FINITE ELEMENT METHOD

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Ali Javili

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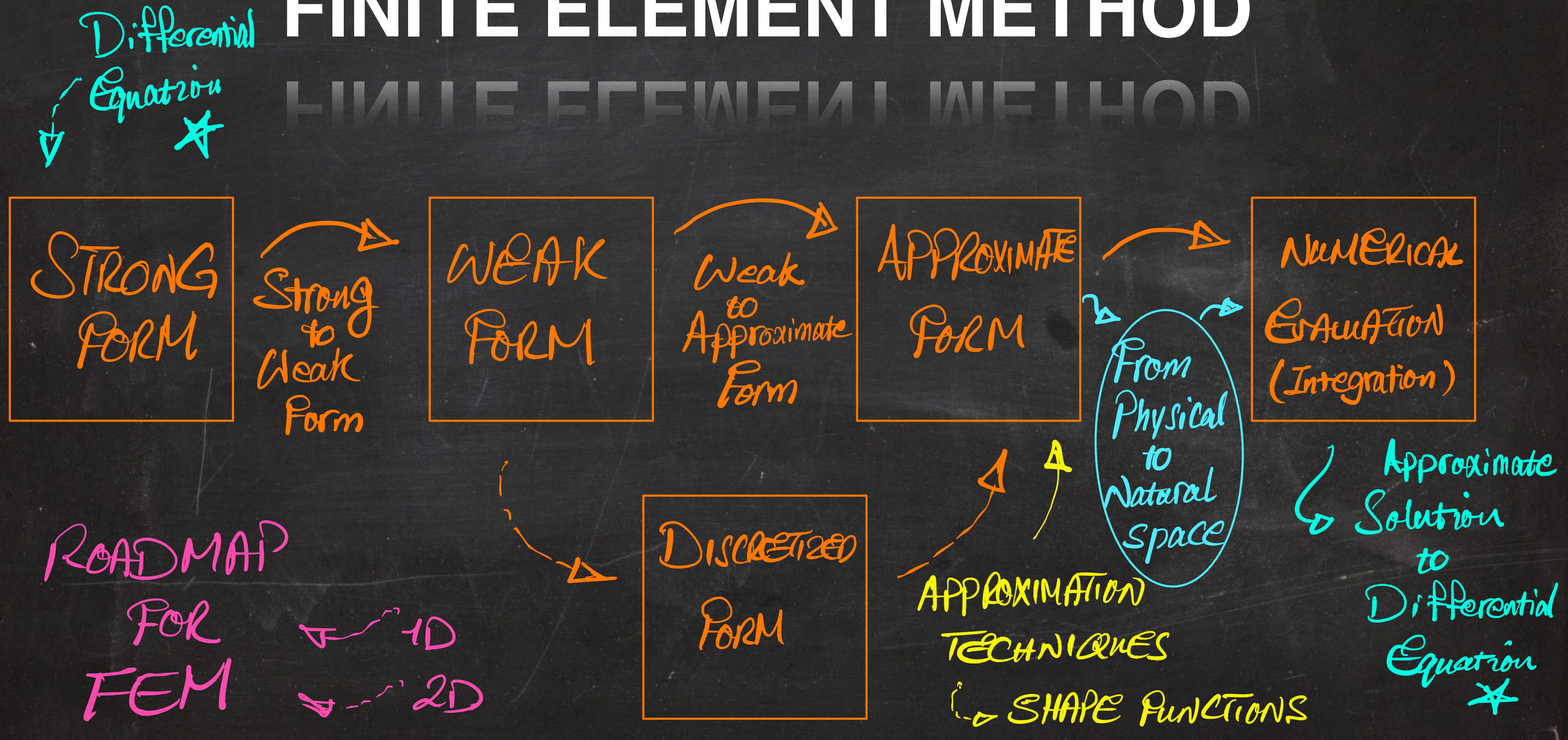
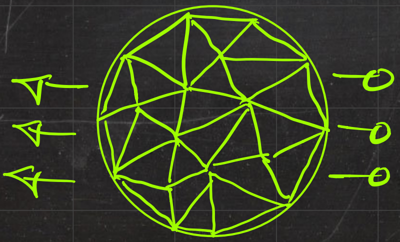
FINITE ELEMENT METHOD



UNDERSTANDING FEM VIA AN ANALOGY (A BRUTAL SIMPLIFICATION)

Approximations in FEM

- Solution Approximation → inherent to numerical techniques using computers
- Equation Approximation → diff equation is transformed to weak form + space approximation
- Input Approximation → space discretization



Discretization (Approximation)

- DOMAIN (X)
- SOLUTION (u)
- TEST (w)

diff. Eq.

STRONG FORM

TO

integral

WEAK FORM

FINITE ELEMENT METHOD IN A NUTSHELL

2ND. O.D.E.

Diff. Eq.

STRONG FORM

$(EAu')' + b = 0$

(I) MULTIPLY BY w

test function

(II) INTEGRATE

WEAK FORM

INTEGRAL FORM

$$\int_0^1 w' u' \, dx = \int_0^1 w \, dx + w(1)\,u'(1) - w(0)\,u'(0)$$

PIECEWISE

DISCRETIZED FORM

Approximation

Approximate Discretized Weak Form

APPROXIMATE FORM

NUMERICAL INTEGRATION

another source of approx...

ELEMENT-WISE QUANTITIES

PIECEWISE INTEGRALS (SOLUTIONS)

GLOBAL SYSTEM

$[\mathbb{K}][u] = [\mathbb{F}]$

ASSEMBLY

SOLVE

PostProcess

FROM GLOBAL TO ELEMENTS

FROM INTEGRAL OVER THE DOMAIN TO SUBINTEGRALS

$$\int_0^1 \dots \, dx = \int_0^a \dots \, dx + \int_a^b \dots \, dx + \dots$$

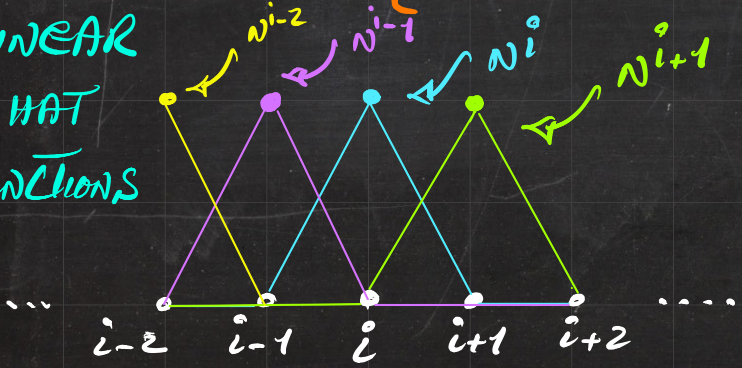
SHAPE FUNCTIONS (HAT FUNCTIONS, TENT FUNCTIONS)

↳ A powerful tool for approximations ⇝ SYSTEMATIC

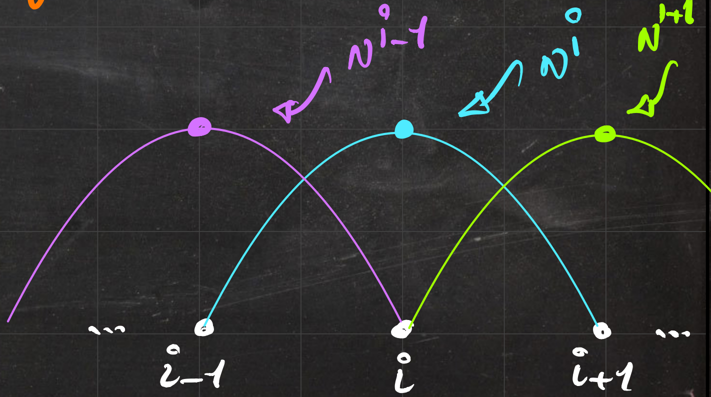
$$N^i(x) \rightsquigarrow \begin{cases} N^i = 1 \ @ \ x^j \ (j=i) \\ N^i = 0 \ @ \ x^j \ (j \neq i) \end{cases}$$

