FINITE ELEMENT METHOD

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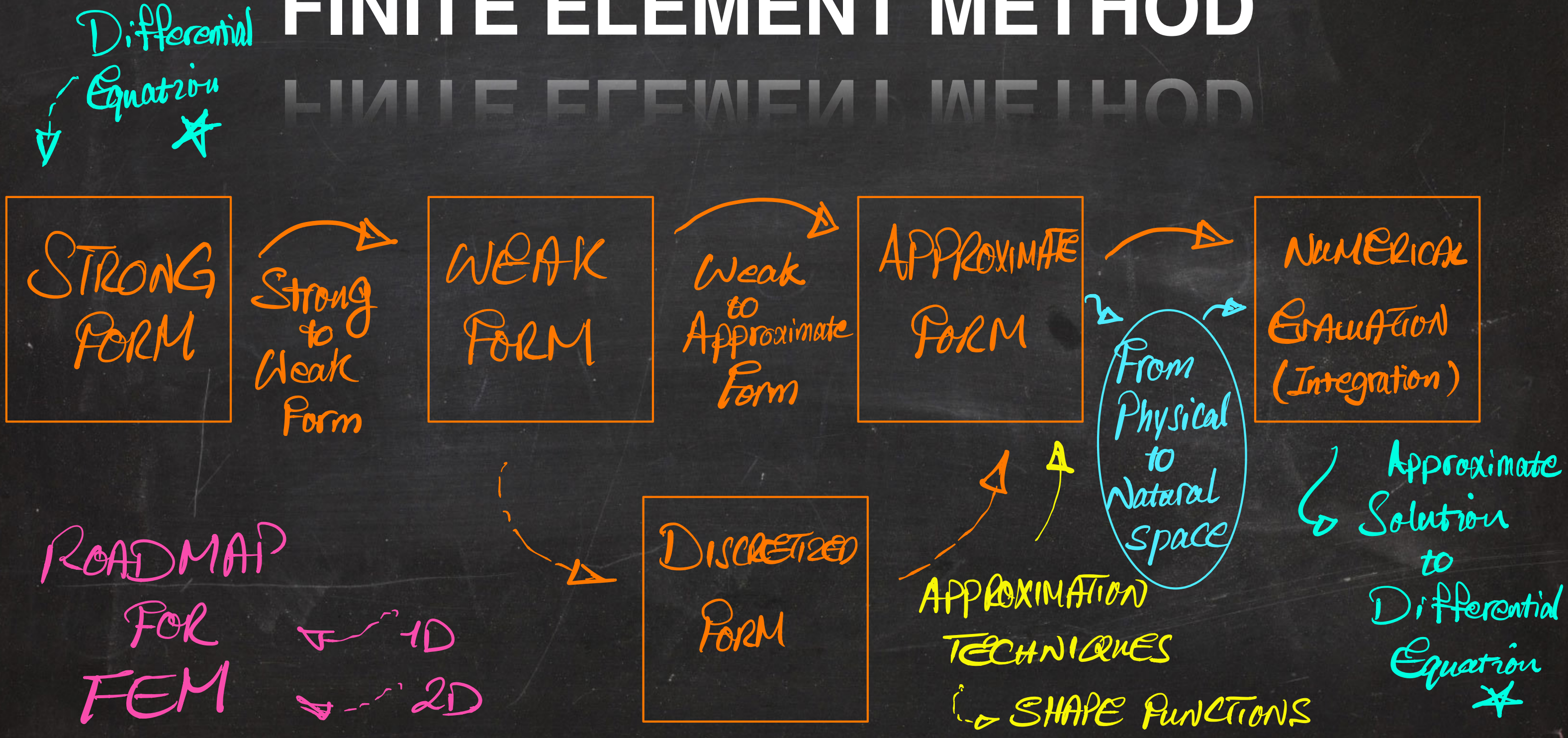
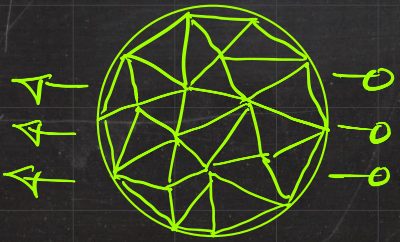


UNDERSTANDING FEM VIA AN ANALOGY (A BRUTAL SIMPLIFICATION)

Approximations in FEM

- → Solution Approximation → inherent to numerical techniques using computers
- → Equation Approximation → diff equation is transformed to weak form + space approximation
- → Input Approximation → space discretization



Discretization (Approximation)
- → DOMAIN (X)
- → SOLUTION (u)
- → TEST (w)

diff. Eq. → STRONG FORM

TO

integral → WEAK FORM

$$\frac{d}{dx}\left(EA\frac{du}{dx}\right)+bA=0 \quad \text{Subject to BCs}$$

$$\hookrightarrow E, A: \text{Const.} \quad \rightsquigarrow EAu'' + bA = 0 \quad \leftarrow f := \frac{b}{E}$$

$$\rightsquigarrow \boxed{\begin{array}{l} u'' + f = 0 \qquad 0 \leqslant x \leqslant 1 \\ D: u(0) = u_0 \\ N: u'(1) = t \end{array}}$$

STRONG FORM

prescribed

FROM STRONG TO WEAK FORM

$$u'' + f = 0 \qquad 0 \leq x \leq 1$$

$$D: \; u(0) = u_0$$

$$N: \; u'(1) = t$$

prescribed

STRONG FORM ← Differential Eq.

(I) MULTIPLY BY TEST FUNCTION w

(II) INTEGRATE OVER THE DOMAIN

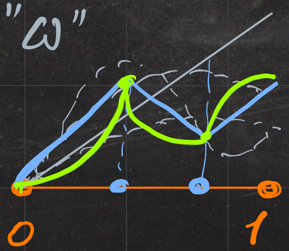
w :
- ARBITRARY
- CONTINUOUS
- $w|_D = 0$ ← ZERO @ DIRICHLET BOUNDARY CONDITIONS

Integral form ← WEAK FORM

→ BECAUSE LOWER ORDER DIFFERENTIATION OF DISPLACEMENT u

STRONG : u''

WEAK : u'



FROM STRONG TO WEAK FORM

$$u'' + f = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = u_0$ ← prescribed

N: $u'(1) = t$ ←

STRONG FORM

(I) MULTIPLY BY TEST FUNCTION w

(II) INTEGRATE OVER THE DOMAIN

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$$

I) $[u'' + f = 0] \times \omega \Rightarrow \omega u'' + \omega f = 0$

II) $\int_0^1 [\omega u'' + \omega f]\, dx = 0$ $\qquad \omega u'' = (\omega u')' - \omega' u'$

$$\int_0^1 (\omega u')'\, dx - \int_0^1 \omega' u'\, dx + \int_0^1 \omega f\, dx = 0$$

FROM STRONG TO WEAK FORM

$$u'' + f = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = u_0$ ← prescribed

N: $u'(1) = t$ ←

I) $[u'' + f = 0] \times \omega \Rightarrow \omega u'' + \omega f = 0$

II) $\int_0^1 [\omega u'' + \omega f]\, dx = 0$ $\qquad \omega u'' = (\omega u')' - \omega' u'$

$$\int_0^1 (\omega u')'\, dx - \int_0^1 \omega' u'\, dx + \int_0^1 \omega f\, dx = 0$$

$$\int_0^1 \omega' u'\, dx = \int_0^1 \omega f\, dx + \omega u' \Big]_0^1$$

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$$

$$\int_0^1 \omega' u'\, dx = \int_0^1 \omega f\, dx + \omega(1)\, u'(1) - \omega(0)\, u'(0)$$

FROM STRONG TO WEAK FORM

$$u'' + f = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = u_0 \leftarrow$ prescribed

N: $u'(1) = t \leftarrow$

I) $[u'' + f = 0] \times w \Rightarrow wu'' + wf = 0$

II) $\int_0^1 [wu'' + wf]\,dx = 0 \qquad wu'' = (wu')' - w'u'$

$$\int_0^1 w'u'\,dx = \int_0^1 wf\,dx + \overbrace{\underset{\substack{\uparrow \\ \text{TEST FUNCTION @ 1}}}{w(1)}\,u'(1) - \underset{\substack{\uparrow \\ \text{TEST FUNCTION @ 0}}}{w(0)}\,u'(0)}$$

w :
- ARBITRARY
- CONTINUOUS
- $w|_D = 0$

BC:

DIRICHLET	u ✓	u' ?
NEUMANN	u ?	u' ✓

FROM STRONG TO WEAK FORM

$$u'' + f = 0 \qquad 0 \leqslant x \leqslant 1$$

$$D: \; u(0) = u_0 \leftarrow \text{prescribed}$$

$$N: \; u'(1) = t \leftarrow$$

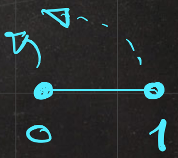
I) $[u'' + f = 0] \times \omega \Rightarrow \omega u'' + \omega f = 0$

II) $\int_0^1 [\omega u'' + \omega f]\, dx = 0$ $\qquad \omega u'' = (\omega u')' - \omega' u'$

WEAK FORM

$$\underbrace{\int_0^1 \omega' u' \, dx}_{\text{INTERNAL CONTRIBUTIONS OVER THE DOMAIN}} = \underbrace{\int_0^1 \omega f \, dx}_{\text{EXTERNAL CONTRIBUTIONS OVER THE DOMAIN}} + \underbrace{\omega(1)\, u'(1) - \omega(0)\, u'(0)}_{\text{EXTERNAL CONTRIBUTIONS OVER THE BOUNDARY OF THE DOMAIN}}$$

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$$



FROM STRONG TO WEAK FORM

$$u'' + 1 = 0 \qquad 0 \leq x \leq 1$$

$$D: \ u(0) = 0$$

$$N: \ u'(1) = 0$$

prescribed

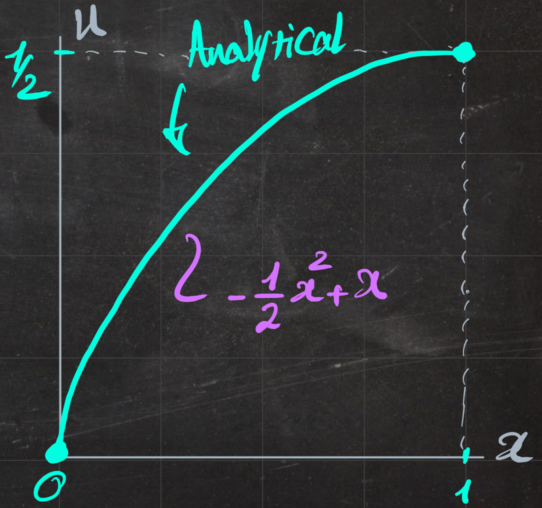
$$u'' = -1 \ \Rightarrow \ u' = -x + C_1$$

$$\Rightarrow u = -\frac{1}{2}x^2 + C_1 x + C_2$$

$$\hookrightarrow u(0) = 0 \Rightarrow C_2 = 0$$

$$\hookrightarrow u'(1) = 0 \Rightarrow C_1 = 1$$

Analytical Solution $\Rightarrow u = -\frac{1}{2}x^2 + x$



FROM STRONG TO WEAK FORM

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

$$\int_0^1 \omega' u' \, dx = \int_0^1 \omega f \, dx + \omega(1) u'(1) - \omega(0) u'(0)$$

$$\Rightarrow \int_0^1 \omega' u' \, dx = \int_0^1 \omega \, dx$$ ← WEAK FORM

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$$

↳ Compute approximate solution

↳ from different spaces

⇓

EXERCISE ↦ ...

UNDERSTANDING THE SIGNIFICANCE OF WEAK FORM BY EXAMPLE

$$\int_0^1 w' u' \, dx = \int_0^1 w \, dx$$

$$u'' + 1 = 0 \qquad 0 \leq x \leq 1$$

$D: \; u(0) = 0$ ← prescribed

$N: \; u'(1) = 0$ ↙

- → 1-PIECE LINEAR APPROXIMATION
- → 2-PIECE LINEAR (UNIFORM) APPROXIMATION
- → 2-PIECE LINEAR (NON-UNIFORM) APPROXIMATION (I)
- → 2-PIECE LINEAR (NON-UNIFORM) APPROXIMATION (II)
- → 2-PIECE LINEAR (GENERAL) APPROXIMATION

$$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$$

UNDERSTANDING THE SIGNIFICANCE OF WEAK FORM BY EXAMPLE

$$\int_0^1 \omega' u' dx = \int_0^1 \omega \, dx$$

$u'' + 1 = 0 \quad 0 \leq x \leq 1$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ← prescribed

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$$

- → 3-PIECE LINEAR (UNIFORM) APPROXIMATION
- → 3-PIECE LINEAR (GENERIC) APPROXIMATION
- → 4-PIECE LINEAR (UNIFORM) APPROXIMATION
- → 4-PIECE LINEAR (GENERIC) APPROXIMATION
- → 1-PIECE QUADRATIC
- → 1-PIECE CUBIC

$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \qquad x \in [0,1]$$

$u'' + 1 = 0 \qquad 0 \leq x \leq 1$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ↙

→ 1-PIECE LINEAR APPROXIMATION

$$w = C_1 x + C_2 \rightarrow C_2 = 0 \qquad u = D_1 x + D_2 \rightarrow D_2 = 0$$

$w(0) = 0 \Rightarrow$ $\quad w|_D = 0 \Leftarrow u(0)$ is GIVEN $\qquad u(0) = 0$

w : ARBITRARY, CONTINUOUS, $w|_D = 0$

$$\int_0^1 C_1 D_1\,dx = \int_0^1 C_1 x\,dx \Rightarrow C_1 D_1 x\Big]_0^1 = \frac{1}{2} C_1 x^2\Big]_0^1$$

$$\Rightarrow D_1 = \frac{1}{2}$$

C_1: Cancels out

$$\int_0^1 w'u'dx = \int_0^1 w\,dx \qquad x\in[0,1]$$

$$u'' + 1 = 0 \qquad 0 \leq x \leq 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

