FINITE ELEMENT METHOD

3

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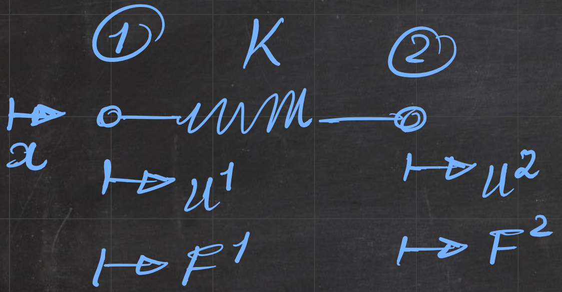
Understanding FEM via

Spring Combinations

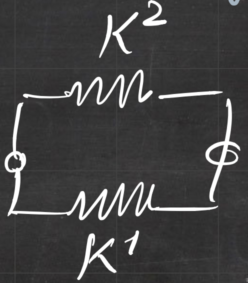
↪ Computer-based Algorithm

Understanding key ingredients of FEM using springs:

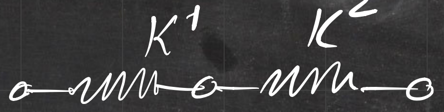


$$\begin{bmatrix} F^1 \\ F^2 \end{bmatrix} = \begin{bmatrix} K & -K \\ -K & K \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix}$$

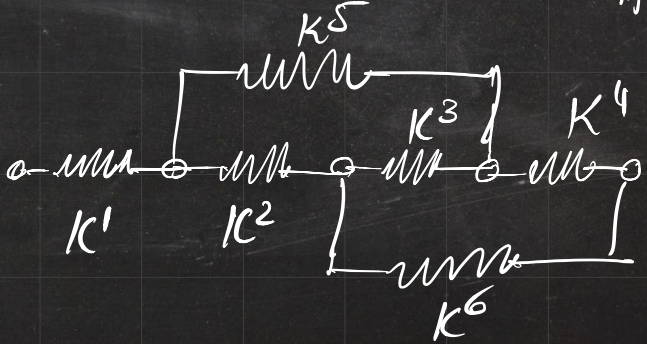
$$\mathbb{F} = \mathbb{K} \cdot \mathbb{u}$$

I) 

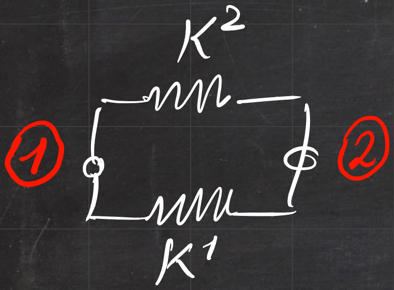
$\rightsquigarrow K_{eff}$?

II) 

$\rightsquigarrow K_{eff}$?

III) 

Understanding key ingredients of FEM using springs:



$F^1 \qquad F^2$

$\hookrightarrow F^{1(1)} + F^{1(2)} \qquad \hookrightarrow F^{2(1)} + F^{2(2)}$

FROM SPRING

NODE

$u^1 \qquad u^2$

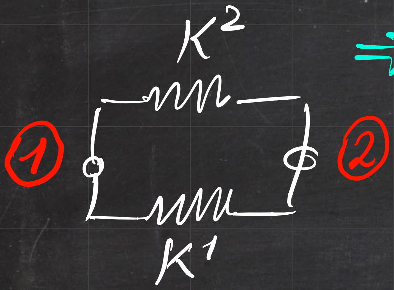
SPRING 1:

$$\begin{bmatrix} F^{1(1)} \\ F^{2(1)} \end{bmatrix} = \begin{bmatrix} K^1 & -K^1 \\ -K^1 & K^1 \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix}$$

SPRING 2:

$$\begin{bmatrix} F^{1(2)} \\ F^{2(2)} \end{bmatrix} = \begin{bmatrix} K^2 & -K^2 \\ -K^2 & K^2 \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix}$$

Understanding key ingredients of FEM using springs:



$$\Rightarrow \boxed{K_{eff} = K^1 + K^2}$$

Parallel Combination

SPRING 1:

$$\begin{bmatrix}F^{1(1)}\\ F^{2(1)}\end{bmatrix} = \begin{bmatrix}K^1 & -K^1\\ -K^1 & K^1\end{bmatrix}\begin{bmatrix}u^1\\ u^2\end{bmatrix}$$

