FINITE ELEMENT METHOD

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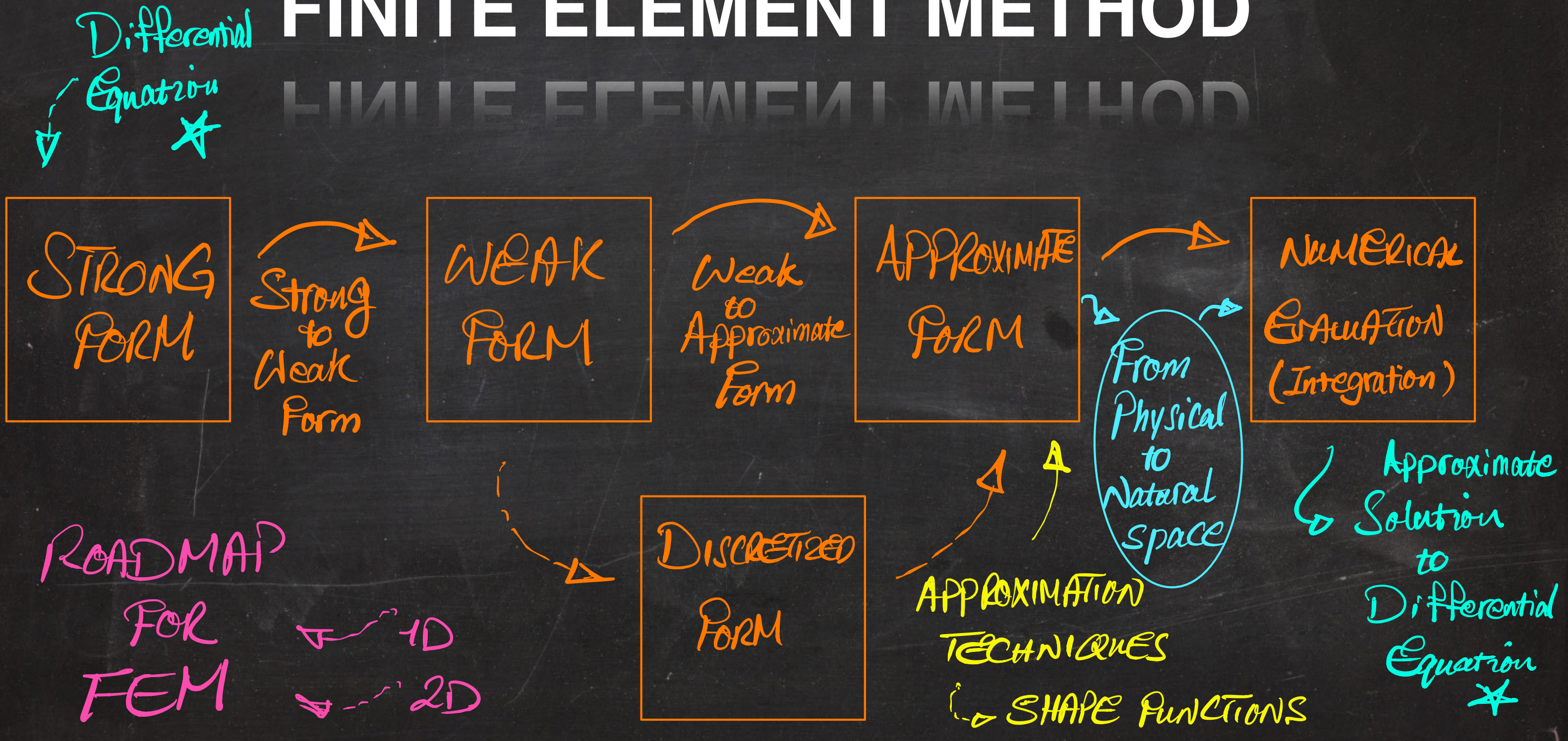
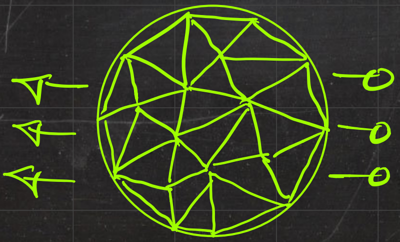
FINITE ELEMENT METHOD