⇝ NEARLY IDENTICAL FOR 2D, 3D

LINEAR HAT FUNCTIONS



QUADRATIC HAT FUNCTIONS



SHAPE FUNCTIONS (HAT FUNCTIONS, TENT FUNCTIONS)

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⟶ NEARLY IDENTICAL FOR 2D / 3D

$$N^i(x) \rightsquigarrow \begin{cases} N^i = 1 \; @ \; x^j \; (j = i) \\ N^i = 0 \; @ \; x^j \; (j \neq i) \end{cases}$$

NODES PER ELEMENT → NPE

$$u \simeq \sum_{i=1}^{NPE} N^i u^i \to \begin{cases} u = N^1 u^1 + N^2 u^2 \\ u = N^1 u^1 + N^2 u^2 + N^3 u^3 \\ u = N^1 u^1 + N^2 u^2 + N^3 u^3 + N^4 u^4 \end{cases}$$

linear approximation

quadratic approximation

cubic approximation

GENERAL STRUCTURE OF STIFFNESS MATRIX FOR 1D FINITE ELEMENTS

NPE: Node Per Element

$$\mathbb{K} = EA \begin{bmatrix} \text{NPE} \times \text{NPE} \end{bmatrix}$$

[NPE x PD] x [NPE x PD]

	1D	2D
LINEAR TRUSS	2x2	4x4
QUADR. TRUSS	3x3	6x6

$$\mathbb{K} = EA \begin{bmatrix} \int_\alpha^\beta N^{1\prime} N^{1\prime}\, dx & \int_\alpha^\beta N^{1\prime} N^{2\prime}\, dx \\ \int_\alpha^\beta N^{2\prime} N^{1\prime}\, dx & \int_\alpha^\beta N^{2\prime} N^{2\prime}\, dx \end{bmatrix}$$

LINEAR TRUSS ELEMENT

$$\Rightarrow K^{ij} = EA \int_\alpha^\beta N^{i\prime} N^{j\prime}\, dx$$

GENERAL STRUCTURE OF STIFFNESS MATRIX FOR 1D FINITE ELEMENTS

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QUADRATIC TRUSS ELEMENT

$$K^{ij} = EA \int_\alpha^\beta N^{i\prime} N^{j\prime}\, dx$$

GENERAL STRUCTURE OF STIFFNESS MATRIX FOR 1D FINITE ELEMENTS

$$\mathbb{K} = EA \begin{bmatrix} NPE \times NPE \end{bmatrix}$$

NPE: Node Per Element

[NPE x PD] x [NPE x PD]

e.g.

	1D	2D
LINEAR TRUSS	2x2	4x4
QUADR. TRUSS	3x3	6x6

$N^i = N^i(x)$ $\quad N^j = N^j(x)$

EVALUATE THIS INTEGRAL

$$K^{ij} = EA \int_\alpha^\beta N^{i\prime} N^{j\prime} \, dx = EA \int_\alpha^\beta f(x)\, dx$$

$N^{i\prime} = N^{i\prime}(x)$ $\quad N^{j\prime} = N^{j\prime}(x)$

NUMERICALLY

NUMERICAL INTEGRATION :

GAUSS QUADRATURE FORMULA

↳ GAUSS POINTS

$$\int_{-1}^{1} g(\xi)\, d\xi = 2 \times g(0)$$

g: LINEAR

$$\int_{-1}^{1} g(\xi)\, d\xi = g\left(\frac{1}{\sqrt{3}}\right) + g\left(-\frac{1}{\sqrt{3}}\right)$$

g: QUADRATIC

$$\int_{-1}^{1} g(\xi)\, d\xi = g\left(\frac{1}{\sqrt{3}}\right) + g\left(-\frac{1}{\sqrt{3}}\right)$$

g: CUBIC

NUMERICAL INTEGRATION : ← QUADRATURE RULE ⟶ GAUSS POINTS

$$\int_{-1}^{1} g(\xi)\, d\xi = \sum_{i=1}^{GPE} \alpha_i \, g(\xi_i)$$

GPE ← NUMBER OF GAUSS POINTS PER ELEMENT

α_i ← weight factor

ξ_i: QUADRATURE POINTS

$$GPE \geq \frac{P+1}{2}$$

P ← ORDER OF POLYNOMIAL

	P	GPE	
LINEAR	1	1	⟶ $\xi_1 = 0$, $\alpha_1 = 2$
QUADRATIC	2	2	⟶ $\xi_1 = -1/\sqrt{3}$, $\alpha_1 = 1$; $\xi_2 = 1/\sqrt{3}$, $\alpha_2 = 1$
CUBIC	3	2	
4th O.	4	3	⟶ $\xi_1 = -\sqrt{0.6}$, $\alpha_1 = 5/9$; $\xi_2 = 0$, $\alpha_2 = 8/9$; $\xi_3 = \sqrt{0.6}$, $\alpha_3 = 5/9$
5th O.	5	3	

$$\sum_{i=1}^{GPE} \alpha_i = 2$$

NUMERICAL INTEGRATION:

$$\int_{-1}^{1} g(\xi)\, d\xi = \sum_{i=1}^{\mathrm{GPE}} \alpha_i\, g(\xi_i)$$

$$\int_a^b f(x)\, dx = ?$$ ↙ TRANSFORM TO $\int_{-1}^{1} g(\xi)\, d\xi$

x_DOMAIN ↙ PHYSICAL SPACE

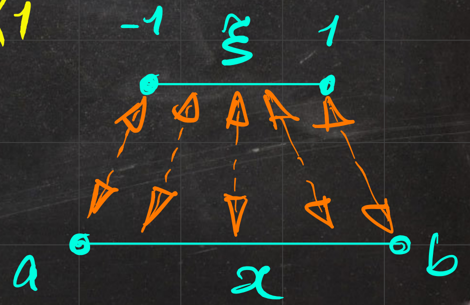
$a \leq x \leq b$

ξ_DOMAIN ↙ NATURAL SPACE

$-1 \leq \xi \leq 1$

MAPPING FROM PHYSICAL TO NATURAL SPACE

$$\hookrightarrow \quad x \overset{?}{=} x(\xi) \Rightarrow f(x) = f(x(\xi)) = g(\xi)$$



NUMERICAL INTEGRATION:

$$\int_{-1}^{1} g(\xi)\,d\xi = \sum_{i=1}^{GPE} \alpha_i\, g(\xi_i)$$

$$\int_a^b f(x)\,dx = \,? \quad \longleftarrow \quad \text{TRANSFORM TO} \quad \int_{-1}^{1} g(\xi)\,d\xi$$

$J = \dfrac{\partial x}{\partial \xi}$ Jacobian

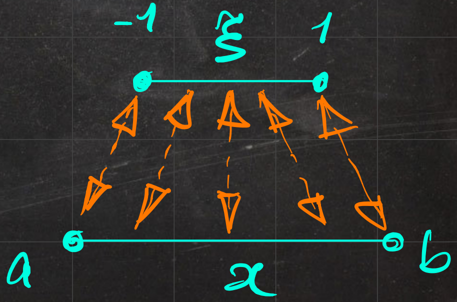
POLYNOMIAL ORDER

$$x \overset{?}{=} x(\xi) \;\Rightarrow\; x = \sum_{i=1}^{P+1} N^i x^i \quad \dots \quad dx = J(\xi)\,d\xi$$