SPRING 2:

$$\begin{bmatrix}F^{1(2)}\\ F^{2(2)}\end{bmatrix} = \begin{bmatrix}K^2 & -K^2\\ -K^2 & K^2\end{bmatrix}\begin{bmatrix}u^1\\ u^2\end{bmatrix}$$

FORCE @ NODE 1

$$\begin{bmatrix}F^1\\ F^2\end{bmatrix} = \begin{bmatrix}F^{1(1)} + F^{1(2)}\\ F^{2(1)} + F^{2(2)}\end{bmatrix}$$

FORCE @ NODE 2

$\xrightarrow{\quad} F^1$ $\qquad$ $\xrightarrow{\quad} F^2$

$\hookrightarrow F^{1(1)} + F^{1(2)}$ $\qquad$ $\hookrightarrow F^{2(1)} + F^{2(2)}$

FROM SPRING

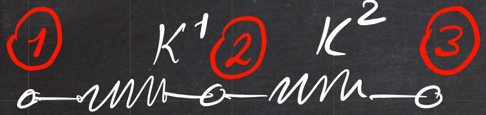
NODE

$\xrightarrow{\quad} u^1$ $\qquad$ $\xrightarrow{\quad} u^2$

$$\Rightarrow \begin{bmatrix}F^1\\ F^2\end{bmatrix} = \begin{bmatrix}[K^1 + K^2] & -[K^1 + K^2]\\ -[K^1 + K^2] & [K^1 + K^2]\end{bmatrix}\begin{bmatrix}u^1\\ u^2\end{bmatrix}$$

$$\Rightarrow \mathbb{F} = \mathbb{K}_{eff} \cdot \mathbb{U} \qquad \mathbb{K}_{eff} = \begin{bmatrix}K_{eff} & -K_{eff}\\ -K_{eff} & K_{eff}\end{bmatrix}$$

Understanding key ingredients of FEM using springs:



$\vec{F}^1 \qquad \vec{F}^2 \qquad \vec{F}^3$

$F^2 = F^{2(1)} + F^{2(2)}$

$\vec{u}^1 \qquad \vec{u}^2 \qquad \vec{u}^3$

$$\frac{1}{K_{eff}} = \frac{1}{K_1} + \frac{1}{K_2}$$

SPRING 1:

$$\begin{bmatrix} F^1 \\ F^{2(1)} \end{bmatrix} = \begin{bmatrix} K^1 & -K^1 \\ -K^1 & K^1 \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix}$$

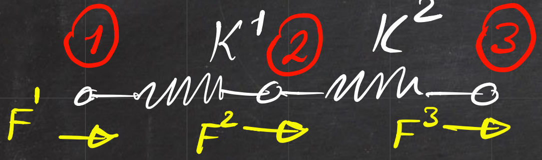
SPRING 2:

$$\begin{bmatrix} F^{2(2)} \\ F^3 \end{bmatrix} = \begin{bmatrix} K^2 & -K^2 \\ -K^2 & K^2 \end{bmatrix} \begin{bmatrix} u^2 \\ u^3 \end{bmatrix}$$

GLOBAL STIFFNESS

$$\begin{bmatrix} F^1 \\ F^2 \\ F^3 \end{bmatrix} = \begin{bmatrix} K^1 & -K^1 & 0 \\ -K^1 & K^1+K^2 & -K^2 \\ 0 & -K^2 & K^2 \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \end{bmatrix}$$

Understanding key ingredients of FEM using springs:



$$-K^1u^1 + [K^1+K^2]u^2 - K^2u^3 = 0 \Rightarrow u^2 = \frac{K^1u^1+K^2u^3}{K^1+K^2}$$

$$\Rightarrow F^1 = \underbrace{\frac{K^1K^2}{K^1+K^2}}_{K_{eff}}[u^1-u^3]$$

$$0 \Leftarrow +$$

$$\Rightarrow F^3 = \underbrace{\frac{K^1K^2}{K^1+K^2}}_{K_{eff}}[u^3-u^1]$$

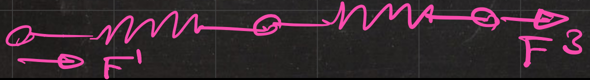
$$F^1+F^3=0$$

$$\begin{bmatrix} F^1 \\ F^2 \\ F^3 \end{bmatrix} = \begin{bmatrix} K^1 & -K^1 & 0 \\ -K^1 & K^1+K^2 & -K^2 \\ 0 & -K^2 & K^2 \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \end{bmatrix}$$