→ 1-PIECE LINEAR APPROXIMATION

$w = C_1 x + C_2 \to C_2 = 0$ $\qquad u = D_1 x + D_2 \to D_2 = 0$

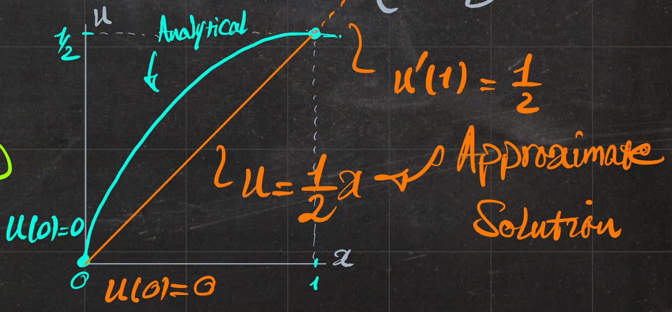
$w(0) = 0$ $\qquad w|_D = 0 \Leftarrow u(0)$ is GIVEN $\qquad u(0) = 0$

$w :$ { ARBITRARY, CONTINUOUS, $w|_D = 0$ }

$\Rightarrow u = \frac{1}{2}x$

DIRICHLET BCs ARE STRONGLY SATISFIED

NEUMANN BCs ARE WEAKLY SATISFIED



$$\int_0^1 \omega' u'\, dx = \int_0^1 \omega\, dx \qquad x \in [0,1]$$

$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ↙

→ 2-PIECE LINEAR (UNIFORM) APPROXIMATION

$0 \leftarrow \omega|_D = 0$

$x \in [0, 0.5] \qquad \omega = C_1 x + \cancel{C_2} \qquad u = E_1 x + \cancel{E_2} \quad 0 \leftarrow u(0) = 0$

$x \in [0.5, 1] \qquad \omega = D_1 x + D_2 \qquad u = F_1 x + F_2$

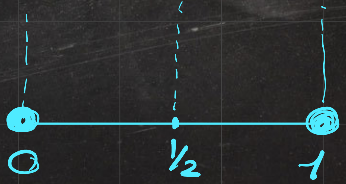
$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$$

$\Rightarrow \frac{1}{2} C_1 + C_2 = \frac{1}{2} D_1 + D_2 \qquad \Rightarrow \frac{1}{2} E_1 + E_2 = \frac{1}{2} F_1 + F_2$

↳ Employ BCs and Continuity Conditions

↳ ω Continuous @ 0.5

↳ u Continuous @ 0.5



$$\int_0^1 w'u'dx = \int_0^1 w\,dx \qquad x\in[0,1]$$

$$u'' + 1 = 0 \qquad 0 \leq x \leq 1$$

D: $u(0) = 0$ ← prescribed

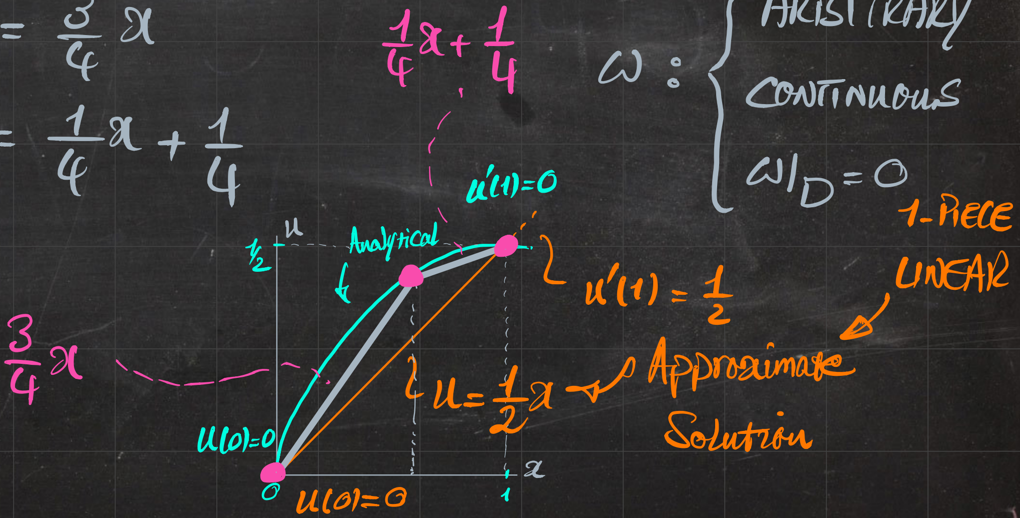
N: $u'(1) = 0$

→ 2-PIECE LINEAR (UNIFORM) APPROXIMATION

$x \in [0, 0.5]$ $\quad u = \frac{3}{4}x$

$x \in [0.5, 1]$ $\quad u = \frac{1}{4}x + \frac{1}{4}$

$$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$$



$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \qquad x \in [0,1]$$

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

→ 2-PIECE LINEAR (NON-UNIFORM) APPROXIMATION

$$x \in [0, 0.6] \qquad w = C_1 x + \overset{0}{C_2} \qquad u = E_1 x + \overset{0}{E_2}$$

$$x \in [0.6, 1] \qquad w = D_1 x + \underset{0.6[C_1 - D_1]}{D_2} \qquad u = F_1 x + \underset{0.6[E_1 - F_1]}{F_2}$$

$$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$$

$$\int_0^{0.6} \underset{C_1}{w'}\,\underset{E_1}{u'}\,dx + \int_{0.6}^{1} \underset{D_1}{w'}\,\underset{F_1}{u'}\,dx = \int_0^{0.6} \underset{C_1 x}{w}\,dx + \int_{0.6}^{1} \underset{D_1 x + 0.6[C_1 - D_1]}{w}\,dx$$

$$\int_0^1 w' u' \, dx = \int_0^1 w \, dx \qquad x \in [0,1]$$

$$u'' + 1 = 0 \qquad 0 \leq x \leq 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

→ 2-PIECE LINEAR (NON-UNIFORM) APPROXIMATION

$x \in [0, 0.6] \qquad u = E_1 x + \overset{0}{\cancel{E_2}}$

$x \in [0.6, 1] \qquad u = F_1 x + F_2$

$0.6[E_1 - F_1]$

$$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$$

$$C_1 [0.6 E_1 - 0.42] + D_1 [0.4 F_1 - 0.08] = 0 \qquad \checkmark C_1, D_1$$

$$\Rightarrow E_1 = 0.7, \; F_1 = 0.2 \quad \Rightarrow \begin{cases} u = 0.7x & 0 \leq x \leq 0.6 \\ u = 0.2x + 0.3 & 0.6 \leq x \leq 1 \end{cases}$$

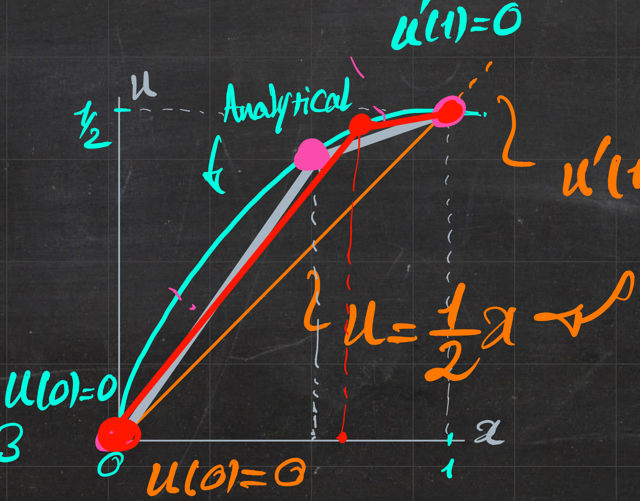
$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \qquad x \in [0,1]$$

$\rightarrow$ 2-PIECE LINEAR (NON-UNIFORM) APPROXIMATION

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←



1-PIECE LINEAR

Approximate Solution

$$\begin{cases} u = 0.7\,x \\ u = \underline{0.2x} + 0.3 \end{cases}$$

$$\int_0^1 w'u'dx = \int_0^1 w\,dx \qquad x\in[0,1]$$

$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

→ 2-PIECE LINEAR (NON-UNIFORM) APPROXIMATION

$x \in [0, 0.4] \qquad u = E_1 x + \overset{0}{\cancel{E_2}}$

$x \in [0.4, 1] \qquad u = F_1 x + F_2$

$\quad 0.4[E_1 - F_1]$

$$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$$

$C_1 [0.4E_1 - 0.32] \qquad \checkmark C_1, D_1$

$+ D_1 [0.6F_1 - 0.18] = 0$

$\Rightarrow E_1 = 0.8, \; F_1 = 0.3 \quad \Rightarrow \begin{cases} u = 0.8x & 0 \leqslant x \leqslant 0.4 \\ u = 0.3x + 0.2 & 0.4 \leqslant x \leqslant 1 \end{cases}$

$$\int_0^1 \omega' u' dx = \int_0^1 \omega \, dx \qquad x \in [0,1]$$

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

→ 2-PIECE LINEAR (NON-UNIFORM) APPROXIMATION

ANALYTICAL

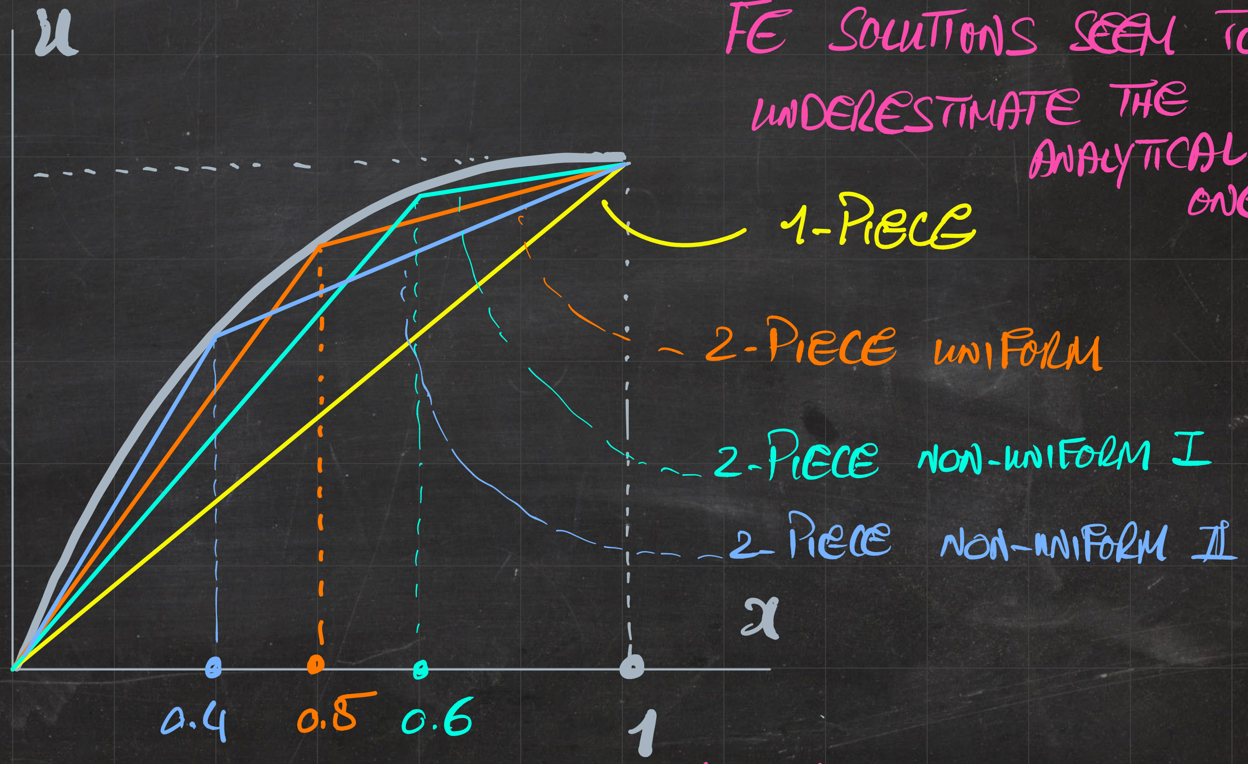
$$\begin{cases} u = 0.8x \\ u = 0.3x + 0.2 \end{cases}$$

$$\begin{cases} u = 0.7x \\ u = \underline{0.2}x + 0.3 \end{cases}$$

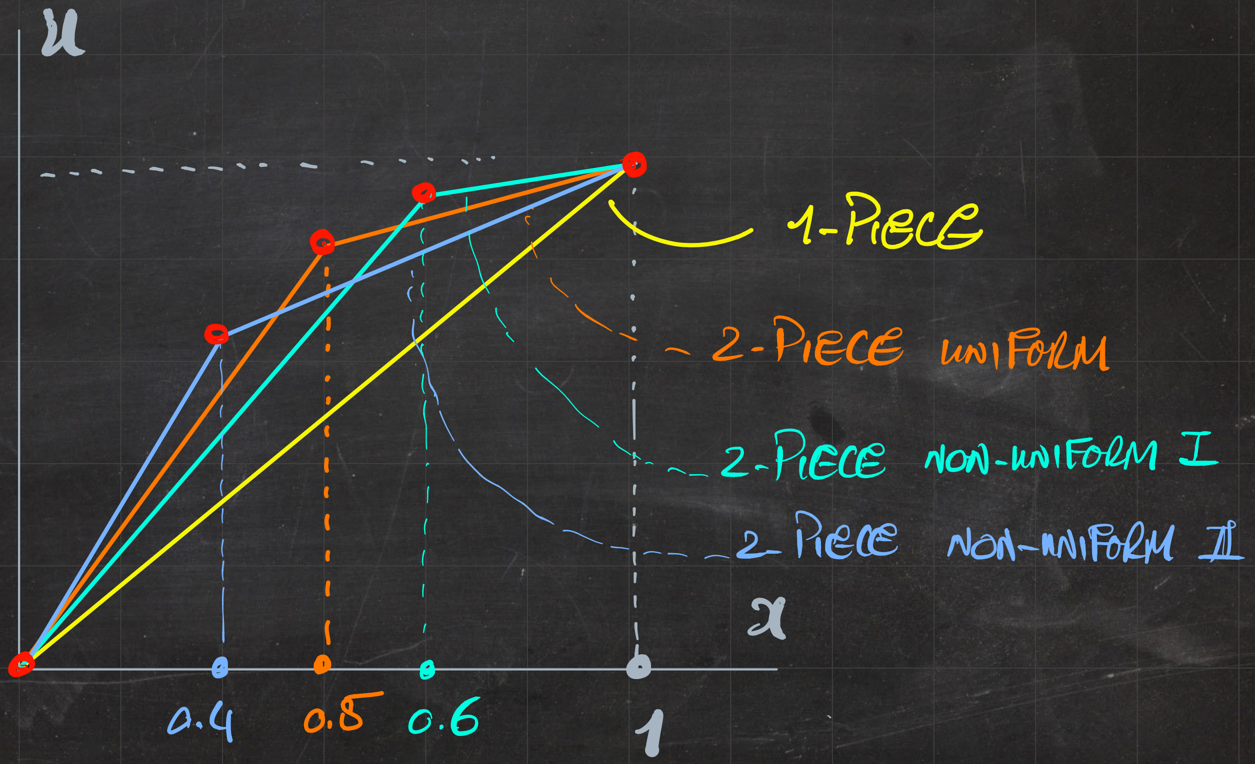
$u = \frac{1}{2}x$ ← Approximate Solution

1-PIECE LINEAR

FE SOLUTIONS SEEM TO UNDERESTIMATE THE ANALYTICAL ONE!