$N^i = N^i(\xi)$

Shape Functions $\longrightarrow$ $\underbrace{u}_{u(x)} = \sum N^i u^i$ values

$N^i(x)$



NUMERICAL INTEGRATION:

$$\int_{-1}^{1} g(\xi)\, d\xi = \sum_{i=1}^{GPE} \alpha_i\, g(\xi_i)$$

$$\int_a^b f(x)\, dx = \, ?$$ TRANSFORM TO $\int_{-1}^{1} g(\xi)\, d\xi$

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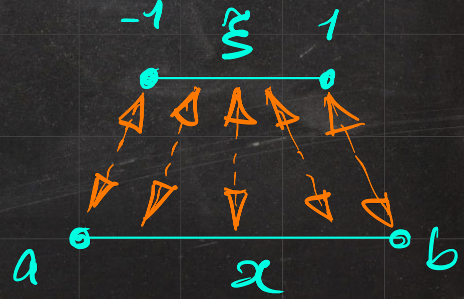
POLYNOMIAL ORDER

$$x \overset{?}{=} x(\xi) \Rightarrow x = \sum_{i=1}^{P+1} N^i x^i \quad \dots \quad dx = J(\xi)\, d\xi$$

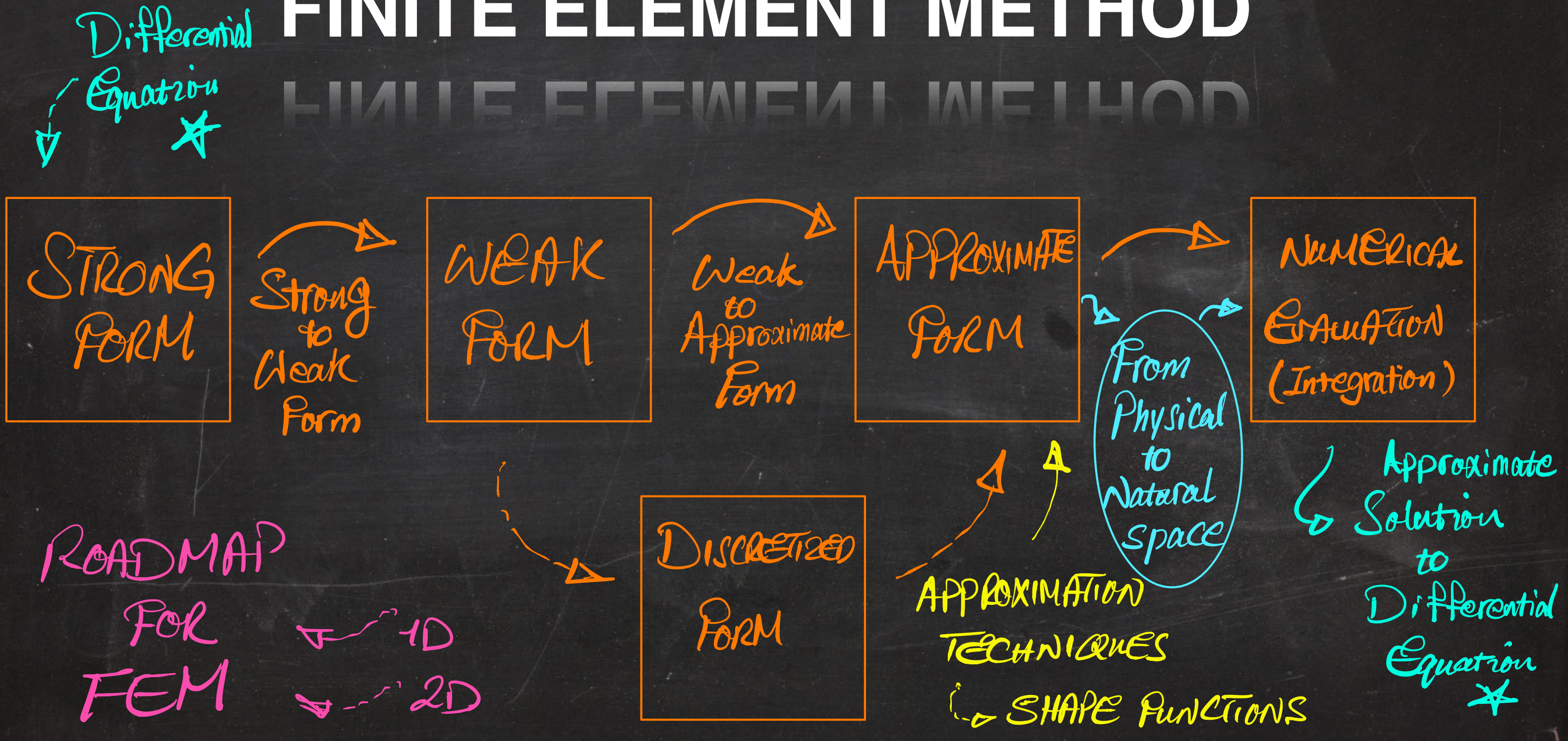
$N^i = N^i(\xi)$

$$\int_a^b f(x)\, dx = \int_{-1}^{1} \underbrace{f(\xi)\, J(\xi)}_{g(\xi)}\, dx = \sum_{i=1}^{GPE} \dots$$

$f(x(\xi))$



FINITE ELEMENT METHOD



1D FEM

Overview and Wrap-up

1D FEM

Overview and Wrap-up

WE FOCUS ON LINEAR APPROXIMATIONS FOR THE SAKE OF CLARITY

↳ immediately relevant to classical truss elements

EINSTEIN SUMMATION CONVENTION

A little definition for notation convenience

A REPEATED INDEX TWICE MEANS SUMMATION OVER THAT INDEX

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A little definition for notation convenience

A REPEATED INDEX TWICE MEANS SUMMATION OVER THAT INDEX

$$\sum_{i=1}^{3} u_i v_i = u_1 v_1 + u_2 v_2 + u_3 v_3 = u_i v_i$$

EINSTEIN SUMMATION CONVENTION

A little definition for notation convenience

A REPEATED INDEX TWICE MEANS SUMMATION OVER THAT INDEX

also, called "dummy index"

$$\sum_{i=1}^{3} u_i v_i = u_1 v_1 + u_2 v_2 + u_3 v_3 = u_i v_i$$

i is summation index

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$$\sum_{i=1}^{3} u_i v_i = u_1 v_1 + u_2 v_2 + u_3 v_3 = u_i v_i$$

i is summation index

$$\sum_{j=1}^{3} A_{ij} u_j \Rightarrow \begin{cases} i=1 \Rightarrow A_{11} u_1 + A_{12} u_2 + A_{13} u_3 \\ i=2 \Rightarrow A_{21} u_1 + A_{22} u_2 + A_{23} u_3 \\ i=3 \Rightarrow A_{31} u_1 + A_{32} u_2 + A_{33} u_3 \end{cases} \Rightarrow \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \equiv A_{ij} u_j$$

$1 \leq i \leq 3$

EINSTEIN SUMMATION CONVENTION

A little definition for notation convenience

A REPEATED INDEX TWICE MEANS SUMMATION OVER THAT INDEX

also, called "dummy index"

$$\sum_{i=1}^{3} u_i v_i = u_1 v_1 + u_2 v_2 + u_3 v_3 \equiv u_i v_i$$

i is summation index

i: free index

$$\sum_{j=1}^{3} A_{ij} u_j \Rightarrow \begin{cases} i=1 \Rightarrow A_{11} u_1 + A_{12} u_2 + A_{13} u_3 \\ i=2 \Rightarrow A_{21} u_1 + A_{22} u_2 + A_{23} u_3 \\ i=3 \Rightarrow A_{31} u_1 + A_{32} u_2 + A_{33} u_3 \end{cases} \Rightarrow \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \equiv A_{ij} u_j$$

$1 \leqslant i \leqslant 3$

j: summation index

SHAPE FUNCTIONS (LINEAR) ← Our key tool for approximations

APPROXIMATE SOLUTION (DISP.)