UNDERSTANDING FEM VIA AN ANALOGY (A BRUTAL SIMPLIFICATION)

Approximations in FEM

- → Solution Approximation → inherent to numerical techniques using computers
- → Equation Approximation → diff equation is transformed to weak form + space approximation
- → Input Approximation → space discretization



Discretization (Approximation)
- → DOMAIN (X)
- → SOLUTION (u)
- → TEST (w)

diff. Eq. → STRONG FORM

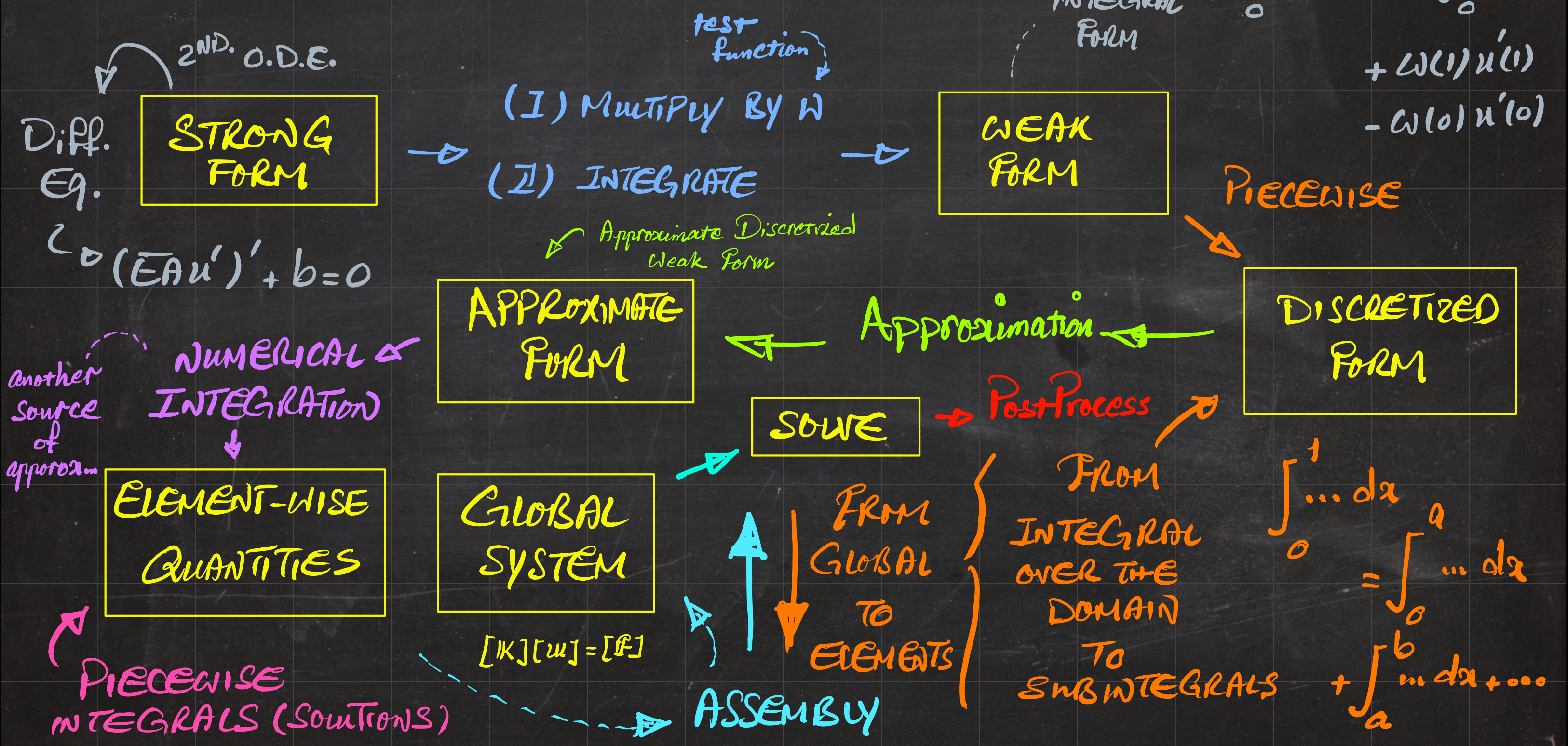
TO

integral → WEAK FORM

1D FEM

Overview and Wrap-up

FINITE ELEMENT METHOD IN A NUTSHELL



FROM STRONG FORM TO ELEMENT STIFFNESS ← IN PHYSICAL SPACE NOW

$EAu''=0$ SUBJECT TO BCs

- DIRICHLET → u IS PRESCRIBED
- NEUMANN → u' IS PRESCRIBED

↓ MULTIPLY BY TEST FUNCTION w

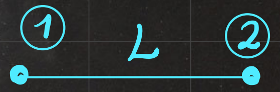
$EAwu''=0$ ← $wu'' = (wu')' - w'u'$

$EA\left[(wu')' - w'u'\right] = 0 \Rightarrow EA\,w'u' = EA\,(wu')'$ ← INTEGRATE

$$\int_L EA\,w'u'\,dx = \int_L EA\,(wu')'\,dx = EA\,wu'\Big]_{①}^{②} = EA\,w^2 u'^2 - EA\,w^1 u'^1$$