FE Solution approaches analytical one from below!


u
1-PIECE
2-PIECE UNIFORM
2-PIECE NON-UNIFORM I
2-PIECE NON-UNIFORM II
x
0.4
0.5
0.6
1

$$\int_0^1 \omega' u' dx = \int_0^1 \omega \, dx \qquad x \in [0,1]$$

$$u'' + 1 = 0 \qquad 0 \leq x \leq 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$

→ 2-PIECE LINEAR (GENERIC) APPROXIMATION

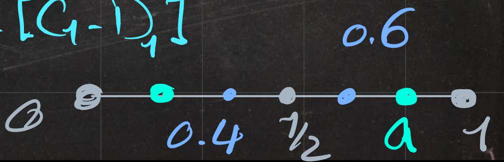
$x \in [0, a]$ $\quad \omega = C_1 x + \overset{0}{C_2} \quad u = E_1 x + \overset{0}{E_2}$

$x \in [a, 1]$ $\quad \omega = D_1 x + \underbrace{D_2}_{a[C_1 - D_1]} \quad u = F_1 x + \underbrace{F_2}_{a[E_1 - F_1]}$

$\omega :$ ARBITRARY, CONTINUOUS, $\omega|_D = 0$

$$\int_0^a \underbrace{\omega'}_{C_1} \underbrace{u'}_{E_1} dx + \int_a^1 \underbrace{\omega'}_{D_1} \underbrace{u'}_{F_1} dx = \int_0^a \underbrace{\omega}_{C_1 x} dx + \int_a^1 \underbrace{\omega}_{D_1 x + a[C_1 - D_1]} dx$$

$$0 \leq a \leq 1$$



$$\int_0^1 \omega' u' dx = \int_0^1 \omega \, dx \qquad x \in [0,1]$$

$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

→ 2-PIECE LINEAR (GENERIC) APPROXIMATION

$x \in [0, a] \Rightarrow u = [1 - \frac{1}{2} a] x$

$x \in [a, 1] \Rightarrow u = [\frac{1}{2} - \frac{1}{2} a] x + \frac{1}{2} a$

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$$

$a = 0.5$

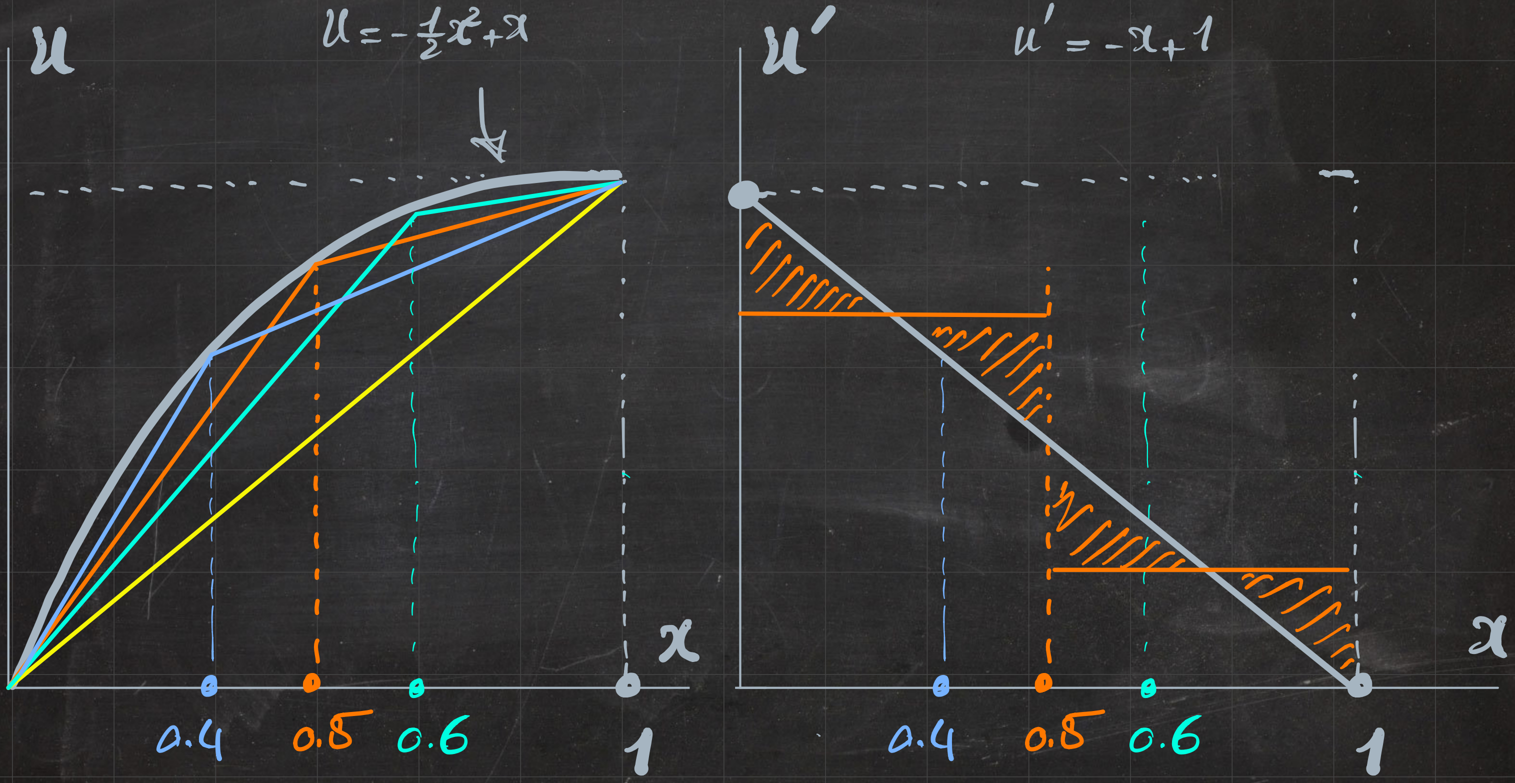
$$\begin{cases} u = 0.75x \\ u = 0.25x + 0.25 \end{cases}$$

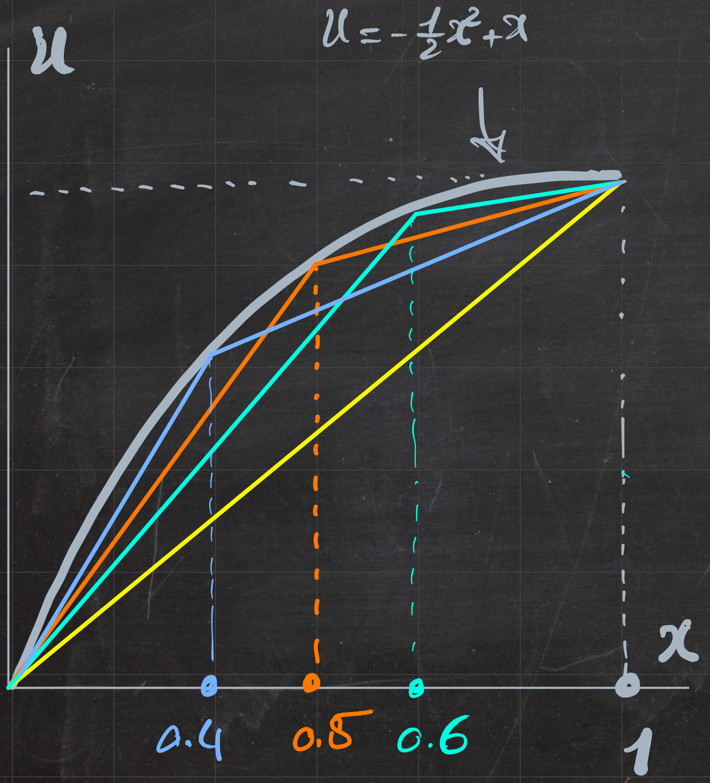
$a = 0.6$

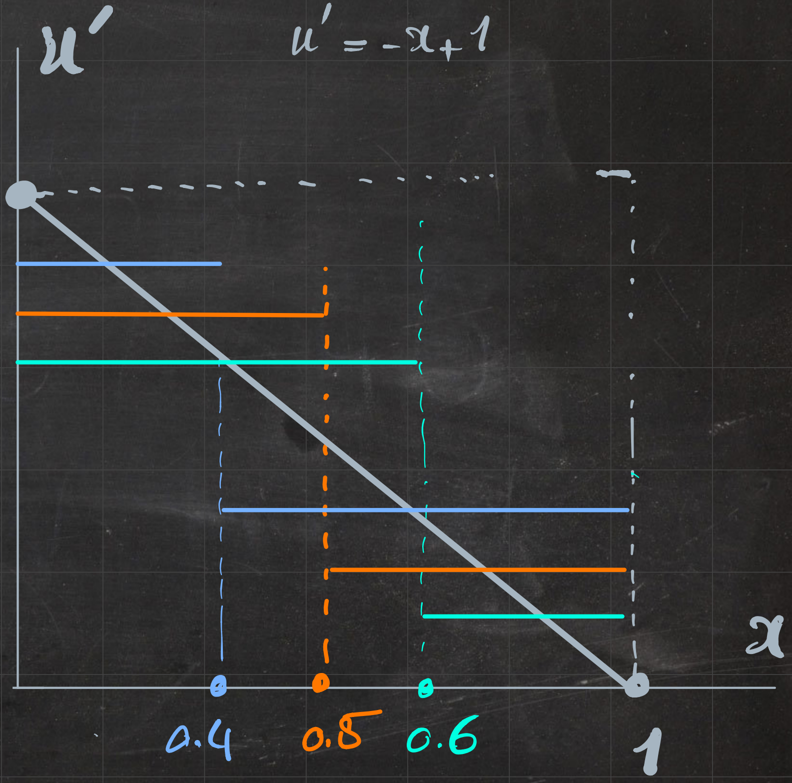
$$\begin{cases} u = 0.7x \\ u = 0.2x + 0.3 \end{cases}$$

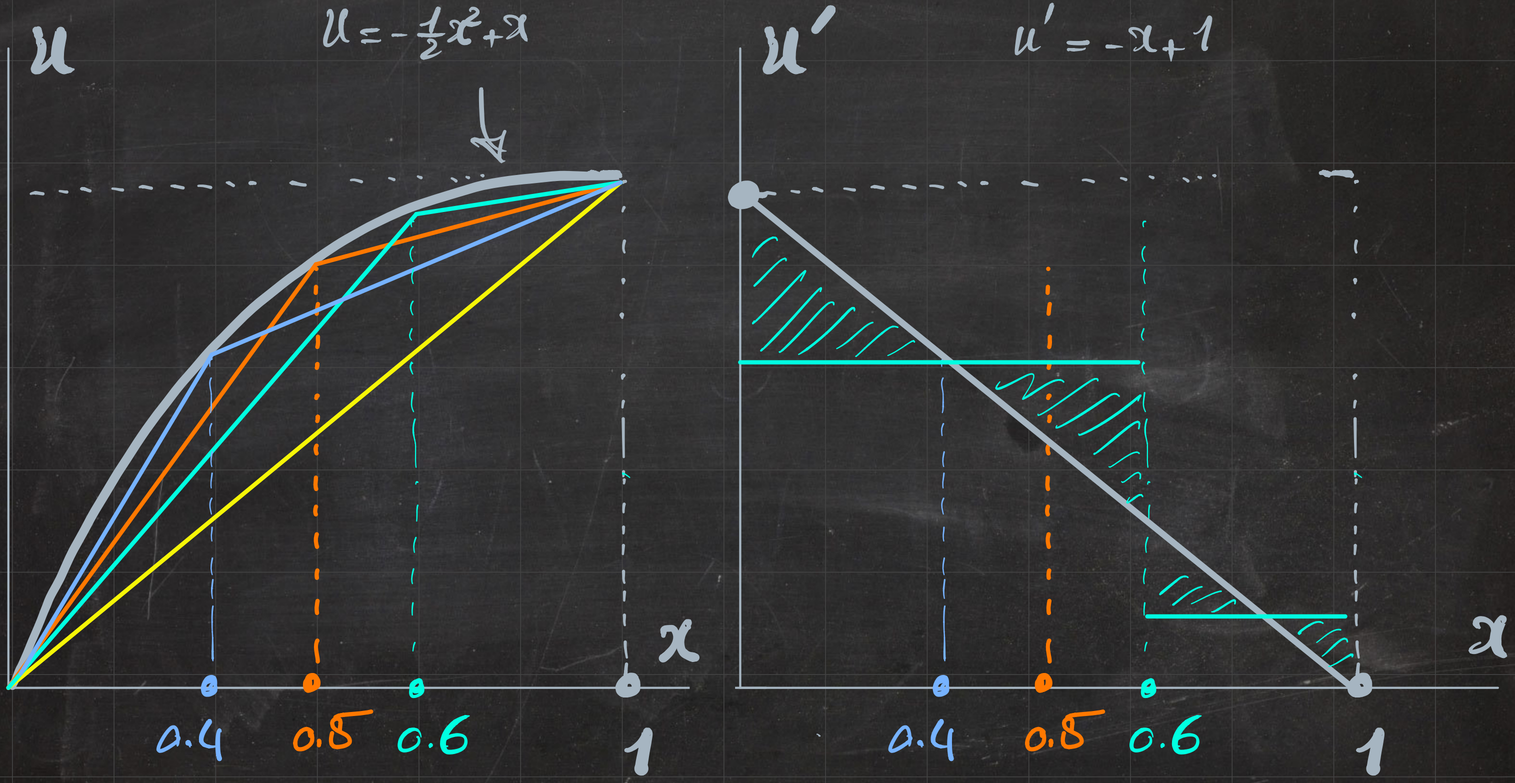
$a = 0.4$

$$\begin{cases} u = 0.8x \\ u = 0.3x + 2 \end{cases}$$


u
$u = -\frac{1}{2}x^2 + x$
x
0.4
0.5
0.6
1
u'
$u' = -x + 1$
x
0.4
0.5
0.6
1


u
$u = -\frac{1}{2}x^2 + x$
x
0.4
0.5
0.6
1


u'
$u' = -x + 1$
x
0.4
0.5
0.6
1


u
$u = -\frac{1}{2}x^2 + x$
x
0.4
0.5
0.6
1
u'
$u' = -x + 1$
x
0.4
0.5
0.6
1

$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \qquad x\in[0,1]$$

→ 3_PIECE LINEAR (GENERIC) APPROXIMATION

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

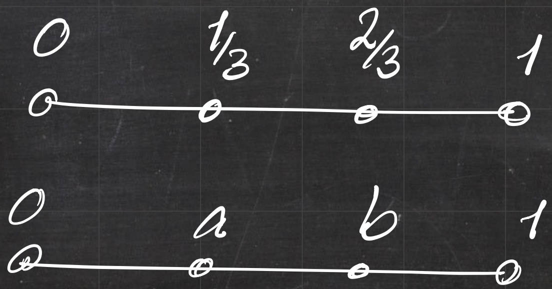
D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