↳ $u = N^i u^i \longrightarrow u(x)$

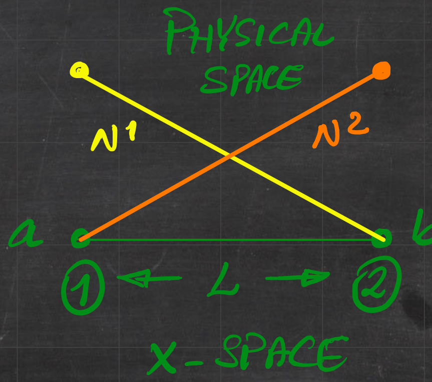
APPROXIMATE TEST FUNCTIONS

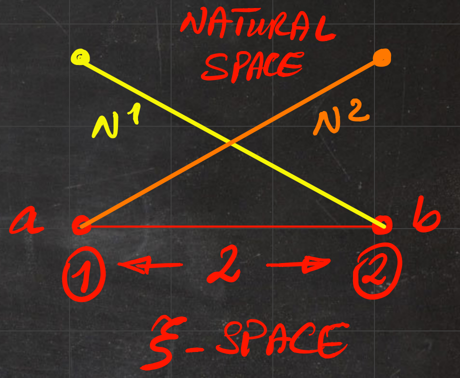
↳ $w = N^j w^j \longrightarrow w(x)$

APPROXIMATE THE SPACE ← PHYSICAL

↳ $x = N^k x^k \longrightarrow x(\xi)$

$\Rightarrow \{\circ\}' = \dfrac{\partial}{\partial x}\{\circ\}$

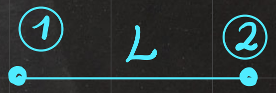




$$N^1 = -\frac{1}{L}[x-b] \qquad N^1 = -\frac{1}{2}[\xi-1]$$

$$N^2 = \frac{1}{L}[x-a] \qquad N^2 = \frac{1}{2}[\xi+1]$$

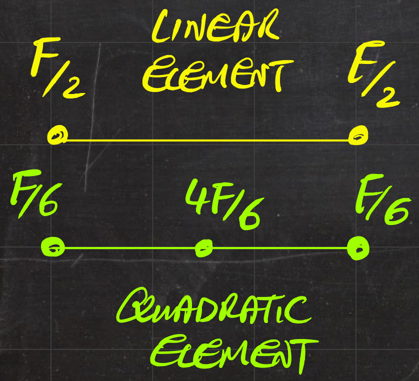
FROM STRONG FORM TO ELEMENT STIFFNESS ← IN PHYSICAL SPACE NOW



$$EAu'' + b = 0$$

↳ NEGLECT THE BODY FORCE DENSITY

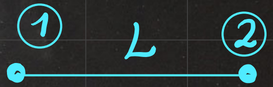
↳ CAN BE TRANSFORMED TO EFFECTIVE NODAL VALUES



↳ e.g. a uniform body force density

$$F = b_0 L \Leftarrow b_0$$

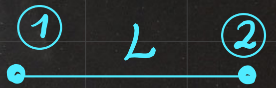
FROM STRONG FORM TO ELEMENT STIFFNESS $\leftarrow$ IN PHYSICAL SPACE NOW



$EA\,u'' = 0$ SUBJECT TO BCs

$\hookrightarrow \begin{cases} \text{DIRICHLET} \rightarrow u \text{ IS PRESCRIBED} \\ \text{NEUMANN} \rightarrow u' \text{ IS PRESCRIBED} \end{cases}$

FROM STRONG FORM TO ELEMENT STIFFNESS ← IN PHYSICAL SPACE NOW



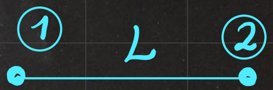
$EAu'' = 0$ SUBJECT TO BCs

- DIRICHLET → u IS PRESCRIBED
- NEUMANN → u' IS PRESCRIBED

MULTIPLY BY TEST FUNCTION ω

$$EA\omega u'' = 0$$

FROM STRONG FORM TO ELEMENT STIFFNESS ← IN PHYSICAL SPACE NOW



$EA\,u'' = 0$ SUBJECT TO BCs

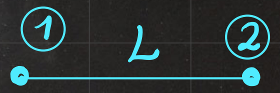
- DIRICHLET → u IS PRESCRIBED
- NEUMANN → u' IS PRESCRIBED

MULTIPLY BY TEST FUNCTION w

$$EA\,w\,u'' = 0 \quad \leftarrow \quad w u'' = (w u')' - w' u'$$

$$EA\left[(w u')' - w' u'\right] = 0 \quad \Rightarrow \quad EA\, w' u' = EA\,(w u')'$$

FROM STRONG FORM TO ELEMENT STIFFNESS IN PHYSICAL SPACE NOW



$EAu'' = 0$ SUBJECT TO BCs

- DIRICHLET $\to u$ IS PRESCRIBED
- NEUMANN $\to u'$ IS PRESCRIBED

MULTIPLY BY TEST FUNCTION w

$$EAwu'' = 0 \quad \leftarrow \quad wu'' = (wu')' - w'u'$$

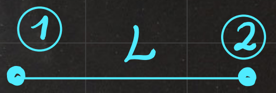
$$EA\left[(wu')' - w'u'\right] = 0 \Rightarrow EA\,w'u' = EA\,(wu')'$$

INTEGRATE

$$\int_L EA\,w'u'\,dx = \int_L EA\,(wu')'\,dx = EA\,wu'\Big]_{1}^{2} = EA\,w^2u'^2 - EA\,w^1u'^1$$

FROM STRONG FORM TO ELEMENT STIFFNESS IN PHYSICAL SPACE now

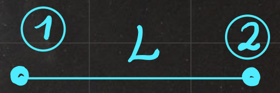
$EA\,u'' = 0$ SUBJECT TO BCs



$$\int_L EA\, w' u' \, dx = \int_L EA\,(w u')' \, dx = EA\, w u' \Big]_{1}^{2} = EA\, w^2 u'^2 - EA\, w^1 u'^1$$

FROM STRONG FORM TO ELEMENT STIFFNESS IN PHYSICAL SPACE NOW

$EA\,u'' = 0$ SUBJECT TO BCs



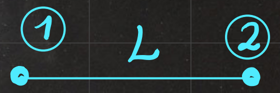
$$\int_L EA\, w' u' \, dx = \int_L EA\, (w u')' \, dx = EA\, w u' \Big]_{1}^{2} = EA\, w^2 u'^2 - EA\, w^1 u'^1$$

$$u = N^i u^i \Rightarrow u' = N^{i\prime} u^i \qquad w = N^j w^j \Rightarrow w' = N^{j\prime} w^j$$

$$\int_L EA\, N^{j\prime} w^j N^{i\prime} u^i \, dx = EA\, w^2 u'^2 - EA\, w^1 u'^1$$

FROM STRONG FORM TO ELEMENT STIFFNESS ← IN PHYSICAL SPACE NOW

$EAu'' = 0$ SUBJECT TO BCs



$$\int_L EA\, w' u'\, dx = \int_L EA\, (w u')'\, dx = EA\, w u' \Big]_{①}^{②} = EA\, w^2 u'^2 - EA\, w^1 u'^1$$

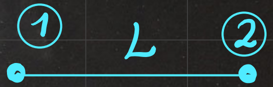
$$u = N^i u^i \Rightarrow u' = N^{i\prime} u^i \qquad w = N^j w^j \Rightarrow w' = N^{j\prime} w^j$$

$$EAu' = EA\varepsilon = A\, E\varepsilon = A\sigma = F$$

$$\int_L EA\, N^{j\prime} w^j\, N^{i\prime} u^i\, dx = \underbrace{EA\, w^2 u'^2}_{F^2} - \underbrace{EA\, w^1 u'^1}_{F^1} = w^2 F^2 + w^1 F^1$$