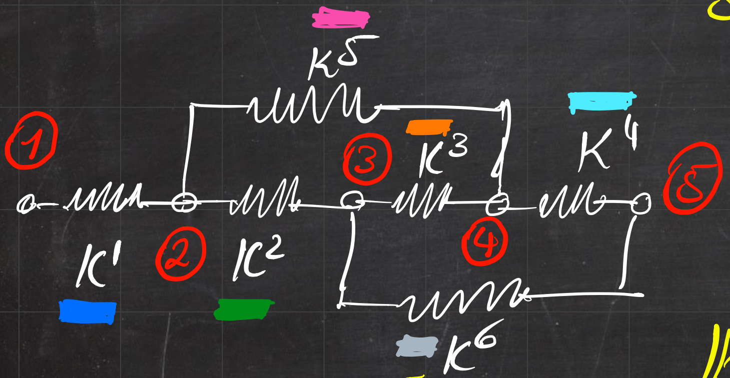
$$\begin{bmatrix} F^1 \\ 0 \\ F^3 \end{bmatrix} = \begin{bmatrix} K^1 & -K^1 & 0 \\ -K^1 & K^1+K^2 & -K^2 \\ 0 & -K^2 & K^2 \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \end{bmatrix}$$

$$\begin{bmatrix} F^1 \\ F^3 \end{bmatrix} = \begin{bmatrix} K_{eff} & -K_{eff} \\ -K_{eff} & K_{eff} \end{bmatrix} \begin{bmatrix} u^1 \\ u^3 \end{bmatrix}$$

SYM, DET = 0



Understanding key ingredients of FEM using springs:



$$
\begin{bmatrix} F^1 \\ F^2 \\ F^3 \\ F^4 \\ F^5 \end{bmatrix}
=
\underbrace{\begin{bmatrix} K^{11} & K^{12} & K^{13} & K^{14} & K^{15} \\ K^{21} & K^{22} & K^{23} & \cdots & \\ K^{31} & \cdots & & & \\ K^{41} & \cdots & & & \\ K^{51} & \cdots & & & K^{55} \end{bmatrix}}_{\mathbb{K}}
\begin{bmatrix} u^1 \\ u^2 \\ u^3 \\ u^4 \\ u^5 \end{bmatrix}
\Rightarrow \mathbb{F} = \mathbb{K} \cdot \mathbb{u}
$$

$0 \leftarrow F^2$, $0 \leftarrow F^3$, $0 \leftarrow F^4$

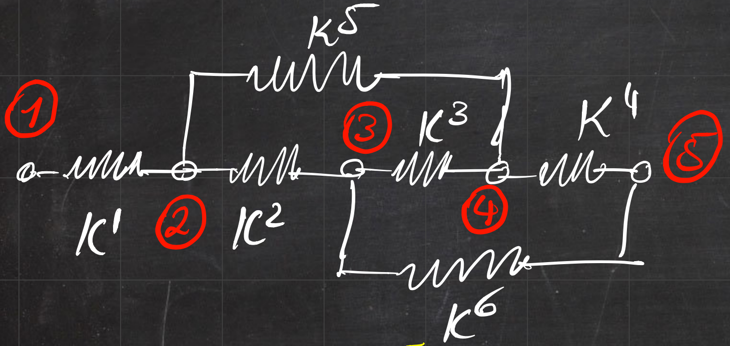
Sym.
&
DET=0

$$
\mathbb{K} =
\begin{array}{c|ccccc}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & K^1 & -K^1 & 0 & 0 & 0 \\
2 & -K^1 & K^1+K^2+K^5 & -K^2 & -K^5 & 0 \\
3 & 0 & -K^2 & K^2+K^3+K^6 & -K^3 & -K^6 \\
4 & 0 & -K^5 & -K^3 & K^3+K^4+K^5 & -K^4 \\
5 & 0 & 0 & -K^6 & -K^4 & K^4+K^6
\end{array}
$$