w : ARBITRARY

CONTINUOUS

ANALYTICAL $w|_D = 0$



$0 \leqslant a < b \qquad a < b \leqslant 1$

$$\int_0^1 w'u'dx = \int_0^1 w\,dx \qquad x\in[0,1]$$

→ 3_PIECE LINEAR (GENERIC) APPROXIMATION

$u''+1=0 \qquad 0\leqslant x\leqslant 1$

D: $u(0)=0$ ← prescribed

N: $u'(1)=0$ ←

$x\in[0,a] \qquad w = C_1x+\cancelto{0}{C_2} \qquad u = F_1x+\cancelto{0}{F_2}$

$a[C_1-D_1] \qquad a[F_1-G_1]$

$x\in[a,b] \qquad w = D_1x+\cancel{D_2} \qquad u = G_1x+\cancel{G_2}$

$x\in[b,1] \qquad w = E_1x+\cancel{E_2} \qquad u = H_1x+\cancel{H_2}$

$w:\begin{cases}\text{ARBITRARY}\\ \text{CONTINUOUS}\\ w|_D=0\end{cases}$ ANALYTICAL

$b[D_1-E_1]+a[C_1-D_1]$

$b[G_1-H_1]+a[F_1-G_1]$

$\forall\; C_1, D_1, E_1$

$F_1, G_1, H_1 = ?$

$$\int_0^1 \omega' u' dx = \int_0^1 \omega \, dx \qquad x \in [0,1]$$

→ 3_PIECE LINEAR (GENERIC) APPROXIMATION

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

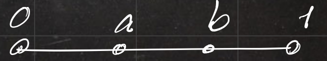
$$\begin{cases} u = [1 - \frac{1}{2}a]x & x \in [0,a] \\ u = [1 - \frac{1}{2}[a+b]]x + \frac{1}{2}ab & x \in [a,b] \\ u = [1 - \frac{1}{2}[b+1]]x + \frac{1}{2}b & x \in [b,1] \end{cases}$$

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$$

ANALYTICAL

$$\hookrightarrow u = [1 - \frac{1}{2}[\alpha,\beta]]x + \frac{1}{2}\alpha\beta \quad \leftarrow \quad 0 \leqslant \alpha < \beta \leqslant 1$$

$\{\alpha,\beta\} \to \{0,a\}$ $\{\alpha,\beta\} \to \{a,b\}$ $\{\alpha,\beta\} \to \{b,1\}$



$$\int_0^1 \omega' u' dx = \int_0^1 \omega \, dx \qquad x \in [0,1]$$

→ 3_PIECE LINEAR (GENERIC) APPROXIMATION

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = 0$ ← prescribed

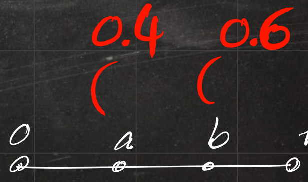
N: $u'(1) = 0$ ↙

$$\begin{cases} u = [1 - \frac{1}{2} a] x & x \in [0,a] \\ u = [1 - \frac{1}{2}[a+b]] x + \frac{1}{2} ab & x \in [a,b] \\ u = [1 - \frac{1}{2}[b+1]] x + \frac{1}{2} b & x \in [b,1] \end{cases}$$

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$$

ANALYTICAL

$$\begin{cases} u = 0.8x & x \in [0, 0.4] \\ u = 0.5x + 0.12 & x \in [0.4, 0.6] \\ u = 0.2x + 0.3 & x \in [0.6, 1] \end{cases}$$



$$\int_0^1 w'u'dx = \int_0^1 w\,dx \qquad x\in[0,1]$$

$\rightarrow$ 3_PIECE LINEAR (GENERIC) APPROXIMATION

$$\begin{cases} u = 0.8x & x\in[0,0.4] \\ u = 0.5x+0.12 & x\in[0.4,0.6] \\ u = 0.2x+0.3 & x\in[0.6,1] \end{cases}$$

$u''+1=0 \qquad 0\leqslant x\leqslant 1$

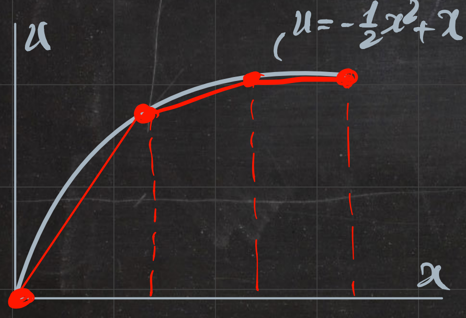
D: $u(0)=0$ ← prescribed

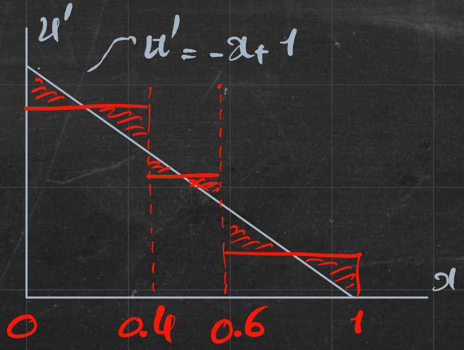
N: $u'(1)=0$ ←

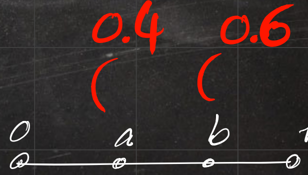
w : ARBITRARY

CONTINUOUS

ANALYTICAL $w|_D=0$







$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \qquad x\in[0,1]$$

→ 4-PIECE LINEAR (GENERIC) APPROXIMATION

$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

$x\in[0,a] \quad w = C_1x + C_2 \qquad u = G_1x + G_2$

$x\in[a,b] \quad w = D_1x + D_2 \qquad u = H_1x + H_2$

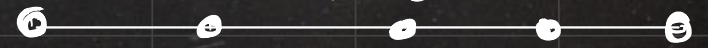
$x\in[b,c] \quad w = E_1x + E_2 \qquad u = I_1x + I_2$

$x\in[c,1] \quad w = F_1x + F_2 \qquad u = J_1x + J_2$

$$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$$

ANALYTICAL

$0 < a < b < c < 1$



$$\int_0^1 \omega' u' dx = \int_0^1 \omega \, dx \qquad x \in [0,1]$$

$$u'' + 1 = 0 \qquad 0 \leq x \leq 1$$

$$D: \; u(0) = 0$$

$$N: \; u'(1) = 0$$

← prescribed ←

→ 4-PIECE LINEAR (GENERIC) APPROXIMATION

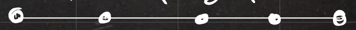
$$\begin{cases} u = [1 - \frac{1}{2} a] x & x \in [0,a] \\ u = [1 - \frac{1}{2}[a+b]] x + \frac{1}{2} ab & x \in [a,b] \\ u = [1 - \frac{1}{2}[b+c]] x + \frac{1}{2} bc & x \in [b,c] \\ u = [1 - \frac{1}{2}[c+1]] x + \frac{1}{2} c & x \in [c,1] \end{cases}$$

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \text{ANALYTICAL} \quad \omega|_D = 0 \end{cases}$$

$\ldots \Rightarrow$ n-piece LINEAR (GENERIC)

$$\hookrightarrow u = \left[1 - \frac{1}{2}[\alpha + \beta]\right] x + \frac{1}{2} \alpha\beta \qquad x \in [\alpha, \beta]$$

$$0 < a < b < c < 1$$



$$\int_0^1 \omega' u' dx = \int_0^1 \omega \, dx \qquad x \in [0,1]$$

→ 4-PIECE LINEAR (GENERIC) APPROXIMATION

$$\begin{cases} u = \frac{7}{8} x & x \in [0, 0.25] \\ u = \frac{5}{8} x + \frac{1}{16} & x \in [0.25, 0.50] \\ u = \frac{3}{8} x + \frac{3}{16} & x \in [0.50, 0.75] \\ u = \frac{1}{8} x + \frac{6}{16} & x \in [0.75, 1.00] \end{cases}$$

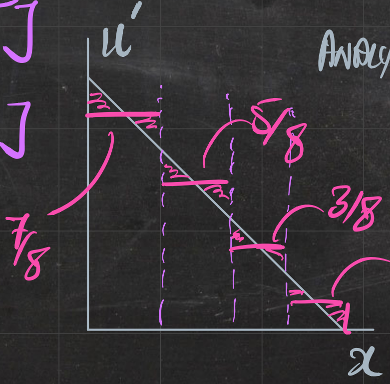
$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

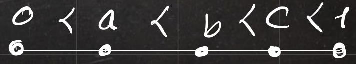
D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$$

ANALYTICAL





$$\int_0^1 \omega' u' dx = \int_0^1 \omega \, dx \qquad x \in [0,1]$$

→ 1-PIECE QUADRATIC APPROXIMATION

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

$\leftarrow \omega|_D = 0$

$$x \in [0,1] \qquad \omega = C_1 x^2 + C_2 x + \cancel{C_3}^{\,0} \qquad u = D_1 x^2 + D_2 x + \cancel{D_3}^{\,0} \leftarrow u(0)=0$$

ω :

- ARBITRARY
- CONTINUOUS
- ANALYTICAL
- $\omega|_D = 0$

$$\int_0^1 \underbrace{\omega'}_{2C_1 x + C_2} \underbrace{u'}_{2D_1 x + D_2} dx = \int_0^1 \underbrace{\omega}_{C_1 x^2 + C_2 x} dx \quad \dots$$

$$\Rightarrow C_1 \Big[\underbrace{\tfrac{4}{3} D_1 - \tfrac{1}{3} + D_2}_{=0} \Big] + C_2 \Big[\underbrace{D_1 - \tfrac{1}{2} + D_2}_{=0} \Big] = 0$$

$$\forall C_1, C_2$$

$$\int_0^1 w'u'dx = \int_0^1 w\,dx \qquad x\in[0,1]$$

$$u''+1=0 \qquad 0\leqslant x\leqslant 1$$

D: $u(0)=0$ ← prescribed

N: $u'(1)=0$ ←

→ 1-PIECE QUADRATIC APPROXIMATION

$$x\in[0,1] \qquad w = C_1x^2+C_2x+\overset{0 \,\leftarrow\, w|_D=0}{\cancel{C_3}} \qquad u = D_1x^2+D_2x+\overset{0\,\leftarrow\, u(0)=0}{\cancel{D_3}}$$

w : ANALYTICAL $\left\{\begin{array}{l}\text{ARBITRARY}\\ \text{CONTINUOUS}\\ w|_D=0\end{array}\right.$

$$\begin{cases}\frac{4}{3}D_1+D_2-\frac{1}{3}=0\\ D_1+D_2-\frac{1}{2}=0\end{cases} \Rightarrow \begin{cases}D_1=-1/2\\ D_2=1\end{cases}$$

$$u=-\frac{1}{2}x^2+x$$

IDENTICAL TO ANALYTICAL SOLUTION

approximation that has zero error

$$\int_0^1 w'u'dx = \int_0^1 w\,dx \qquad x \in [0,1]$$

$$u'' + 1 = 0 \qquad 0 \leq x \leq 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

→ 1-PIECE QUADRATIC APPROXIMATION

$$x \in [0,1] \qquad w = C_1x^2 + C_2x + \overset{0}{\cancel{C_3}} \xleftarrow{} w|_D = 0 \qquad u = D_1x^2 + D_2x + \overset{0}{\cancel{D_3}} \xleftarrow{} u(0)=0$$

w : ARBITRARY

CONTINUOUS

ANALYTICAL

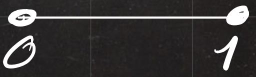
$w|_D = 0$

IF THE APPROXIMATION SPACE IS LARGE ENOUGH, IT CAN INCLUDE THE EXACT SOLUTION !