FROM STRONG FORM TO ELEMENT STIFFNESS ← IN PHYSICAL SPACE NOW

$EA\,u'' = 0$ SUBJECT TO BCs



$$\int_L EA\, N^{j\prime} w^j\, N^{i\prime} u^i \, dx = w^2 F^2 + w^1 F^1$$

FROM STRONG FORM TO ELEMENT STIFFNESS IN PHYSICAL SPACE NOW

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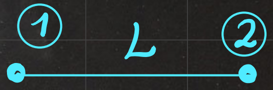


$$\int_L EA\, N^{j\prime} w^j\, N^{i\prime} u^i \,dx = w^2 F^2 + w^1 F^1$$

$$w^j \left[\int_L EA\, N^{j\prime} N^{i\prime} \,dx \right] u^i = w^2 F^2 + w^1 F^1$$

FROM STRONG FORM TO ELEMENT STIFFNESS   IN PHYSICAL SPACE NOW

$EAu'' = 0$ SUBJECT TO BCs

$$\int_L EA\, N^{j\prime} w^j\, N^{i\prime} u^i \, dx = w^2 F^2 + w^1 F^1$$

$$w^j \left[\int_L EA\, N^{j\prime} N^{i\prime} \, dx \right] u^i = w^2 F^2 + w^1 F^1$$

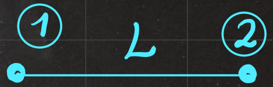
w: ARBITRARY

e.g. $w^1 = 1$, $w^2 = 0$

e.g. $w^1 = 0$, $w^2 = 1$

FROM STRONG FORM TO ELEMENT STIFFNESS — IN PHYSICAL SPACE NOW

$EAu'' = 0$ SUBJECT TO BCs



$$\int_L EA\, N^{j\prime} w^j\, N^{i\prime} u^i \, dx = w^2 F^2 + w^1 F^1$$

$$w^j \left[\int_L EA\, N^{j\prime} N^{i\prime} \, dx \right] u^i = w^2 F^2 + w^1 F^1$$

w: ARBITRARY

$$\left[\int_L EA\, N^{1\prime} N^{i\prime} \, dx \right] u^i = F^1 \quad \Leftarrow$$

e.g. $w^1 = 1,\ w^2 = 0$

e.g. $w^1 = 0,\ w^2 = 1$

FROM STRONG FORM TO ELEMENT STIFFNESS $\leftarrow$ IN PHYSICAL SPACE NOW

$EA\,u'' = 0$ SUBJECT TO BCs

$$\int_L EA\, N^{j\prime} w^j\, N^{i\prime} u^i \, dx = w^2 F^2 + w^1 F^1$$

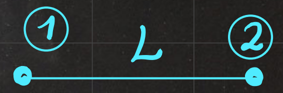
$$w^j \left[\int_L EA\, N^{j\prime} N^{i\prime} \, dx \right] u^i = w^2 F^2 + w^1 F^1 \quad \leftarrow \quad w: \text{ ARBITRARY}$$

$$\left[\int_L EA\, N^{1\prime} N^{i\prime} \, dx \right] u^i = F^1 \quad \Leftarrow \quad \text{e.g. } w^1 = 1,\ w^2 = 0$$

$$\left[\int_L EA\, N^{2\prime} N^{i\prime} \, dx \right] u^i = F^2 \quad \Leftarrow \quad \text{e.g. } w^1 = 0,\ w^2 = 1$$

FROM STRONG FORM TO ELEMENT STIFFNESS IN PHYSICAL SPACE NOW

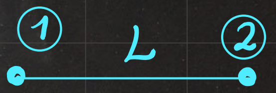
$EAu'' = 0$ SUBJECT TO BCs



$$\left[\int_L EA\, N^{1\prime} N^{i\prime}\, dx\right] u^i = F^1$$
$$\left[\int_L EA\, N^{2\prime} N^{i\prime}\, dx\right] u^i = F^2$$

FROM STRONG FORM TO ELEMENT STIFFNESS IN PHYSICAL SPACE NOW

$EA u'' = 0$ SUBJECT TO BCs



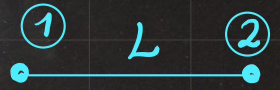
$$\left[\int_L EA N^{1\prime} N^{i\prime} dx \right] u^i = F^1$$

$$\left[\int_L EA N^{2\prime} N^{i\prime} dx \right] u^i = F^2$$

$$\int_L EA N^{1\prime} N^{1\prime} dx\, u^1 + \int_L EA N^{1\prime} N^{2\prime} dx\, u^2 = F^1$$

FROM STRONG FORM TO ELEMENT STIFFNESS ← IN PHYSICAL SPACE NOW

$EAu'' = 0$ SUBJECT TO BCs



$$\left[\int_L EA\, N^{1\prime} N^{i\prime}\, dx\right] u^i = F^1$$

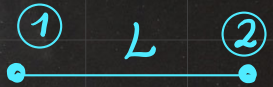
$$\left[\int_L EA\, N^{2\prime} N^{i\prime}\, dx\right] u^i = F^2$$

$$\int_L EA\, N^{1\prime} N^{1\prime}\, dx\, u^1 + \int_L EA\, N^{1\prime} N^{2\prime}\, dx\, u^2 = F^1$$

$$\int_L EA\, N^{2\prime} N^{1\prime}\, dx\, u^1 + \int_L EA\, N^{2\prime} N^{2\prime}\, dx\, u^2 = F^2$$

FROM STRONG FORM TO ELEMENT STIFFNESS IN PHYSICAL SPACE NOW

$EA u'' = 0$ SUBJECT TO BCs

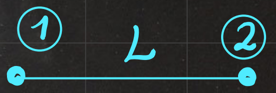


$$\int_L EA N^{1\prime} N^{1\prime} dx \, u^1 + \int_L EA N^{1\prime} N^{2\prime} dx \, u^2 = F^1$$

$$\int_L EA N^{2\prime} N^{1\prime} dx \, u^1 + \int_L EA N^{2\prime} N^{2\prime} dx \, u^2 = F^2$$

FROM STRONG FORM TO ELEMENT STIFFNESS IN PHYSICAL SPACE NOW

$EA\,u'' = 0$ SUBJECT TO BCs



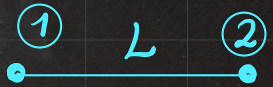
$$\int_L EA\,N^{1\prime} N^{1\prime}\,dx\; u^1 + \int_L EA\,N^{1\prime} N^{2\prime}\,dx\; u^2 = F^1$$

$$\int_L EA\,N^{2\prime} N^{1\prime}\,dx\; u^1 + \int_L EA\,N^{2\prime} N^{2\prime}\,dx\; u^2 = F^2$$

$$EA \begin{bmatrix} \int_L N^{1\prime} N^{1\prime}\,dx & \int_L N^{1\prime} N^{2\prime}\,dx \\ \int_L N^{2\prime} N^{1\prime}\,dx & \int_L N^{2\prime} N^{2\prime}\,dx \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} = \begin{bmatrix} F^1 \\ F^2 \end{bmatrix}$$

FROM STRONG FORM TO ELEMENT STIFFNESS IN PHYSICAL SPACE NOW

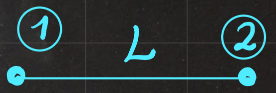
$EA\,u'' = 0$ SUBJECT TO BCs



$$EA \begin{bmatrix} \int_L N^{1\prime} N^{1\prime}\, dx & \int_L N^{1\prime} N^{2\prime}\, dx \\ \int_L N^{2\prime} N^{1\prime}\, dx & \int_L N^{2\prime} N^{2\prime}\, dx \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} = \begin{bmatrix} F^1 \\ F^2 \end{bmatrix}$$

FROM STRONG FORM TO ELEMENT STIFFNESS IN PHYSICAL SPACE NOW

$EA\,u'' = 0$ SUBJECT TO BCs



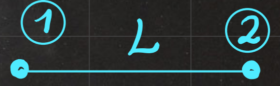
$$EA \begin{bmatrix} \int_L N^{1\prime} N^{1\prime}\,dx & \int_L N^{1\prime} N^{2\prime}\,dx \\ \int_L N^{2\prime} N^{1\prime}\,dx & \int_L N^{2\prime} N^{2\prime}\,dx \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} = \begin{bmatrix} F^1 \\ F^2 \end{bmatrix}$$

NEXT, WE WRITE $x = x(\xi)$ USING SHAPE FUNCTIONS.