FROM STRONG FORM TO ELEMENT STIFFNESS ← IN PHYSICAL SPACE now

$EAu'' = 0$ SUBJECT TO BCs



$$EA\begin{bmatrix}\int_L N^{1'}N^{1'}dx & \int_L N^{1'}N^{2'}dx\\ \int_L N^{2'}N^{1'}dx & \int_L N^{2'}N^{2'}dx\end{bmatrix}\begin{bmatrix}u^1\\ u^2\end{bmatrix}=\begin{bmatrix}F^1\\ F^2\end{bmatrix}$$

$$\begin{bmatrix}K^{11} & K^{12}\\ K^{21} & K^{22}\end{bmatrix}\begin{bmatrix}u^1\\ u^2\end{bmatrix}=\begin{bmatrix}F^1\\ F^2\end{bmatrix} \leftarrow K^{ij} = EA\int_L N^{i'}N^{j'}dx$$

$$K^{ij} = EA \int_L N^{i\prime} N^{j\prime} \, dx$$

PHYSICAL

RECALL:

$$\int_{-1}^{1} g(\xi)\, d\xi = \sum_{gp=1}^{GPE} g(\xi_{gp})\, \alpha_{gp}$$

NATURAL

$$= EA \int_{-1}^{1} \frac{\partial N^i}{\partial \xi} \frac{\partial N^j}{\partial \xi} J^{-1} \, d\xi$$

$$= EA \sum_{gp=1}^{GPE} \left\{ \left[\frac{\partial N^i}{\partial \xi} \frac{\partial N^j}{\partial \xi} J^{-1} \right]_{gp} \times \alpha_{gp} \right\}$$