$$u = -\frac{1}{2}x^2 + x$$

IDENTICAL TO ANALYTICAL SOLUTION

approximation that has zero error



$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \qquad x\in[0,1]$$

→ 1-PIECE CUBIC APPROXIMATION

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

$$x\in[0,1] \qquad w = C_1x^3 + C_2x^2 + C_3x + \cancelto{0}{C_4}$$

$$u = D_1x^3 + D_2x^2 + D_3x + \cancelto{0}{D_4}$$

w : ARBITRARY

CONTINUOUS

ANALYTICAL $w|_D = 0$

$$\int_0^1 [3C_1x^2 + 2C_2x + C_3][3D_1x^2 + 2D_2x + D_3]\,dx = \int_0^1 [C_1x^3 + C_2x^2 + C_3x]\,dx$$

$$\int_0^1 \omega' u' \, dx = \int_0^1 \omega \, dx \qquad x \in [0,1]$$

→ 1-PIECE CUBIC APPROXIMATION

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

$$x \in [0,1] \qquad u = D_1 x^3 + D_2 x^2 + D_3 x + \cancelto{0}{D_4}$$

$$\ldots \Rightarrow C_1\left[\tfrac{9}{5} D_1 + \tfrac{6}{4} D_2 + D_3 - \tfrac{1}{4}\right] \qquad \checkmark\, C_1, C_2, C_3$$

$$+ C_2\left[\tfrac{6}{4} D_1 + \tfrac{4}{3} D_2 + D_3 - \tfrac{1}{3}\right] + C_3\left[D_1 + D_2 + D_3 - \tfrac{1}{2}\right] = 0$$

ω : ARBITRARY

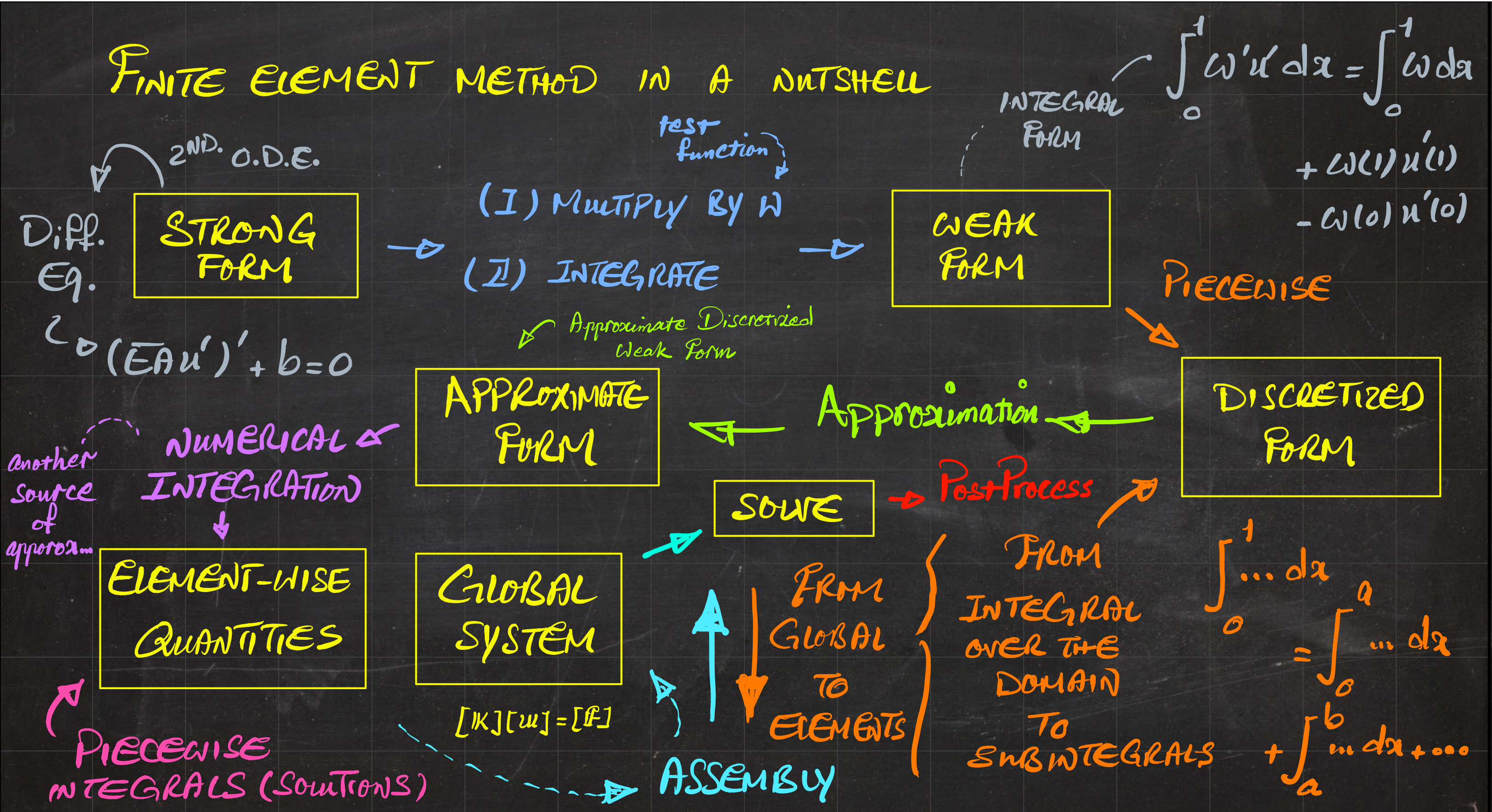
CONTINUOUS

ANALYTICAL $\omega|_D = 0$

$$\begin{cases} \left[\tfrac{9}{5} D_1 + \tfrac{6}{4} D_2 + D_3 - \tfrac{1}{4}\right] = 0 \\ \left[\tfrac{6}{4} D_1 + \tfrac{4}{3} D_2 + D_3 - \tfrac{1}{3}\right] = 0 \\ \left[D_1 + D_2 + D_3 - \tfrac{1}{2}\right] = 0 \end{cases} \Rightarrow \begin{cases} D_1 = 0 \\ D_2 = -\tfrac{1}{2} \\ D_3 = 1 \end{cases} \Rightarrow u = -\tfrac{1}{2} x^2 + x$$

if approximation space is large enough, we recover the exact solution!

FINITE ELEMENT METHOD IN A NUTSHELL



STRONG FORM

Diff. Eq.

2ND. O.D.E.

$-(EAu')'+b=0$

(I) MULTIPLY BY W

test function

(II) INTEGRATE

WEAK FORM

INTEGRAL FORM

$\int_0^1 w'u'\,dx = \int_0^1 w\,dx + w(1)u'(1) - w(0)u'(0)$

PIECEWISE

DISCRETIZED FORM

Approximation

APPROXIMATE FORM

Approximate Discretized Weak Form

NUMERICAL INTEGRATION

another source of approx...

ELEMENT-WISE QUANTITIES

PIECEWISE INTEGRALS (SOLUTIONS)

GLOBAL SYSTEM

$[\mathbb{K}][u]=[\mathbb{F}]$

ASSEMBLY

SOLVE

PostProcess

FROM GLOBAL TO ELEMENTS

FROM INTEGRAL OVER THE DOMAIN TO SUBINTEGRALS

$\int_0^1 \dots dx = \int_0^a \dots dx + \int_a^b \dots dx + \dots$

APPROXIMATION: UNDERSTANDING VIA EXAMPLES

(I)

$u(0) = 0$

$u(1) = 1$

$u(0.5) = ? \approx 0.5$

$u(0.8) = ? \approx 0.8$

(II)

$u(0) = 1$

$u(1.6) = 5$

$u(0.8) = ? \approx 3$

$u(1) = ? \approx 3.5$

(III)

$u(0) = 1$

$u(0.5) = 2$

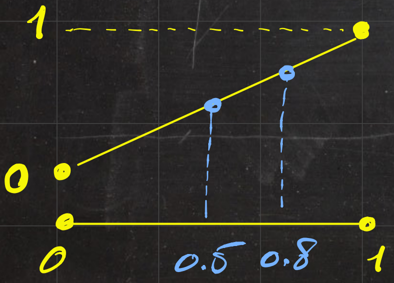
$u(1) = 4$

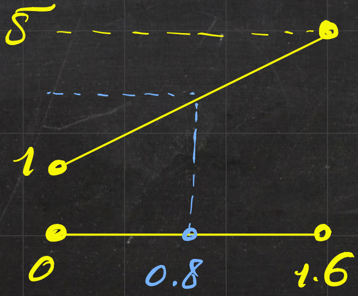
$u(0.8) = ? \approx 3.08$

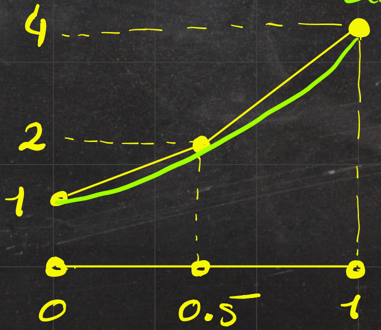
LINEAR APPROXIMATION

3.2

QUADRATIC APPROXIMATION







APPROXIMATION: UNDERSTANDING VIA EXAMPLES

QUADRATIC APPROXIMATION?

(III) $u(0)=1$

$u(0.5)=2$

$u(1)=4$

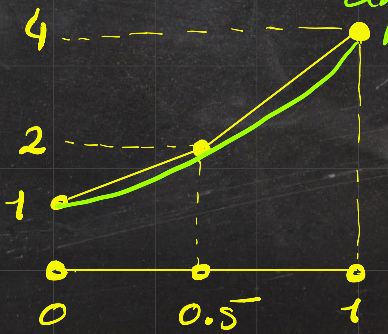
$u(0.8)=\ ?\ \approx 3.08$

LINEAR APPROXIMATION $\rightarrow$ 3.2

QUADRATIC APPROXIMATION

$$f(x) = ax^2 + bx + c \Rightarrow f(x) = 2x^2 + x + 1$$

$$\left.\begin{array}{l} f(0) = 1 \\ f(0.5) = 2 \\ f(1) = 4 \end{array}\right\} \Rightarrow \begin{array}{l} 3 \text{ EQNs} \\ 3 \text{ UNKNOWNS} \end{array} \Rightarrow \begin{array}{l} a = 2 \\ b = 1 \\ c = 1 \end{array}$$



APPROXIMATION: UNDERSTANDING VIA EXAMPLES

(IV)

$$\left.\begin{array}{l} u(0)=1 \\ u(0.2)=2 \\ u(0.6)=4 \\ u(1)=8 \end{array}\right\} \Rightarrow u(0.8)=?$$

$$f(x)=ax^3+bx^2+cx+d$$

$$\Sigma \Downarrow$$

$$\left.\begin{array}{l} \text{4 Equations} \\ \text{4 Unknowns} \end{array}\right\} \Rightarrow \begin{array}{l} a \\ b \\ c \\ d \end{array}$$

$$\Leftarrow$$

$$\checkmark \; f(x)$$

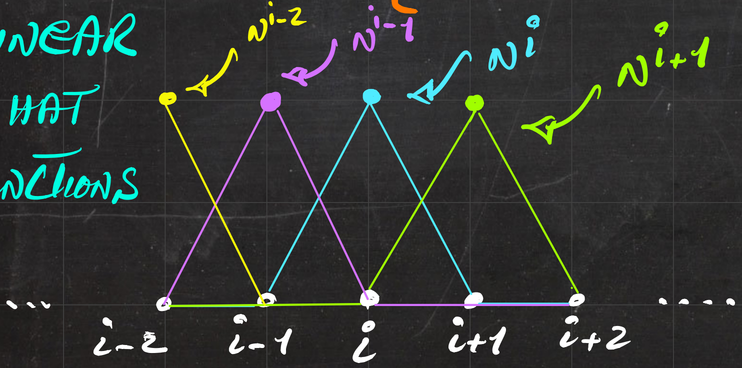
SHAPE FUNCTIONS (HAT FUNCTIONS, TENT FUNCTIONS)

↳ A powerful tool for approximations ⇝ SYSTEMATIC

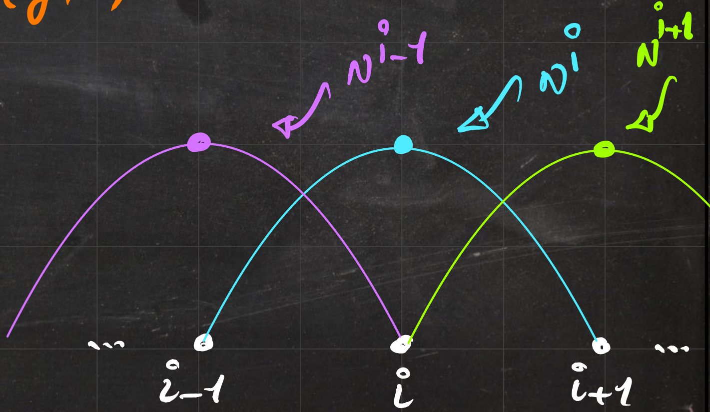
⇝ NEARLY IDENTICAL FOR 2D 3D

$$N^i(x) \rightsquigarrow \begin{cases} N^i = 1 \ @ \ x^j \ (j=i) \\ N^i = 0 \ @ \ x^j \ (j \neq i) \end{cases}$$

LINEAR HAT FUNCTIONS



QUADRATIC HAT FUNCTIONS



SHAPE FUNCTIONS (HAT FUNCTIONS , TENT FUNCTIONS)

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$$N^i(x) \rightsquigarrow \begin{cases} N^i = 1 \quad @ \; x^j \; (j=i) \\ N^i = 0 \quad @ \; x^j \; (j \neq i) \end{cases}$$

NODES PER ELEMENT → NPE

$$u = \sum_{i=1}^{NPE} N^i u^i \rightarrow \begin{cases} u = N^1 u^1 + N^2 u^2 \\ u = N^1 u^1 + N^2 u^2 + N^3 u^3 \\ u = N^1 u^1 + N^2 u^2 + N^3 u^3 + N^4 u^4 \end{cases}$$

linear approximation

quadratic approximation

cubic approximation

SHAPE FUNCTIONS (HAT FUNCTIONS, TENT FUNCTIONS)

↳ A powerful tool for approximations $\rightsquigarrow$ SYSTEMATIC

$\rightsquigarrow$ NEARLY IDENTICAL FOR 2D 3D

$$N^i(x) \rightsquigarrow \begin{cases} N^i = 1 \ @ \ x^j \ (j = i) \\ N^i = 0 \ @ \ x^j \ (j \neq i) \end{cases}$$

$$u = N^1 u^1 + N^2 u^2 \qquad (I) \qquad \Rightarrow \begin{cases} N^1 = 1 - x \\ N^2 = x \end{cases}$$

$$\left.\begin{array}{l} u(0) = 0 \rightsquigarrow u^1 = 0 \\ u(1) = 1 \rightsquigarrow u^2 = 1 \end{array}\right\} \quad u = [1-x] \times 0 + [x] \times 1 = x$$

$$\Uparrow$$

$$u \simeq x$$

SHAPE FUNCTIONS (HAT FUNCTIONS, TENT FUNCTIONS)