(ISOPARAMETRIC CONCEPT)

FROM STRONG FORM TO ELEMENT STIFFNESS ← IN PHYSICAL SPACE NOW

$EA\,u'' = 0$ SUBJECT TO BCs



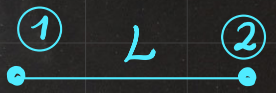
$$EA \begin{bmatrix} \int_L N^{1\prime} N^{1\prime}\, dx & \int_L N^{1\prime} N^{2\prime}\, dx \\ \int_L N^{2\prime} N^{1\prime}\, dx & \int_L N^{2\prime} N^{2\prime}\, dx \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} = \begin{bmatrix} F^1 \\ F^2 \end{bmatrix}$$

↳ NEXT, WE WRITE $x = x(\xi)$ USING SHAPE FUNCTIONS. (ISOPARAMETRIC CONCEPT)

↳ ALTERNATIVELY, WE COULD HAVE WRITTEN THE SHAPE FUNCTIONS IN NATURAL SPACE ξ FROM SCRATCH!

FROM STRONG FORM TO ELEMENT STIFFNESS IN PHYSICAL SPACE NOW

$EA\,u'' = 0$ SUBJECT TO BCs



$$EA \begin{bmatrix} \int_L N^{1\prime} N^{1\prime}\, dx & \int_L N^{1\prime} N^{2\prime}\, dx \\ \int_L N^{2\prime} N^{1\prime}\, dx & \int_L N^{2\prime} N^{2\prime}\, dx \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} = \begin{bmatrix} F^1 \\ F^2 \end{bmatrix}$$

$$\begin{bmatrix} K^{11} & K^{12} \\ K^{21} & K^{22} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} = \begin{bmatrix} F^1 \\ F^2 \end{bmatrix} \leftarrow K^{ij} = EA \int_L N^{i\prime} N^{j\prime}\, dx$$

$$K^{ij} = EA \int_L N^{i\prime} N^{j\prime} \, dx$$

$$K^{ij} = EA \int_L N^{i\prime} N^{j\prime} \, dx = EA \int_L \frac{\partial N^i}{\partial x} \frac{\partial N^j}{\partial x} \, dx$$

$$K^{ij} = EA \int_L N^{i\prime} N^{j\prime} \, dx = EA \int_L \frac{\partial N^i}{\partial x} \frac{\partial N^j}{\partial x} \, dx$$

$$x = x(\xi) \Rightarrow dx = \frac{\partial x}{\partial \xi} d\xi \Rightarrow dx = J \, d\xi \quad \text{with } J = \frac{\partial x}{\partial \xi}$$

$$K^{ij} = EA\int_L N^{i\prime} N^{j\prime}\,dx = EA\int_L \frac{\partial N^i}{\partial x}\frac{\partial N^j}{\partial x}\,dx \quad J\,d\xi$$

$$x = x(\xi) \Rightarrow dx = \frac{\partial x}{\partial \xi}\,d\xi \Rightarrow dx = J\,d\xi \quad \text{with } J = \frac{\partial x}{\partial \xi}$$

$$= EA\int_{-1}^{1} \frac{\partial N^i}{\partial x}\frac{\partial N^j}{\partial x}\,J\,d\xi$$

$$K^{ij} = EA \int_L N^{i\prime} N^{j\prime}\, dx = EA \int_L \frac{\partial N^i}{\partial x} \frac{\partial N^j}{\partial x}\, dx$$

$$J\, d\xi$$

$$x = x(\xi) \Rightarrow dx = \frac{\partial x}{\partial \xi}\, d\xi \Rightarrow dx = J\, d\xi \quad \text{with } J = \frac{\partial x}{\partial \xi}$$

$$= EA \int_{-1}^{1} \frac{\partial N^i}{\partial x} \frac{\partial N^j}{\partial x}\, J\, d\xi$$

$$\frac{\partial N^i}{\partial x} = \frac{\partial N^i}{\partial \xi} \frac{\partial \xi}{\partial x} = \frac{\partial N^i}{\partial \xi} J^{-1}$$

$$K^{ij} = EA\int_L N^{i\prime} N^{j\prime}\, dx = EA\int_L \frac{\partial N^i}{\partial x}\frac{\partial N^j}{\partial x}\, dx$$

$$J\, d\xi$$

$$x = x(\xi) \Rightarrow dx = \frac{\partial x}{\partial \xi} d\xi \Rightarrow dx = J\, d\xi \text{ with } J = \frac{\partial x}{\partial \xi}$$

$$= EA\int_{-1}^{1} \frac{\partial N^i}{\partial x}\frac{\partial N^j}{\partial x} J\, d\xi$$

$$\frac{\partial N^i}{\partial x} = \frac{\partial N^i}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{\partial N^i}{\partial \xi} J^{-1}$$

$$\frac{\partial N^j}{\partial x} = \frac{\partial N^j}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{\partial N^j}{\partial \xi} J^{-1}$$

$$K^{ij} = EA\int_L N^{i\prime} N^{j\prime}\, dx = EA\int_L \frac{\partial N^i}{\partial x}\frac{\partial N^j}{\partial x}\, dx \quad J\, d\xi$$

$$x = x(\xi) \Rightarrow dx = \frac{\partial x}{\partial \xi}\, d\xi \Rightarrow dx = J\, d\xi \quad \text{with } J = \frac{\partial x}{\partial \xi}$$

$$= EA\int_{-1}^{1} \frac{\partial N^i}{\partial x}\frac{\partial N^j}{\partial x}\, J\, d\xi \qquad \frac{\partial N^i}{\partial x} = \frac{\partial N^i}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{\partial N^i}{\partial \xi} J^{-1}$$

$$= EA\int_{-1}^{1} \frac{\partial N^i}{\partial \xi}\frac{\partial N^j}{\partial \xi}\, J^{-1} J^{-1} J\, d\xi \qquad \frac{\partial N^j}{\partial x} = \frac{\partial N^j}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{\partial N^j}{\partial \xi} J^{-1}$$

$$K^{ij} = EA\int_L N^{i\prime} N^{j\prime}\, dx = EA\int_L \frac{\partial N^i}{\partial x}\frac{\partial N^j}{\partial x}\, \underbrace{dx}_{J\,d\xi}$$

$$x = x(\xi) \Rightarrow dx = \frac{\partial x}{\partial \xi}\, d\xi \Rightarrow dx = J\, d\xi \quad \text{with } J = \frac{\partial x}{\partial \xi}$$

$$= EA\int_{-1}^{1} \frac{\partial N^i}{\partial x}\frac{\partial N^j}{\partial \xi}\, J\, d\xi \qquad \frac{\partial N^i}{\partial x} = \frac{\partial N^i}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{\partial N^i}{\partial \xi} J^{-1}$$

$$= EA\int_{-1}^{1} \frac{\partial N^i}{\partial \xi}\frac{\partial N^j}{\partial \xi}\, J^{-1}\underbrace{J^{-1} J}_{1}\, d\xi \qquad \frac{\partial N^j}{\partial x} = \frac{\partial N^j}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{\partial N^j}{\partial \xi} J^{-1}$$

$$K^{ij} = EA\int_L N^{i\prime} N^{j\prime}\, dx = EA\int_L \frac{\partial N^i}{\partial x}\frac{\partial N^j}{\partial x}\, dx \quad J\, d\xi$$

$$x = x(\xi) \Rightarrow dx = \frac{\partial x}{\partial \xi}\, d\xi \Rightarrow dx = J\, d\xi \quad \text{with } J = \frac{\partial x}{\partial \xi}$$

$$= EA\int_{-1}^{1} \frac{\partial N^i}{\partial x}\frac{\partial N^j}{\partial x}\, J\, d\xi = EA\int_{-1}^{1} \frac{\partial N^i}{\partial \xi}\frac{\partial N^j}{\partial \xi}\, J^{-1}\, d\xi$$

$$K^{ij} = EA \int_L N^{i\prime} N^{j\prime} \, dx = EA \int_L \frac{\partial N^i}{\partial x} \frac{\partial N^j}{\partial x} \, dx$$