5×5

Understanding key ingredients of FEM using springs:

$$k^1 = k^2 = k^3 = k^4 = k^5 = k^6 = 1$$



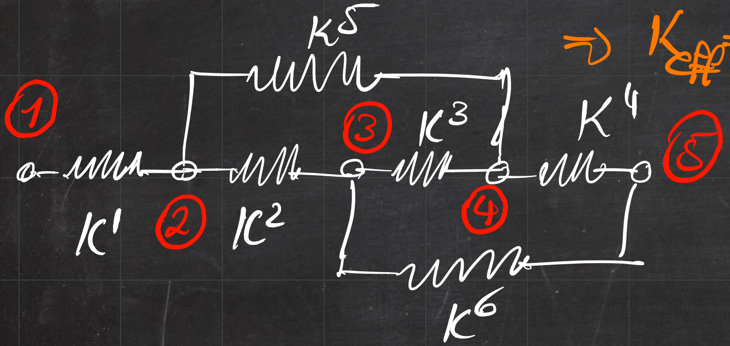
$$\begin{bmatrix} F^1 \\ F^2 \\ F^3 \\ F^4 \\ F^5 \end{bmatrix} = \underbrace{\begin{bmatrix} K^{11} & K^{12} & K^{13} & K^{14} & K^{15} \\ K^{21} & K^{22} & K^{23} & \dots & \\ K^{31} & \dots & & & \\ K^{41} & \dots & & & \\ K^{51} & \dots & & & K^{55} \end{bmatrix}}_{\mathbb{K}} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \\ u^4 \\ u^5 \end{bmatrix} \Rightarrow \mathbb{F} = \mathbb{K} \cdot \mathbb{u}$$

$0 \leftarrow F^2$, $0 \leftarrow F^3$, $0 \leftarrow F^4$

$$\mathbb{K} = \underbrace{\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ 0 & -1 & 3 & -1 & -1 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}}_{5 \times 5}$$

Understanding key ingredients of FEM using springs:

$K^1 = K^2 = K^3 = K^4 = K^5 = K^6 = 1$



$\Rightarrow K_{eff} = 1.5$

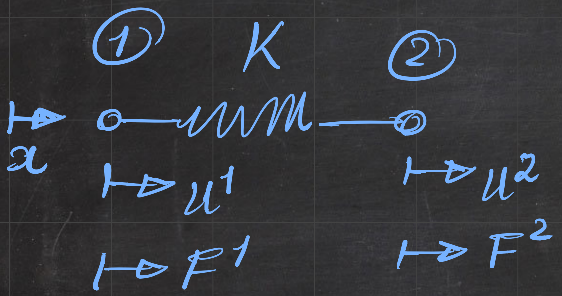
$$\begin{bmatrix} F^1 \\ F^5 \end{bmatrix} = \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} \begin{bmatrix} u^1 \\ u^5 \end{bmatrix}$$

$\Rightarrow F^5 = 0.5\,[u^5 - u^1]$

$$\mathbb{K} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ 0 & -1 & 3 & -1 & -1 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}_{5\times5}$$

$\Rightarrow F = Ku$

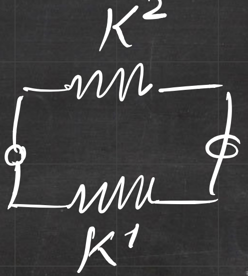
Understanding key ingredients of FEM using springs:



$$\begin{bmatrix} F^1 \\ F^2 \end{bmatrix} = \begin{bmatrix} K & -K \\ -K & K \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix}$$

$$\mathbb{F} = \mathbb{K} \cdot \mathbb{U}$$

I)

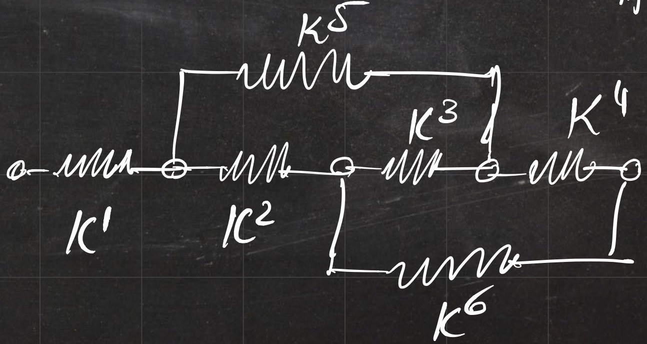


$\rightsquigarrow K_{eff}$?