↳ A powerful tool for approximations ⇝ SYSTEMATIC

⇝ NEARLY IDENTICAL FOR 2D 3D

$$N^i(x) \rightsquigarrow \begin{cases} N^i = 1 \; @ \; x^j \; (j=i) \\ N^i = 0 \; @ \; x^j \; (j \neq i) \end{cases}$$

APPROXIMATION OF ORDER M ⇝

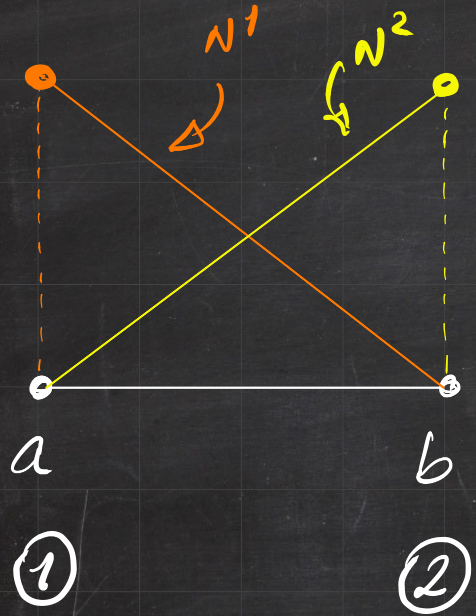
$$u = \sum_{i=1}^{M+1} N^i u^i \Rightarrow$$

linear approximation ⇝ $\sum_1^2$

quadratic " ⇝ $\sum_1^3$

cubic " ⇝ $\sum_1^4$

↳ NPE ≡ M+1



$$N^1 = \frac{x-b}{a-b}$$

$$N^2 = \frac{x-a}{b-a}$$

LINEAR
SHAPE
FUNCTIONS

LINEAR APPROXIMATION

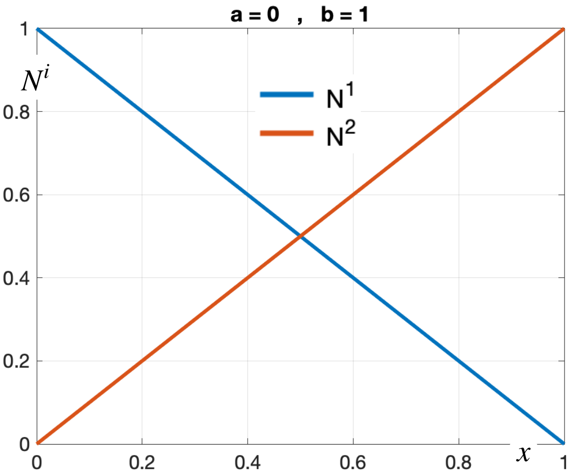
QUADRATIC APPROXIMATION

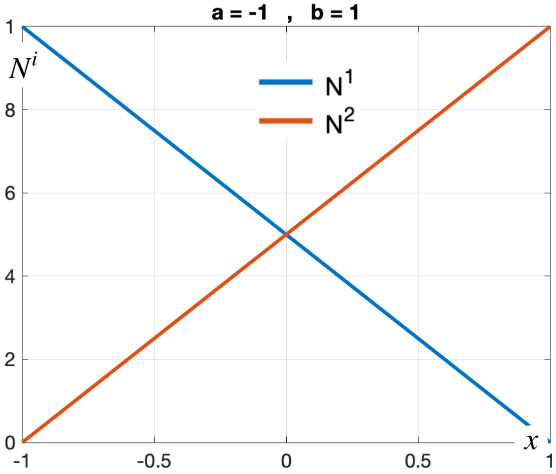
CUBIC APPROXIMATION

1D Linear Shape Functions

$$N^1 = \frac{[x - b]}{[a - b]}$$

$$N^2 = \frac{[x - a]}{[b - a]}$$



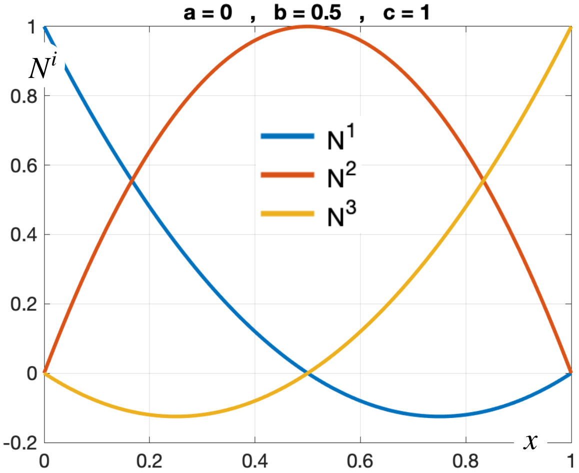


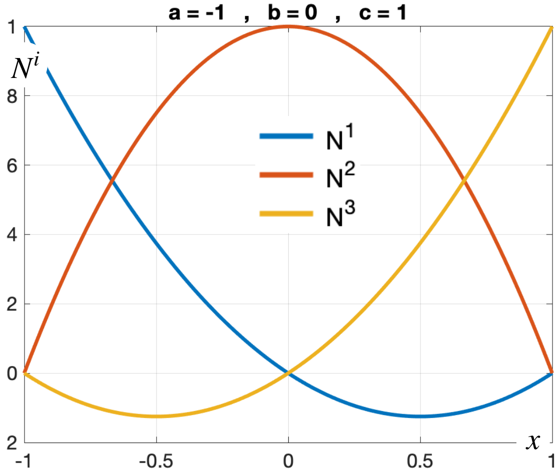
1D Quadratic Shape Functions

$$N^1 = \frac{[x-b][x-c]}{[a-b][a-c]}$$

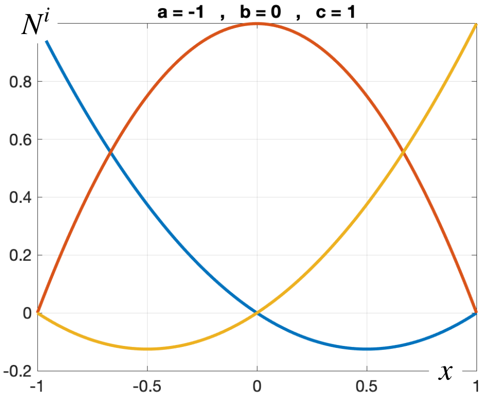
$$N^2 = \frac{[x-a][x-c]}{[b-a][b-c]}$$

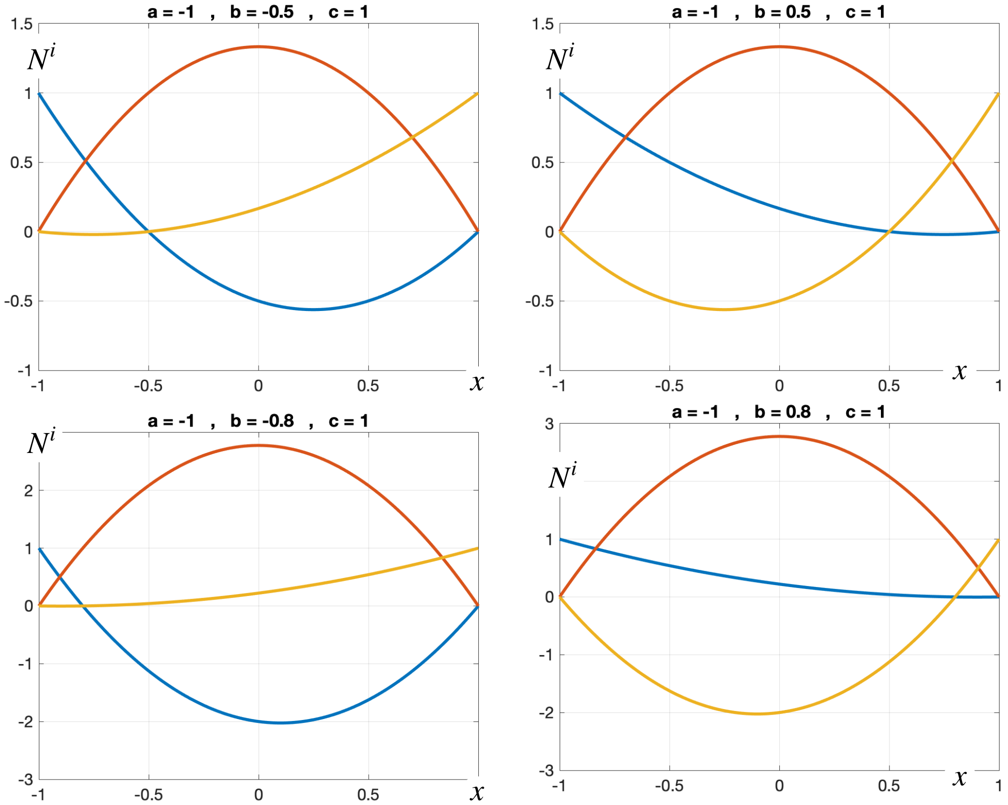
$$N^3 = \frac{[x-a][x-b]}{[c-a][c-b]}$$





1D Quadratic Shape Functions





$$N^1 = \frac{[x-b][x-c]}{[a-b][a-c]}$$

$$N^2 = \frac{[x-a][x-c]}{[b-a][b-c]}$$

$$N^3 = \frac{[x-a][x-b]}{[c-a][c-b]}$$

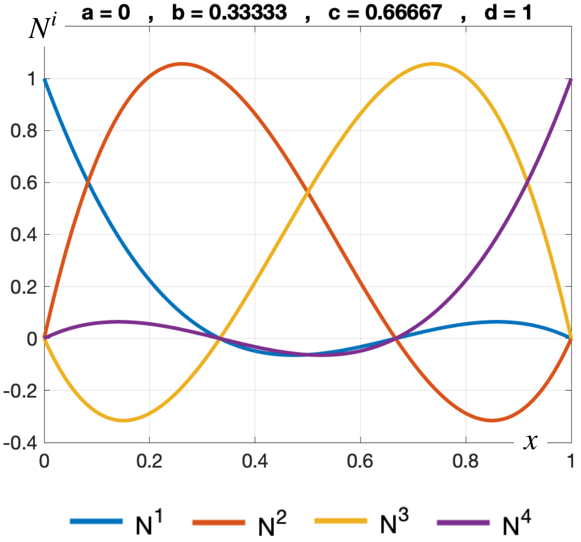
1D Cubic Shape Functions

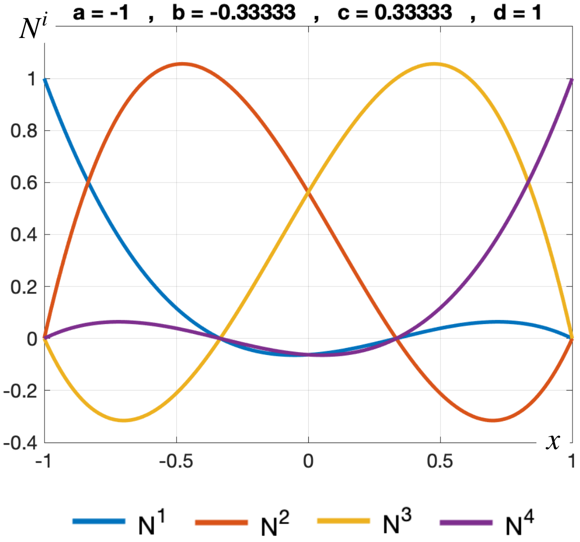
$$N^1 = \frac{[x-b][x-c][x-d]}{[a-b][a-c][a-d]}$$

$$N^2 = \frac{[x-a][x-c][x-d]}{[b-a][b-c][b-d]}$$

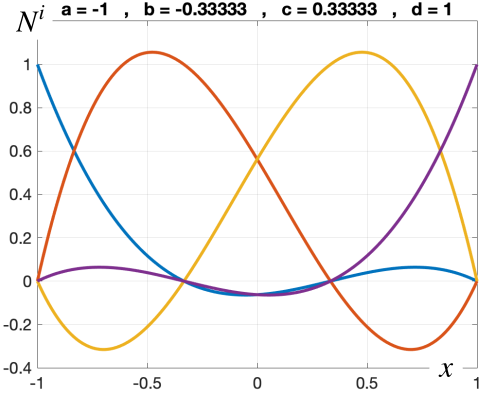
$$N^3 = \frac{[x-a][x-b][x-d]}{[c-a][c-b][c-d]}$$

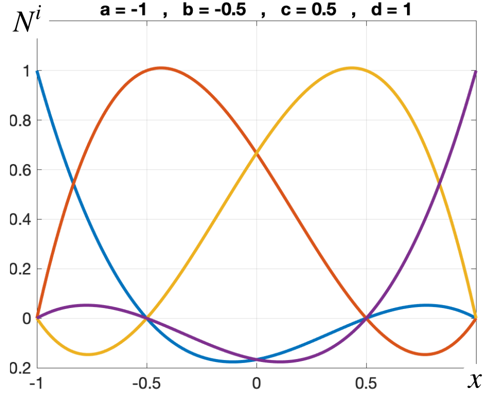
$$N^4 = \frac{[x-a][x-b][x-c]}{[d-a][d-b][d-c]}$$

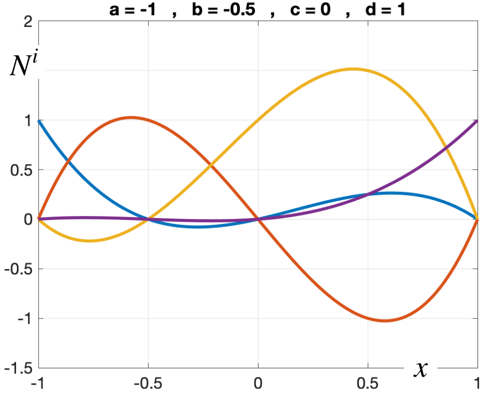




1D Cubic Shape Functions


a = -1 , b = -0.33333 , c = 0.33333 , d = 1


a = -1 , b = -0.5 , c = 0.5 , d = 1


a = -1 , b = -0.5 , c = 0 , d = 1

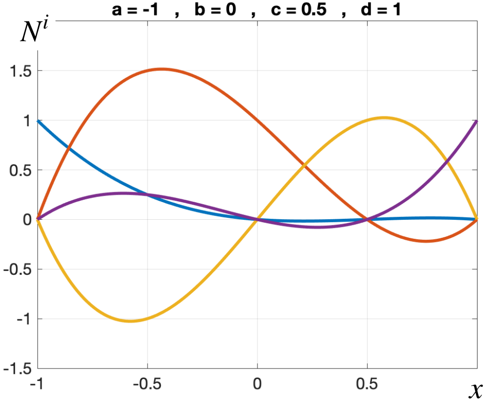
$$N^1 = \frac{[x-b][x-c][x-d]}{[a-b][a-c][a-d]}$$