$J \, d\xi$

$$x = x(\xi) \Rightarrow dx = \frac{\partial x}{\partial \xi} d\xi \Rightarrow dx = J \, d\xi \quad \text{with } J = \frac{\partial x}{\partial \xi}$$

$$= EA \int_{-1}^{1} \frac{\partial N^i}{\partial x} \frac{\partial N^j}{\partial x} J \, d\xi = EA \int_{-1}^{1} \underbrace{\frac{\partial N^i}{\partial \xi} \frac{\partial N^j}{\partial \xi} J^{-1}}_{g(\xi)} \, d\xi$$

NUMERICAL INTEGRATION USING GAUSS QUADRATURE

$$\int_{-1}^{1} g(\xi) \, d\xi$$

$$K^{ij} = EA \int_L N^{i\prime} N^{j\prime} \, dx$$ PHYSICAL

RECALL:

$$\int_{-1}^{1} g(\xi)\, d\xi = \sum_{gp=1}^{GPE} g(\xi_{gp})\, \alpha_{gp}$$

NATURAL

$$= EA \int_{-1}^{1} \frac{\partial N^i}{\partial \xi} \frac{\partial N^j}{\partial \xi} J^{-1} \, d\xi$$

$$= EA \sum_{gp=1}^{GPE} \left\{ \left[\frac{\partial N^i}{\partial \xi} \frac{\partial N^j}{\partial \xi} J^{-1} \right]_{gp} \times \alpha_{gp} \right\}$$

Loop over gp

⋮

END

e.g. in MATLAB

WHAT YOU SEE IN THE CODE ?

For gp=1:GPE

⋮

End

FINITE ELEMENT METHOD IN A NUTSHELL

2ND. O.D.E.

Diff. Eq.

$\rightarrow (EAu')' + b = 0$

STRONG FORM → (I) MULTIPLY BY w (test function) (II) INTEGRATE → **WEAK FORM**

INTEGRAL FORM

$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx + w(1)u'(1) - w(0)u'(0)$$

PIECEWISE → **DISCRETIZED FORM**

DISCRETIZED FORM → Approximation → **APPROXIMATE FORM**

Approximate Discretized Weak Form

APPROXIMATE FORM → NUMERICAL INTEGRATION → **ELEMENT-WISE QUANTITIES**

another source of approx...

PIECEWISE INTEGRALS (SOLUTIONS)

GLOBAL SYSTEM

$[K][u] = [F]$

ASSEMBLY

FROM GLOBAL TO ELEMENTS

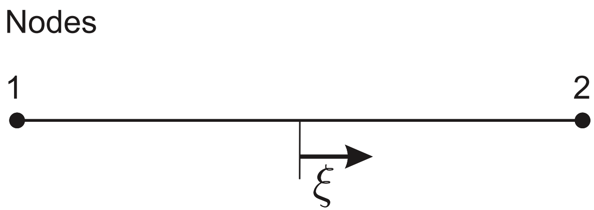
→ **SOLVE** → PostProcess

FROM INTEGRAL OVER THE DOMAIN TO SUBINTEGRALS

$$\int_0^1 \dots dx = \int_0^a \dots dx + \int_a^b \dots dx + \dots$$

- one-dimensional 2-noded element (D1CU2N)
- one-dimensional 3-noded element (D1CU3N)
- one-dimensional 4-noded element (D1CU4N)
- one-dimensional 5-noded element (D1CU5N)
- one-dimensional 6-noded element (D1CU6N)
- one-dimensional 7-noded element (D1CU7N)
- one-dimensional quadrature rule
- one-dimensional constant body force nodal distribution

one-dimensional 2-noded element (D1CU2N)



Node Number	Coordinate
	ξ
1	-1
2	1

$$N^1 = -\frac{1}{2}(\xi - 1) \quad , \quad N^2 = \frac{1}{2}(\xi + 1)$$

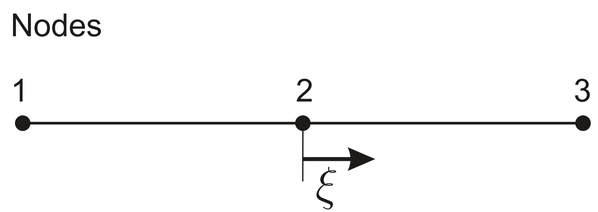
$$N^1_{,\xi} = -\frac{1}{2} \quad , \quad N^2_{,\xi} = \frac{1}{2}$$

1D Finite Element Library

one-dimensional 3-noded element (D1CU3N)

Numerical Analysis and High-level Applications Lab

NAHAL

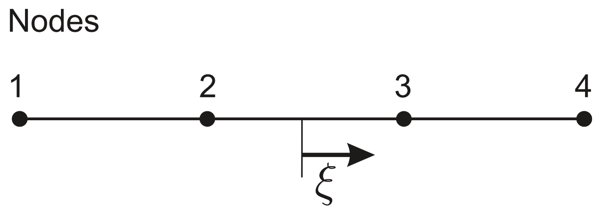


Node Number	Coordinate
	ξ
1	-1
2	0
3	1

$$N^1 = \frac{1}{2}\xi\left(\xi - 1\right) \quad , \quad N^2 = -\left(\xi - 1\right)\left(\xi + 1\right) \quad , \quad N^3 = \frac{1}{2}\xi\left(\xi + 1\right)$$

$$N^1_{,\xi} = \frac{1}{2}\left(2\xi - 1\right) \quad , \quad N^2_{,\xi} = -2\xi \quad , \quad N^1_{,\xi} = \frac{1}{2}\left(2\xi + 1\right)$$

one-dimensional 4-noded element (D1CU4N)



Node Number	Coordinate ξ
1	-1
2	-1/3
3	1/3
4	1

$$N^1 = -\frac{9}{16}\left(\xi + 1/3\right)\left(\xi - 1/3\right)\left(\xi - 1\right)$$

$$N^2 = \frac{27}{16}\left(\xi + 1\right)\left(\xi - 1/3\right)\left(\xi - 1\right)$$

$$N^3 = -\frac{27}{16}\left(\xi + 1\right)\left(\xi + 1/3\right)\left(\xi - 1\right)$$

$$N^4 = \frac{9}{16}\left(\xi + 1\right)\left(\xi + 1/3\right)\left(\xi - 1/3\right)$$

1D Finite Element Library

one-dimensional 5-noded element (D1CU5N)

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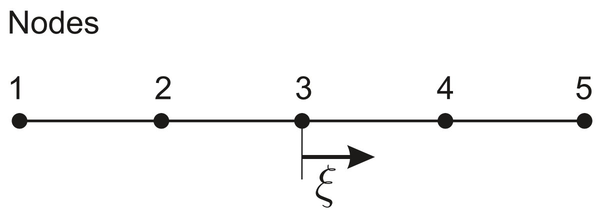


Node Number	Coordinate ξ
1	-1
2	-1/2
3	0
4	1/2
5	1

$$N^1 = \frac{2}{3}\left(\xi + 1/2\right)\xi\left(\xi - 1/2\right)\left(\xi - 1\right)$$

$$N^2 = -\frac{8}{3}\left(\xi + 1\right)\xi\left(\xi - 1/2\right)\left(\xi - 1\right)$$