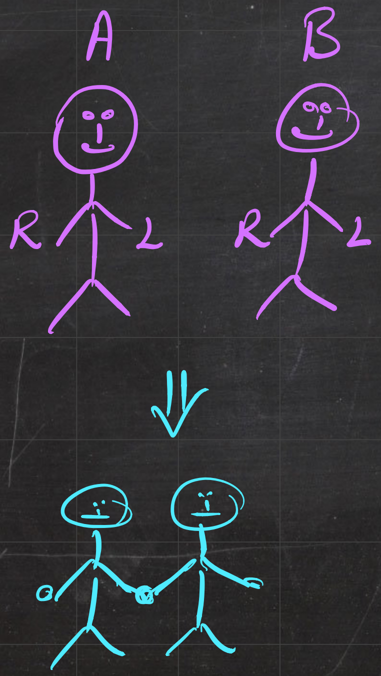
II)

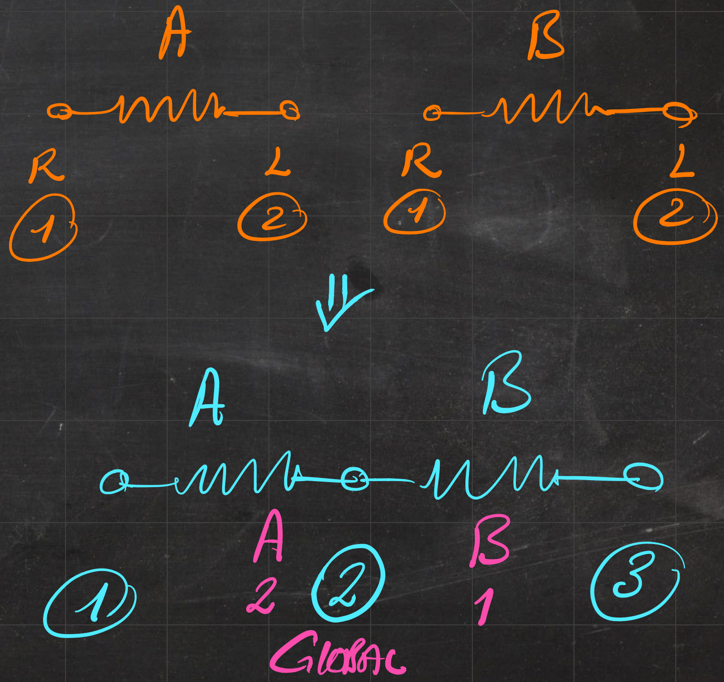
$\rightsquigarrow K_{eff}$?

III)

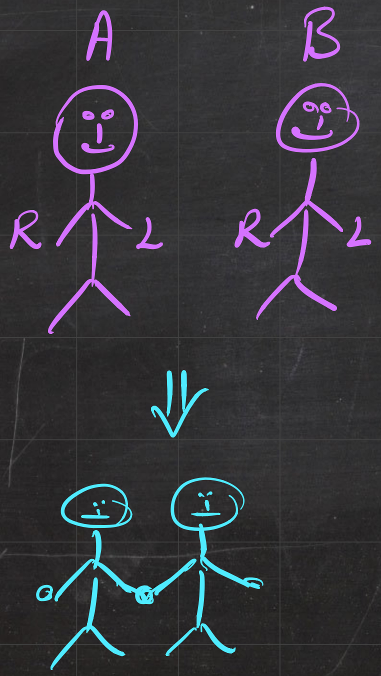


TOWARDS AN ALGORITHMIC APPROACH TO ASSEMBLY:





TOWARDS AN ALGORITHMIC APPROACH TO ASSEMBLY:

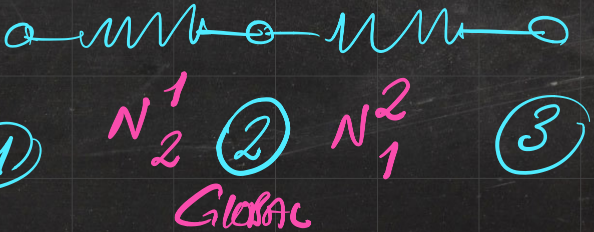


Superscript: GLOBAL

subscript: LOCAL

GLOBAL NODE

$$N^2 = N_2^1 = N_1^2$$



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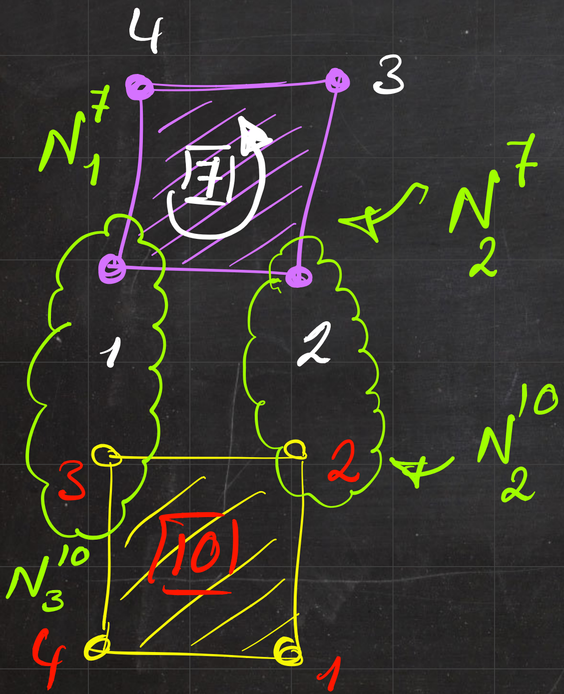
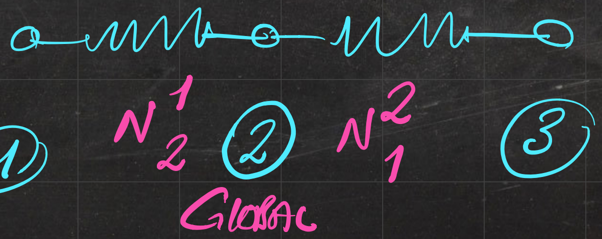


Superscript : GLOBAL

subscript : LOCAL

GLOBAL NODE

$$N^2 = N_2^1 = N_1^2$$



TOWARDS AN ALGORITHMIC APPROACH TO ASSEMBLY:

* ELEMENTS DO NOT HAVE LEFT AND RIGHT

↳ ALTERNATIVELY, THEY HAVE "ORIENTATION"

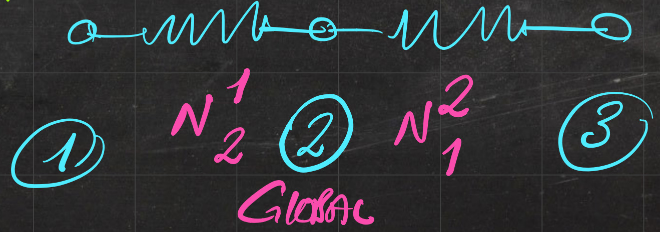
↘ LOCAL C.C.W. NODE NUMBERING

Superscript: GLOBAL

subscript: LOCAL

GLOBAL NODE

$$N^2 = N_2^1 = N_1^2$$

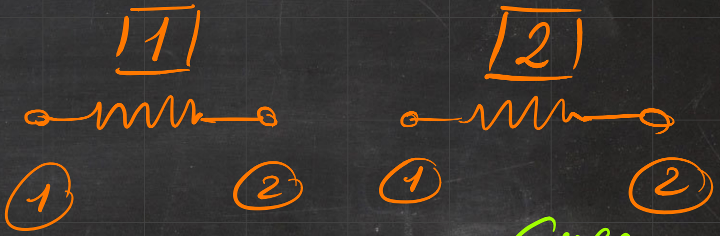


TOWARDS AN ALGORITHMIC APPROACH TO ASSEMBLY:

* ELEMENTS DO NOT HAVE NAMES

↳ ALTERNATIVELY, THEY HAVE NUMBERS TO IDENTIFY THEM

GLOBAL



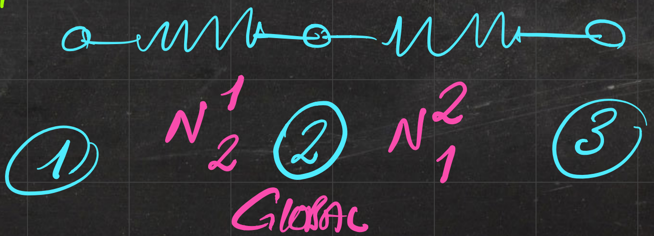
GLOBAL ELEMENT

Superscript: GLOBAL

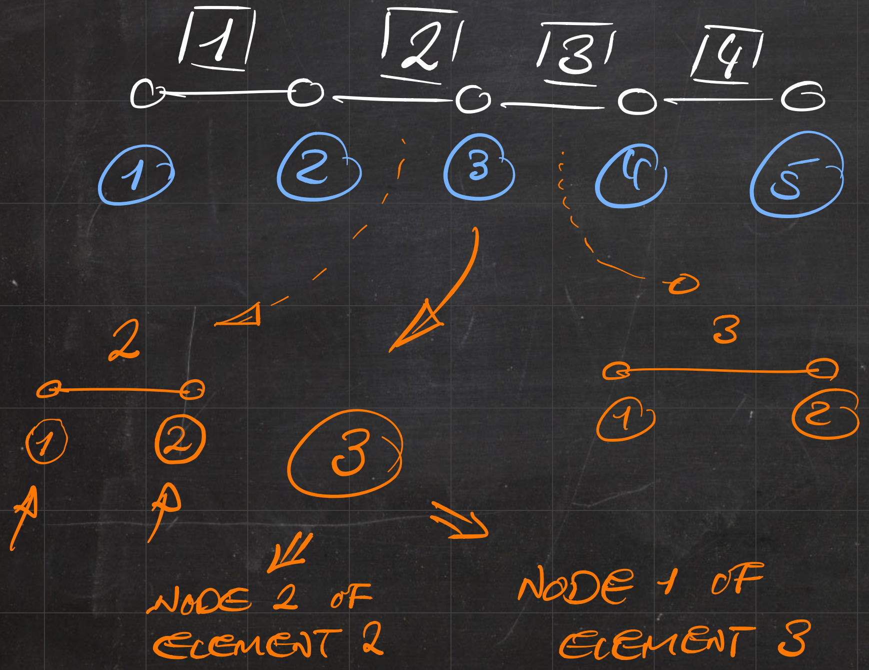
subscript: LOCAL

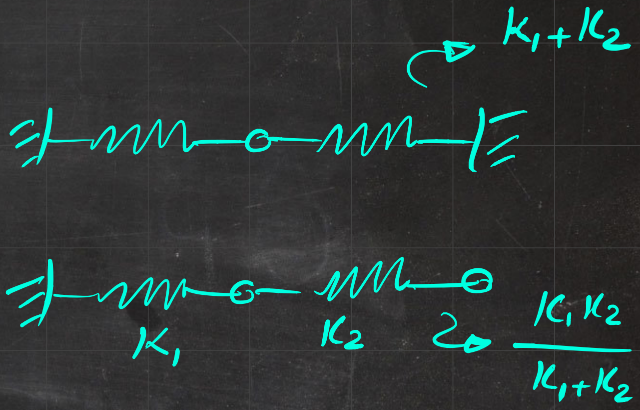
GLOBAL NODE

$$N^2 = N_2^1 = N_1^2$$



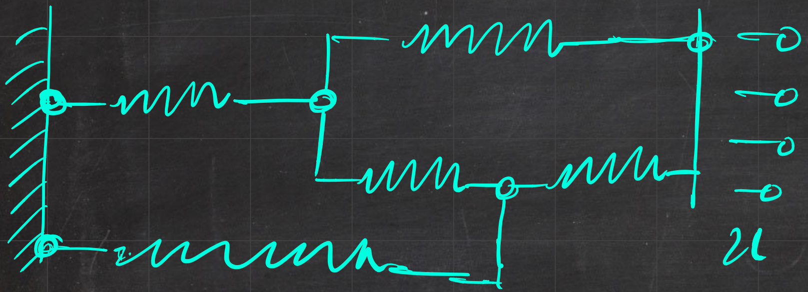
TOWARDS AN ALGORITHMIC APPROACH TO ASSEMBLY:

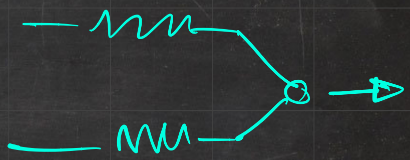


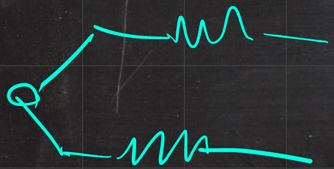


$$\Rightarrow K^{3} = K^{2}_{2} + K^{3}_{1}$$

TOWARDS AN ALGORITHMIC APPROACH TO ASSEMBLY:







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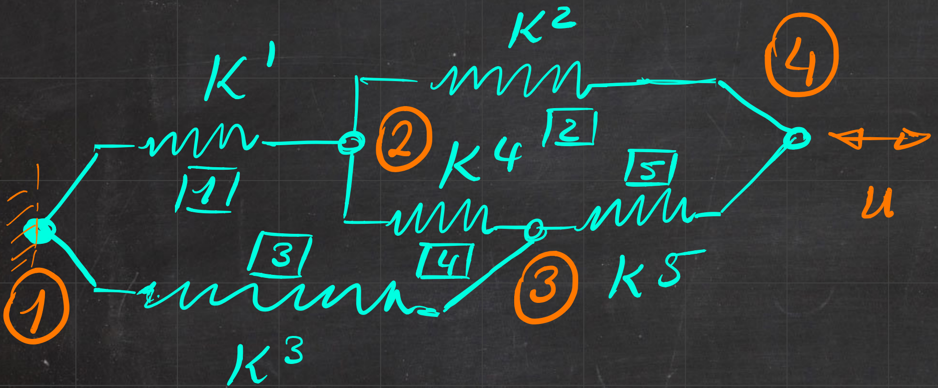
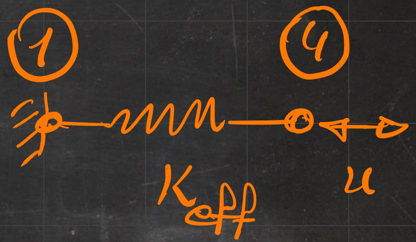




ELEMENT 1

$$\mathbb{K}^1 = \begin{bmatrix} K^1 & -K^1 \\ -K^1 & K^1 \end{bmatrix}$$

BOLD

ELEMENT 2

$$\mathbb{K}^2 = \begin{bmatrix} K^2 & -K^2 \\ -K^2 & K^2 \end{bmatrix}$$

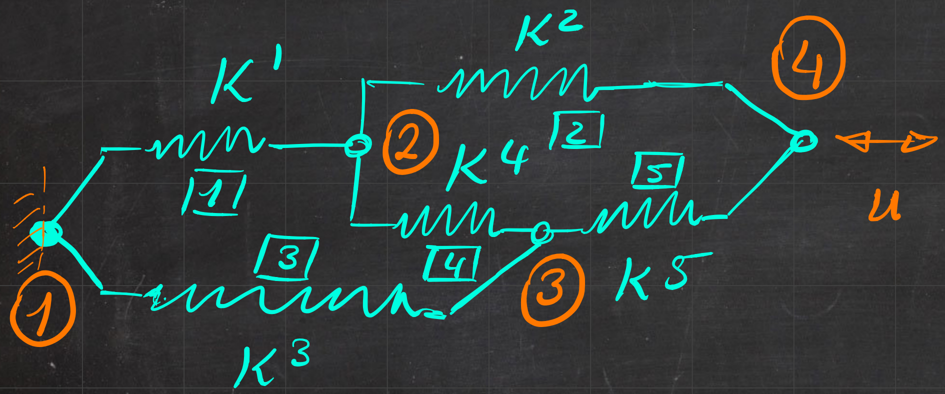
ELEMENT 3

$$\mathbb{K} = \begin{bmatrix} K^3 & -K^3 \\ -K^3 & K^3 \end{bmatrix}$$

ELEMENT

$$\mathbb{K} = \begin{bmatrix} K & -K \\ -K & K \end{bmatrix}$$

TOWARDS AN ALGORITHMIC APPROACH TO ASSEMBLY:



ELEMENT 5

$$\mathbb{K}^5 = \begin{bmatrix} k^5 & -k^5 \\ -k^5 & k^5 \end{bmatrix}$$

ELEMENT 1

$$\mathbb{K}^1 = \begin{bmatrix} k^1 & -k^1 \\ -k^1 & k^1 \end{bmatrix}$$

BOUND

ELEMENT 2

$$\mathbb{K}^2 = \begin{bmatrix} k^2 & -k^2 \\ -k^2 & k^2 \end{bmatrix}$$

ELEMENT 3

$$\mathbb{K}^3 = \begin{bmatrix} k^3 & -k^3 \\ -k^3 & k^3 \end{bmatrix}$$

ELEMENT 4

$$\mathbb{K}^4 = \begin{bmatrix} k^4 & -k^4 \\ -k^4 & k^4 \end{bmatrix}$$

$$
\overset{\text{GLOBAL}}{\mathbb{K}} = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cccc} & & & \\ & & & \\ & & & \\ & & & \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
$$