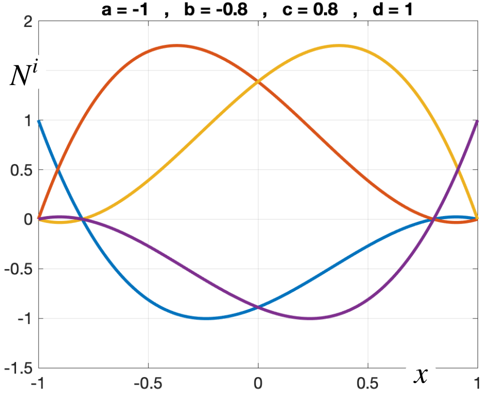
$$N^2 = \frac{[x-a][x-c][x-d]}{[b-a][b-c][b-d]}$$

$$N^3 = \frac{[x-a][x-b][x-d]}{[c-a][c-b][c-d]}$$

$$N^4 = \frac{[x-a][x-b][x-c]}{[d-a][d-b][d-c]}$$

🟦 N^1
🟥 N^2
🟨 N^3
🟪 N^4


a = -1 , b = 0 , c = 0.5 , d = 1


a = -1 , b = -0.8 , c = 0.8 , d = 1

APPROXIMATION: UNDERSTANDING VIA EXAMPLES

(I) $u(0) = 0$ ← u^1

$u(1) = 1$ ← u^2

$u(0.5) = ?\ \approx 0.5$

$u(0.8) = ?\ \approx 0.8$

(II) $u(0) = 1$ ← u^1

$u(1.6) = 5$ ← u^0

$u(0.8) = ?\ \approx 3$

$u(1) = ?\ \approx 3.5$

(III) $u(0) = 1$

$u(0.5) = 2$

$u(1) = 4$

$u(0.8) = ?$

$$u = N^1 u^1 + N^2 u^2$$

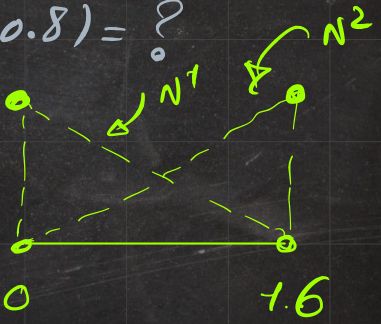
$$= [1-x]\,\overset{0}{u^1} + x\,\overset{1}{u^2}$$

$$= x \Rightarrow u(x) = x\ ✓$$

$$u = N^1 \overset{1}{u^1} + N^2 \overset{5}{u^2}$$

$$= \frac{[x-1.6]}{-1.6}\,u^1 + \frac{[x-0]}{1.6}\,u^2$$

$$= \frac{4}{1.6}x + 1 \Rightarrow u(x) = 2.5x + 1$$



APPROXIMATION: UNDERSTANDING VIA EXAMPLES

(I)

$u^1 \nearrow u(0) = 0$

$u^2 \nearrow u(1) = 1$

$u(0.5) = ? \approx 0.5$

$u(0.8) = ? \approx 0.8$

$$u = N^1 u^1 + N^2 u^2$$

$$= [1-x]\,\overset{0}{u^1} + x\,\overset{1}{u^2}$$

$$= x \Rightarrow u(x) = x \checkmark$$

(II)

$u^1 \nearrow u(0) = 1$

$u^0 \nearrow u(1.6) = 5$

$u(0.8) = ? \approx 3$

$u(1) = ? \approx 3.5$

$$u = N^1 \overset{1}{u^1} + N^2 \overset{5}{u^2}$$

$$= \frac{[x-1.6]}{-1.6} u^1 + \frac{[x-0]}{1.6} u^2$$

$$\Rightarrow u(x) = 2.5x + 1$$

(III)

$u^1 \nearrow u(0) = 1$

$u^2 \nearrow u(0.5) = 2$

$u^3 \nearrow u(1) = 4$

$u(0.8) = ?$

$$N^1 = \frac{[x-0.5][x-1]}{0.5}$$

$$N^2 = \frac{[x-0][x-1]}{-0.25}$$

$$N^3 = \frac{[x-0][x-0.5]}{0.5}$$

$$u = N^1 u^1 + N^2 u^2 + N^3 u^3$$

$$= 2[x^2 - 1.5x + 0.5] - 8[x^2 - x] + 8[x^2 - 0.5x]$$

$$\Rightarrow u(x) = 2x^2 + x + 1$$

APPROXIMATION : UNDERSTANDING VIA EXAMPLES

(IV)

$$\left.\begin{array}{l} u(0)=1 \\ u(0.2)=2 \\ u(0.6)=4 \\ u(1)=8 \end{array}\right\} \Rightarrow u(0.8)=?$$

$$f(x)=ax^3+bx^2+cx+d$$

$$\overset{\Sigma}{\Downarrow}$$

$$\left.\begin{array}{l}\text{4 Equations}\\ \text{4 Unknowns}\end{array}\right\} \Rightarrow \begin{array}{c}a\\b\\c\\d\end{array}$$

$$\swarrow$$

$$\checkmark \; f(x)$$

APPROXIMATION : UNDERSTANDING VIA EXAMPLES

(IV)

$$\overbrace{\underbrace{\quad}^{-0.2}\quad\underbrace{\quad}^{-0.6}\quad\underbrace{\quad}^{-1}}^{-0.12}$$

① → $u(0) = 1$ $N^1 = [x-0.2][x-0.6][x-1] \,/\, [-0.12]$

② → $u(0.2) = 2$ $N^2 = [x-0][x-0.6][x-1] \,/\, [0.064]$

③ → $u(0.6) = 4$ $N^3 = [x-0][x-0.2][x-1] \,/\, [-0.096]$

④ → $u(1) = 8$ $N^4 = [x-0][x-0.2][x-0.6] \,/\, [0.32]$

FROM COMPUTER PERSPECTIVE

↓

UNNECESSARY

↓

ADDITIONAL STEP

$$\Rightarrow u = N^1u^1 + N^2u^2 + N^3u^3 + N^4u^4 \;\ldots\; \Rightarrow \overbrace{u(x) = \alpha x^3 + \beta x^2 + \gamma x + \xi}$$

UNDERSTANDING THE SIGNIFICANCE OF WEAK FORM BY EXAMPLE

$$\int_0^1 w' u' \, dx = \int_0^1 w \, dx$$

$u'' + 1 = 0 \qquad 0 \leq x \leq 1$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ↙

- → 1-PIECE LINEAR APPROXIMATION
- → 2-PIECE LINEAR (UNIFORM) APPROXIMATION
- → 1-PIECE QUADRATIC APPROXIMATION

$$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$$

REMEMBER HOW WE DID BEFORE

$w = C_1 x + C_2 \qquad u = D_1 x + D_2$

$\dots \int_0^1 w' u' \, dx = \int_0^1 w \, dx \dots \quad \forall w$

$\dots \Rightarrow D_1 \; \& \; D_2$ ✓

UNDERSTANDING THE SIGNIFICANCE OF WEAK FORM BY EXAMPLE

$$\int_0^1 w'u'dx = \int_0^1 w\,dx$$

$$u'' + 1 = 0 \qquad 0 \leq x \leq 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

→ 1-PIECE LINEAR APPROXIMATION

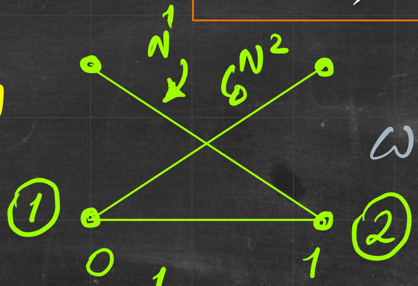
$$w = N^1 w^1 + N^2 w^2 \qquad u = N^1 u^1 + N^2 u^2$$

w^1: w @ node 1

w^2: w @ node 2

u^1: u @ node 1

u^2: u @ node 2



$N^1 = 1 - x$

$N^2 = x$

w : ARBITRARY

CONTINUOUS

$w|_D = 0$

REMEMBER HOW WE DID BEFORE

$$w = C_1 x + C_2 \qquad u = D_1 x + D_2$$

$$\ldots \int_0^1 w'u'dx = \int_0^1 w\,dx \ldots \quad \forall w$$

$$\ldots \Rightarrow D_1 \,\&\, D_2 \checkmark$$

UNDERSTANDING THE SIGNIFICANCE OF WEAK FORM BY EXAMPLE

$$\int_0^1 w'u'dx = \int_0^1 w\,dx$$

$u''+1=0 \qquad 0\leqslant x\leqslant 1$

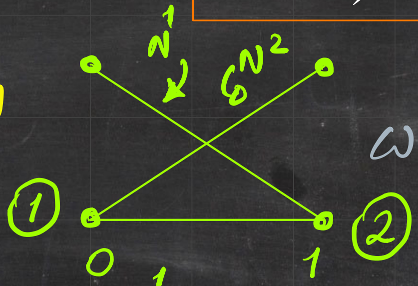
D: $u(0)=0$ ← prescribed

N: $u'(1)=0$ ←

→ 1-PIECE LINEAR APPROXIMATION

$w = N^1\cancel{w^1} + N^2w^2 \qquad u = N^1\cancel{u^1} + N^2u^2$

$w^1=0 \leftarrow w|_D=0 \qquad u^1=0 \leftarrow u(0)=0$



$N^1 = 1-x$

$N^2 = x$

w : ARBITRARY, CONTINUOUS, $w|_D=0$

$w = N^2w^2 = xw^2 \qquad u = N^2u^2 = xu^2$

$$\int_0^1 w^2u^2\,dx = \int_0^1 xw^2\,dx \to w^2u^2 = w^2\,\underbrace{\tfrac{1}{2}x^2\Big]_0^1}_{\leftarrow \frac{1}{2}}$$

REMEMBER HOW WE DID BEFORE

$w = C_1x + C_2 \qquad u = D_1x + D_2$

$\dots \int_0^1 w'u'dx = \int_0^1 w\,dx \dots \forall w$

$\dots \Rightarrow D_1 \,\&\, D_2$ ✓

UNDERSTANDING THE SIGNIFICANCE OF WEAK FORM BY EXAMPLE

$$\int_0^1 w'u'dx = \int_0^1 w\,dx$$

$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ↙

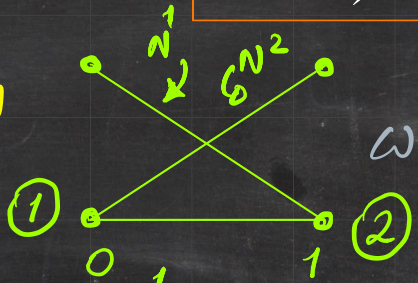
→ 1_PIECE LINEAR APPROXIMATION

$w = N^1 \cancel{w^1} + N^2 w^2 \qquad u = N^1 \cancel{u^1} + N^2 u^2$

$w^1 = 0 \leftarrow w|_D = 0 \qquad u^1 = 0 \leftarrow u(0) = 0$

$w = N^2 w^2 = x w^2 \qquad u = N^2 u^2 = x u^2$

$$\int_0^1 w^2 u^2 \, dx = \int_0^1 x w^2 \, dx \rightarrow w^2 u^2 = w^2 \frac{1}{2} \quad \forall w^2 \Rightarrow u^2 = \frac{1}{2}$$



$N^1 = 1 - x$

$N^2 = x$

w : { ARBITRARY, CONTINUOUS, $w|_D = 0$

REMEMBER HOW WE DID BEFORE

$w = C_1 x + C_2 \qquad u = D_1 x + D_2$

$\dots \int_0^1 w'u'dx = \int_0^1 w\,dx \dots \forall w$

$\dots \Rightarrow D_1 \& D_2$ ✓

UNDERSTANDING THE SIGNIFICANCE OF WEAK FORM BY EXAMPLE

$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx$$

$$u'' + 1 = 0 \qquad 0 \leq x \leq 1$$

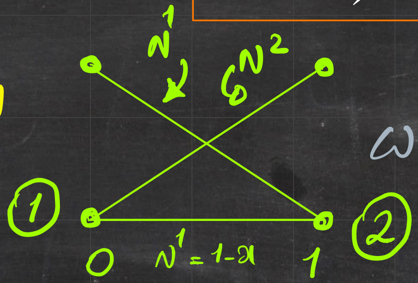
D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

→ 1-PIECE LINEAR APPROXIMATION

$$w = N^1 \cancel{w^1} + N^2 w^2 \qquad u = N^1 \cancel{u^1} + N^2 u^2$$

$$w^1 = 0 \leftarrow w|_D = 0 \qquad u^1 = 0 \leftarrow u(0) = 0$$



$N^1 = 1-x$

$N^2 = x$

$$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$$

$$w = N^2 w^2 = x w^2 \qquad u = N^2 u^2 = x u^2 \Rightarrow u = \frac{1}{2} x \checkmark$$

$$\int_0^1 w^2 u^2\,dx = \int_0^1 x w^2\,dx \rightarrow w^2 u^2 = w^2 \frac{1}{2} \quad \forall w^2 \Rightarrow u^2 = \frac{1}{2}$$