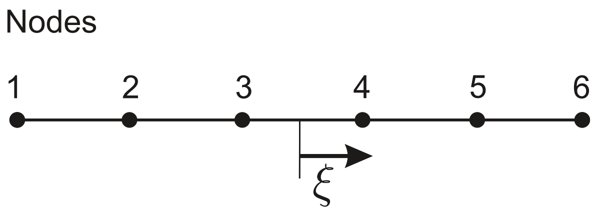
$$N^3 = 4\left(\xi + 1\right)\left(\xi + 1/2\right)\left(\xi - 1/2\right)\left(\xi - 1\right)$$

$$N^4 = -\frac{8}{3}\left(\xi + 1\right)\left(\xi + 1/2\right)\xi\left(\xi - 1\right)$$

$$N^5 = \frac{2}{3}\left(\xi + 1\right)\left(\xi + 1/2\right)\xi\left(\xi - 1/2\right)$$

1D Finite Element Library

one-dimensional 6-noded element (D1CU6N)



Node Number	Coordinate ξ
1	-1
2	-3/5
3	-1/5
4	1/5
5	3/5
6	1

$$N^1 = -\frac{625}{768}\left(\xi+3/5\right)\left(\xi+1/5\right)\left(\xi-1/5\right)\left(\xi-3/5\right)\left(\xi-1\right)$$

$$N^2 = \frac{3125}{768}\left(\xi+1\right)\left(\xi+1/5\right)\left(\xi-1/5\right)\left(\xi-3/5\right)\left(\xi-1\right)$$

$$N^3 = -\frac{3125}{384}\left(\xi+1\right)\left(\xi+3/5\right)\left(\xi-1/5\right)\left(\xi-3/5\right)\left(\xi-1\right)$$

$$N^4 = \frac{3125}{384}\left(\xi+1\right)\left(\xi+3/5\right)\left(\xi+1/5\right)\left(\xi-3/5\right)\left(\xi-1\right)$$

$$N^5 = -\frac{3125}{768}\left(\xi+1\right)\left(\xi+3/5\right)\left(\xi+1/5\right)\left(\xi-1/5\right)\left(\xi-1\right)$$

$$N^6 = \frac{625}{768}\left(\xi+1\right)\left(\xi+3/5\right)\left(\xi+1/5\right)\left(\xi-1/5\right)\left(\xi-3/5\right)$$

1D Finite Element Library

one-dimensional 7-noded element (D1CU7N)

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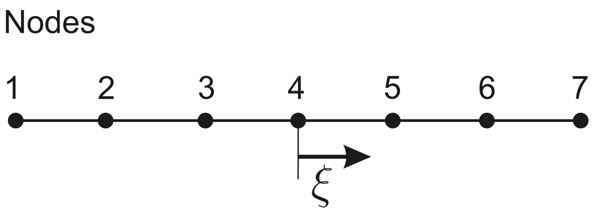

Node Number	Coordinate ξ
1	-1
2	-2/3
3	-1/3
4	0
5	1/3
6	2/3
7	1

$$N^1 = \frac{81}{80}\left(\xi + 2/3\right)\left(\xi + 1/3\right)\xi\left(\xi - 1/3\right)\left(\xi - 2/3\right)\left(\xi - 1\right)$$

$$N^2 = -\frac{243}{40}\left(\xi + 1\right)\left(\xi + 1/3\right)\xi\left(\xi - 1/3\right)\left(\xi - 2/3\right)\left(\xi - 1\right)$$

$$N^3 = \frac{243}{16}\left(\xi + 1\right)\left(\xi + 2/3\right)\xi\left(\xi - 1/3\right)\left(\xi - 2/3\right)\left(\xi - 1\right)$$

$$N^4 = -\frac{81}{4}\left(\xi + 1\right)\left(\xi + 2/3\right)\left(\xi + 1/3\right)\left(\xi - 1/3\right)\left(\xi - 2/3\right)\left(\xi - 1\right)$$

$$N^5 = \frac{243}{16}\left(\xi + 1\right)\left(\xi + 2/3\right)\left(\xi + 1/3\right)\xi\left(\xi - 2/3\right)\left(\xi - 1\right)$$

$$N^6 = -\frac{243}{40}\left(\xi + 1\right)\left(\xi + 2/3\right)\left(\xi + 1/3\right)\xi\left(\xi - 1/3\right)\left(\xi - 1\right)$$

$$N^7 = \frac{81}{80}\left(\xi + 1\right)\left(\xi + 2/3\right)\left(\xi + 1/3\right)\xi\left(\xi - 1/3\right)\left(\xi - 2/3\right)$$

one-dimensional quadrature rule (i)

$$\int_{-1}^{1} \{\bullet\}\, \mathrm{d}\xi \approx \sum_{i=1}^{N_{\text{Gauss Points}}} \alpha_i \, \{\bullet\}\big|_{\text{Gauss Point}^i}$$

Gauss Point Number	Coordinate ξ	Weight Factor α
1	0	2

Gauss Point Number	Coordinate ξ	Weight Factor α
1	$-1/\sqrt{3}$	1
2	$1/\sqrt{3}$	1

Gauss Point Number	Coordinate ξ	Weight Factor α
1	$-\sqrt{0.6}$	5/9
2	0	8/9
3	$\sqrt{0.6}$	5/9

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one-dimensional quadrature rule (ii)

Gauss Point Number	Coordinate ξ	Weight Factor α
1	-0.861 136 311 594 953	0.347 854 845 137 454
2	-0.339 981 043 584 856	0.652 145 154 862 546
3	0.339 981 043 584 856	0.652 145 154 862 546
4	0.861 136 311 594 953	0.347 854 845 137 454

Gauss Point Number	Coordinate ξ	Weight Factor α
1	-0.906 179 845 938 664	0.236 926 885 056 189
2	-0.538 469 310 105 683	0.478 628 670 499 366
3	0.000 000 000 000 000	0.568 888 888 888 889
4	0.538 469 310 105 683	0.478 628 670 499 366
5	0.906 179 845 938 664	0.236 926 885 056 189

Gauss Point Number	Coordinate ξ	Weight Factor α
1	-0.932 469 514 203 152	0.171 324 492 379 170
2	-0.661 209 386 466 265	0.360 761 573 048 139
3	-0.238 619 186 083 197	0.467 913 934 572 691
4	0.238 619 186 083 197	0.467 913 934 572 691
5	0.661 209 386 466 265	0.360 761 573 048 139
6	0.932 469 514 203 152	0.171 324 492 379 170

one-dimensional quadrature rule (iii)

Gauss Point Number	Coordinate ξ	Weight Factor α
1	-0.949 107 912 342 759	0.129 484 966 168 870
2	-0.741 531 185 599 394	0.279 705 391 489 277
3	-0.405 845 151 377 397	0.381 830 050 505 119
4	0.000 000 000 000 000	0.417 959 183 673 469
5	0.405 845 151 377 397	0.381 830 050 505 119
6	0.741 531 185 599 394	0.279 705 391 489 277
7	0.949 107 912 342 759	0.129 484 966 168 870

one-dimensional constant body force nodal distribution

$$f^i = \frac{\int_{-1}^{1} N^i \mathrm{d}\xi}{\int_{-1}^{1} \mathrm{d}\xi}$$

Node number	1	2
Weight factor	1/2	1/2

Node number	1	2	3
Weight factor	1/6	4/6	1/6

Node number	1	2	3	4
Weight factor	1/8	3/8	3/8	1/8

Node number	1	2	3	4	5
Weight factor	7/90	32/90	12/90	32/90	7/90

Node number	1	2	3	4	5	6
Weight factor	19/288	25/96	25/144	25/144	25/96	19/288

Node number	1	2	3	4	5	6	7
Weight factor	41/840	18/70	9/280	68/210	9/280	18/70	41/840