1$

$L = 1$



$$\mathbb{K}_\theta = \frac{EA}{L}\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ \sin\theta\cos\theta & \sin^2\theta & -\sin\theta\cos\theta & -\sin^2\theta \\ -\cos^2\theta & -\cos\theta\sin\theta & \cos^2\theta & \cos\theta\sin\theta \\ -\sin\theta\cos\theta & -\sin^2\theta & \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

EXAMPLE:

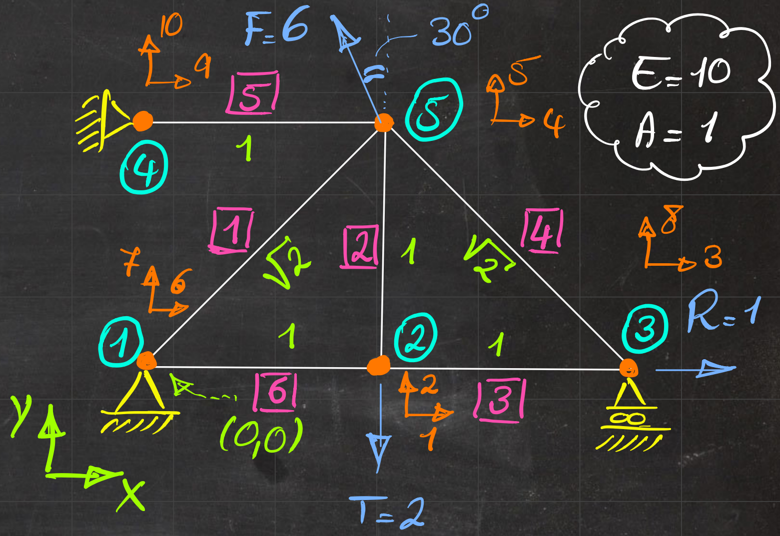
BETWEEN 1 - 2

$$\mathbb{K} = \frac{EA}{1}\begin{array}{c} \begin{array}{cccc} 6 & 7 & 1 & 2 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}\begin{array}{c} \\ 6 \\ 7 \\ 1 \\ 2 \end{array}$$

$E = 10$

$A = 1$

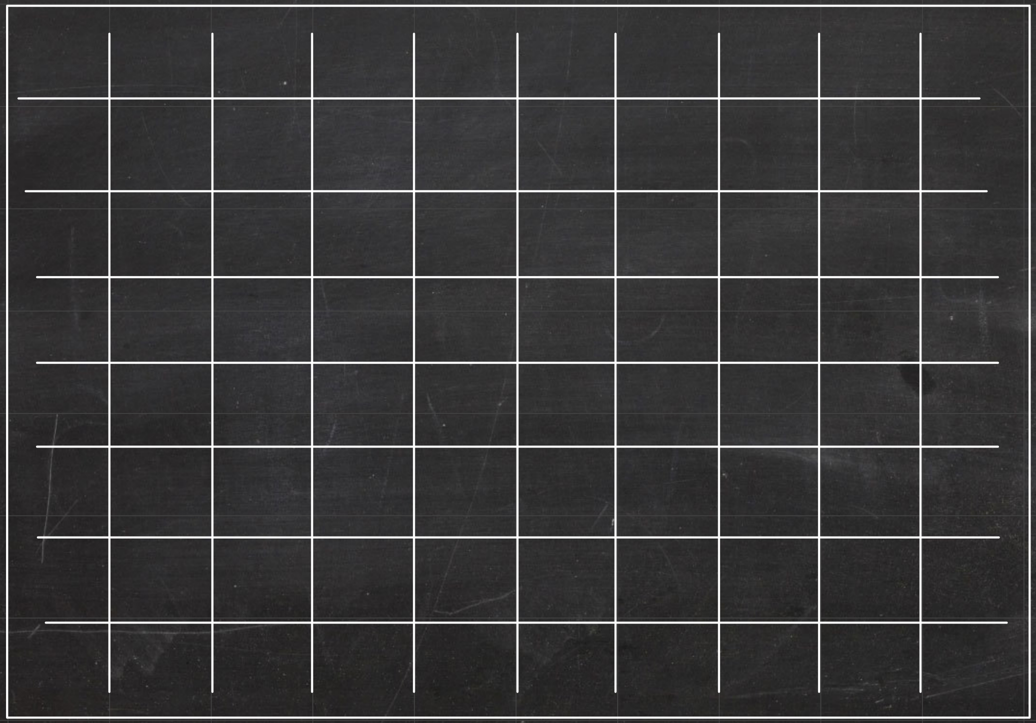
$L = 1$



E=10

A=1

$$\mathbb{K}_\theta = \frac{EA}{L}\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ \sin\theta\cos\theta & \sin^2\theta & -\sin\theta\cos\theta & -\sin^2\theta \\ -\cos^2\theta & -\cos\theta\sin\theta & \cos^2\theta & \cos\theta\sin\theta \\ -\sin\theta\cos\theta & -\sin^2\theta & \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$



$$
EA \begin{array}{c|cccccccccc|c}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \\
\hline
 & 1 & 0 & -1 & & & & & 0 & & & 1 \\
 & 0 & 0 & 0 & & & & & 0 & & & 2 \\
 & -1 & 0 & 1 & & & & & 0 & & & 3 \\
 & & & & & & & & & & & 4 \\
 & & & & & & & & & & & 5 \\
 & & & & & & & & & & & 6 \\
 & & & & & & & & & & & 7 \\
 & 0 & 0 & 0 & & & & & 0 & \cdots & & 8 \\
 & & & & & & & & & & & 9 \\
 & & & & & & & & & & & 10 \\
\hline
\end{array}
$$

$$
\mathbb{K}^3 = \frac{EA}{1}
\begin{array}{c}
\begin{array}{cccc} 1 & 2 & 3 & 8 \end{array} \\
\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{array}
\begin{array}{c} 1 \\ 2 \\ 3 \\ 8 \end{array}
$$

$$
EA\;
\begin{array}{c|cccccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline
1 & 1 & 0 & -1 & & & & & 0 & & \\
2 & 0 & 0 & 0 & & & & & 0 & & \\
3 & -1 & 0 & 1 & & & & & 0 & & \\
4 & & & & & & & & & & \\
5 & & & & & & & & & & \\
6 & & & & & & & & & & \\
7 & & & & & & & & & & \\
8 & 0 & 0 & 0 & & & & & 0 & \cdots & \\
9 & & & & & & & & & & \\
10 & & & & & & & & & &
\end{array}
$$

$$
\mathbb{K}^3 = \frac{EA}{1}
\begin{array}{c}
\begin{array}{cccc} 1 & 2 & 3 & 8 \end{array} \\
\left[\begin{array}{cccc}
1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{array}\right]
\end{array}
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 8 \end{array}
$$

$$
\begin{array}{c}
\begin{array}{cccc} 1 & 2 & 3 & 8 \end{array} \\
\left[\begin{array}{cccc}
K^G_{11} & K^G_{12} & K^G_{13} & K^G_{18} \\
K^G_{21} & K^G_{22} & K^G_{23} & K^G_{28} \\
 & & & \\
 & & &
\end{array}\right]
\end{array}
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 8 \end{array}
$$

STATIC CONDENSATION

$$\left[\begin{array}{c|c} K^{uu} & K^{up} \\ \hline K^{pu} & K^{pp} \end{array}\right] \left[\begin{array}{c} u^{u} \\ \hline u^{p} \end{array}\right] = \left[\begin{array}{c} F^{p} \\ \hline F^{u} \end{array}\right]$$

$$\Rightarrow \quad u^{u} = \checkmark$$