REMEMBER HOW WE DID BEFORE

$$w = C_1 x + C_2 \qquad u = D_1 x + D_2$$

$$\ldots \int_0^1 w'u'\,dx = \int_0^1 w\,dx \ldots \forall w$$

$$\ldots \Rightarrow D_1 \;\&\; D_2 \checkmark$$

2. PIECE LINEAR UNIFORM APPROXIMATION

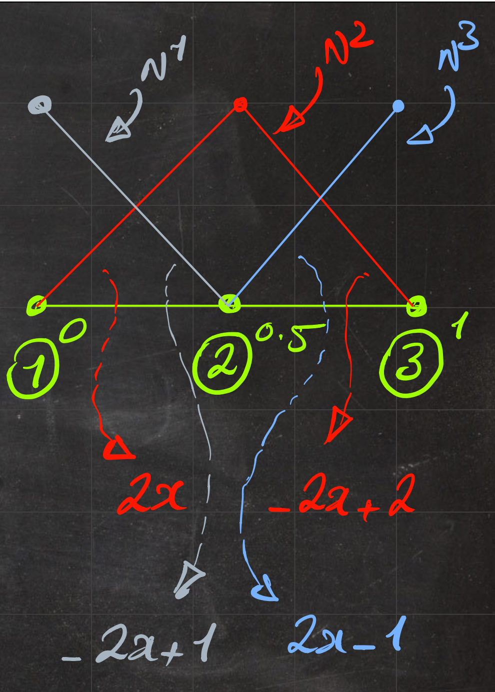
$$w = N^1 \overset{0}{\cancel{w^1}} + N^2 w^2 + N^3 w^3 \qquad u = N^1 \overset{0}{\cancel{u^1}} + N^2 u^2 + N^3 u^3$$

$$\Rightarrow w = N^2 w^2 + N^3 w^3 \qquad \Rightarrow u = N^2 u^2 + N^3 u^3$$

$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \Rightarrow \int_0^{1/2} \cdots + \int_{1/2}^1 \cdots = \cdots$$

$$\int_0^{1/2} w'u'\,dx + \int_{1/2}^1 w'u'\,dx = \int_0^{1/2} w\,dx + \int_{1/2}^1 w\,dx$$

$$w' = N^{2\prime} w^2 + N^{3\prime} w^3 \qquad u' = N^{2\prime} u^2 + N^{3\prime} u^3$$



$u'' + 1 = 0 \qquad 0 \leq x \leq 1$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

2- PIECE LINEAR UNIFORM APPROXIMATION

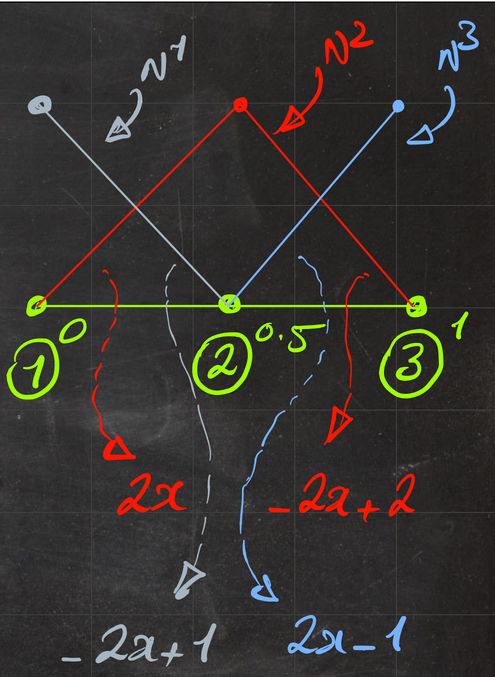
$$w = N^1 \cancelto{0}{w^1} + N^2 w^2 + N^3 w^3 \qquad u = N^1 \cancelto{0}{u^1} + N^2 u^2 + N^3 u^3$$

$$\Rightarrow w = N^2 w^2 + N^3 w^3 \qquad \Rightarrow u = N^2 u^2 + N^3 u^3$$

$$\int_0^{1/2} [N^{2\prime} w^2 + N^{3\prime} w^3][N^{2\prime} u^2 + N^{3\prime} u^3]\, dx$$

$$+ \int_{1/2}^{1} [N^{2\prime} w^2 + N^{3\prime} w^3][N^{2\prime} u^2 + N^{3\prime} u^3]\, dx$$

$$= \int_0^{1/2} [N^2 w^2 + N^3 w^3]\, dx + \int_{1/2}^{1} [N^2 w^2 + N^3 w^3]\, dx$$



$u'' + 1 = 0 \qquad 0 \leq x \leq 1$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ↙

2- PIECE LINEAR UNIFORM APPROXIMATION

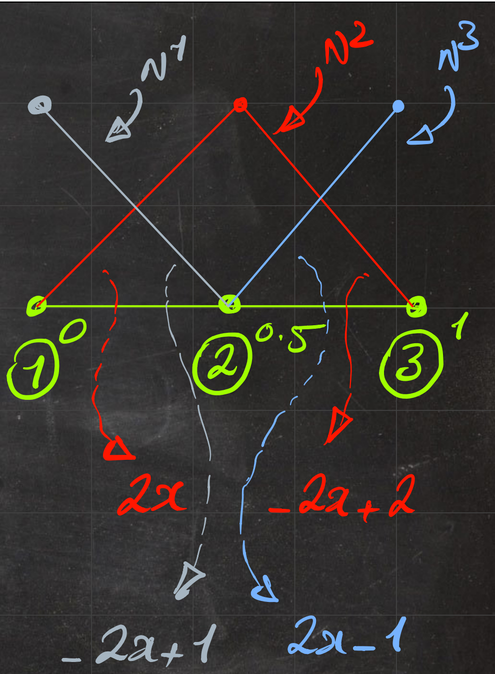
$$w = N^1 \overset{0}{\cancel{w^1}} + N^2 w^2 + N^3 w^3 \qquad u = N^1 \overset{0}{\cancel{u^1}} + N^2 u^2 + N^3 u^3$$

$$\Rightarrow w = N^2 w^2 + N^3 w^3 \qquad \Rightarrow u = N^2 u^2 + N^3 u^3$$

$$\int_0^{1/2} [\underset{2}{\cancel{N^{2\prime}}} w^2 + \underset{0}{\cancel{N^{3\prime}}} w^3][\underset{2}{\cancel{N^{2\prime}}} u^2 + \underset{0}{\cancel{N^{3\prime}}} u^3]\, dx$$

$$+ \int_{1/2}^{1} [\underset{-2}{\cancel{N^{2\prime}}} w^2 + \underset{2}{\cancel{N^{3\prime}}} w^3][\underset{-2}{\cancel{N^{2\prime}}} u^2 + \underset{2}{\cancel{N^{3\prime}}} u^3]\, dx$$

$$= \int_0^{1/2} [\underset{2x}{N^2} w^2 + \overset{0}{N^3} w^3]\, dx + \int_{1/2}^{1} [\underset{-2x+2}{N^2} w^2 + \overset{2x-1}{N^3} w^3]\, dx$$



$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$

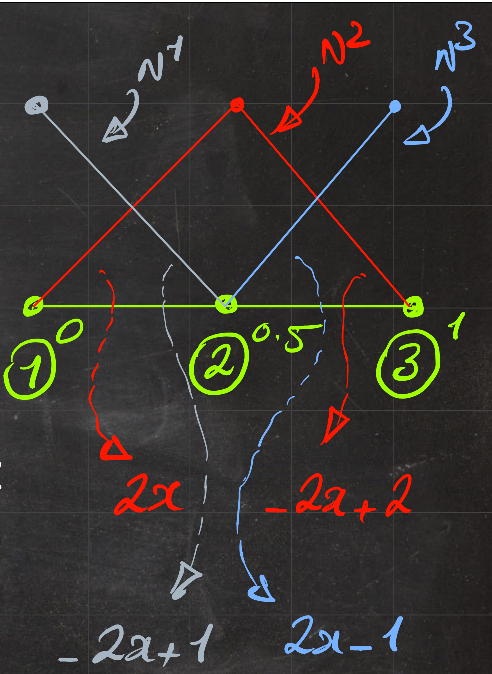
D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

2- PIECE LINEAR UNIFORM APPROXIMATION

$$\omega = N^1 \overset{0}{\cancel{\omega^1}} + N^2\omega^2 + N^3\omega^3 \qquad u = N^1 \overset{0}{\cancel{u^1}} + N^2 u^2 + N^3 u^3$$

$$\Rightarrow \omega = N^2\omega^2 + N^3\omega^3 \qquad \Rightarrow u = N^2 u^2 + N^3 u^3$$



$$\int_0^{1/2} 2\omega^2 \times 2u^2\,dx + \int_{1/2}^{1} [-2\omega^2 + 2\omega^3][-2u^2 + 2u^3]\,dx$$

$$= \int_0^{1/2} 2x\omega^2\,dx + \int_{1/2}^{1} [-2x+2]\,\omega^2\,dx$$

$$+ \int_{1/2}^{1} [2x-1]\,\omega^3\,dx$$

$u'' + 1 = 0 \qquad 0 \leq x \leq 1$

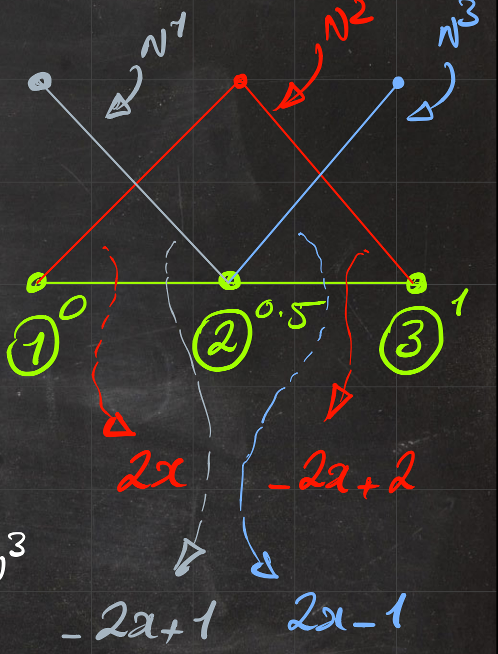
D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

2- PIECE LINEAR UNIFORM APPROXIMATION

$$w = N^1 \cancelto{0}{w^1} + N^2 w^2 + N^3 w^3 \qquad u = N^1 \cancelto{0}{u^1} + N^2 u^2 + N^3 u^3$$

$$\Rightarrow w = N^2 w^2 + N^3 w^3 \qquad \Rightarrow u = N^2 u^2 + N^3 u^3$$



INTEGRATION

$$\ldots \Longrightarrow 2w^2u^2 + 2w^2u^2 - 2w^2u^3 - 2w^3u^2$$
$$+ 2w^3u^3 = \frac{1}{2} w^2 + \frac{1}{4} w^3 \quad \forall w^2, w^3$$

$$\Downarrow$$

$$w^2\left[4u^2 - 2u^3 - \frac{1}{2}\right] + w^3\left[2u^3 - 2u^2 - \frac{1}{4}\right] = 0 \quad \forall w^2, w^3$$

$u'' + 1 = 0 \qquad 0 \leq x \leq 1$

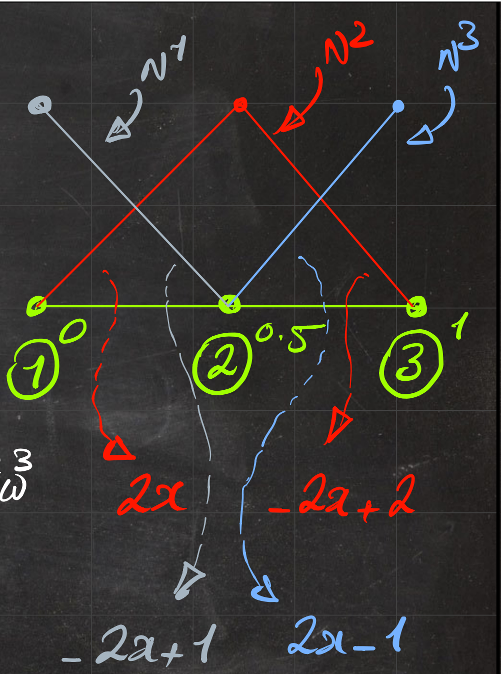
D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

2- PIECE LINEAR UNIFORM APPROXIMATION

$$w = N^1 \overset{\nearrow 0}{\cancel{w^1}} + N^2 w^2 + N^3 w^3 \qquad u = N^1 \overset{\nearrow 0}{\cancel{u^1}} + N^2 u^2 + N^3 u^3$$

$$\Rightarrow w = N^2 w^2 + N^3 w^3 \qquad \Rightarrow u = N^2 u^2 + N^3 u^3$$



$$w^2\left[4u^2 - 2u^3 - \frac{1}{2}\right] + w^3\left[2u^3 - 2u^2 - \frac{1}{4}\right] = 0 \quad \forall w^2, w^3$$

$$\begin{cases} 4u^2 - 2u^3 - \frac{1}{2} = 0 \\ 2u^3 - 2u^2 - \frac{1}{4} = 0 \end{cases} \Rightarrow \begin{cases} u^2 = \frac{3}{8} \\ u^3 = \frac{1}{2} \end{cases} \Rightarrow u = u(x) \ ✓$$

$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

1-PIECE QUADRATIC APPROXIMATION

$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$

$D: \; u(0) = 0$ ← prescribed

$N: \; u'(1) = 0$ ←

Summary:

in the previous approach $\rightsquigarrow$ we had $\begin{cases} u = \alpha_1 x + \beta_1 & 0 \leq x \leq \frac{1}{2} \\ u = \alpha_2 x + \beta_2 & \frac{1}{2} \leq x \leq 1 \end{cases}$

... $\Rightarrow$ WE CALCULATED $\alpha_1, \alpha_2, \beta_1, \beta_2$ $\rightsquigarrow$ $\underbrace{\text{THEN COMPUTE NODAL VALUES}}_{\text{NECESSARY}}$

in the current approach $\rightsquigarrow$ we have $u = N^1 u^1 + N^2 u^2 + N^3 u^3 \quad \begin{cases} 0 \leq x \leq \frac{1}{2} \\ \frac{1}{2} \leq x \leq 1 \end{cases}$

... $\Rightarrow$ WE CALCULATE u^1, u^2, u^3 $\rightsquigarrow$ $\overbrace{\text{THEN COMPUTE}}^{\text{UNNECESSARY}}$ $\begin{cases} u = u(x) & 0 \leq x \leq \frac{1}{2} \\ u = u(x) & \frac{1}{2} \leq x \leq 1 \end{cases}$