Loop over gp

⋮

END

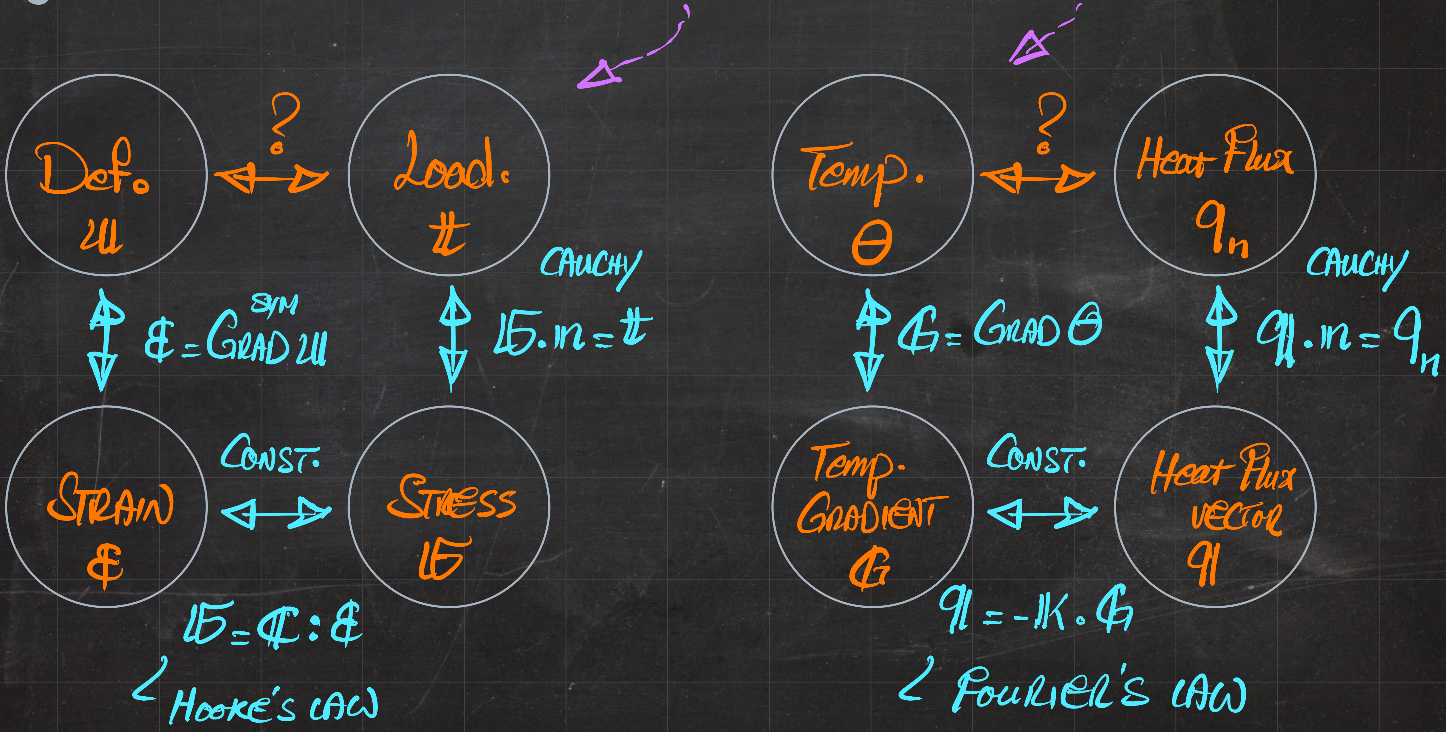
e.g. in MATLAB

WHAT YOU SEE IN THE CODE !

```
For gp=1:GPE
⋮
End
```

2D FEM

Big Picture of Mechanics (Mechanical Problems & Thermal Problems)



STRONG FORM (GENERIC FORM) $\rightsquigarrow \operatorname{Div} \boldsymbol{\sigma} + \mathbf{b} = \mathbf{0}$, $\operatorname{Div} \mathbf{q} + c = 0$

$$\frac{\partial \sigma_{jk}}{\partial x_k} + b_j = 0$$

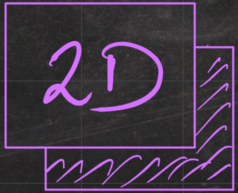
$$\frac{\partial q_i}{\partial x_i} + c = 0$$

$$\begin{cases} \dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{xy}}{\partial y} + b_x = 0 \\[2ex] \dfrac{\partial \sigma_{yx}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} + b_y = 0 \end{cases}$$

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + c = 0$$

2D $\rightsquigarrow$ Plane STRAIN
Plane STRESS

$$\left.\begin{array}{l} \text{Plane STRAIN} \\ \text{Plane STRESS} \end{array}\right\} \boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\varepsilon}$$



2D Finite Element Library

two-dimensional finite elements library

- two-dimensional 4-noded quadrilateral element (D2QU4N)

  a.k.a. bilinear quadrilateral element

- two-dimensional 9-noded quadrilateral element (D2QU9N)

  a.k.a. Lagrange biquadratic quadrilateral element

- two-dimensional 8-noded quadrilateral element (D2QU8N)

  a.k.a. serendipity biquadratic quadrilateral element

- two-dimensional 3-noded triangular element (D2TR3N)

  a.k.a. constant strain triangle

- two-dimensional 6-noded triangular element (D2TR6N)

  a.k.a. quadratic triangle

- two-dimensional quadrature rule

2D Finite Element Library

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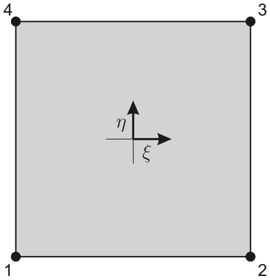
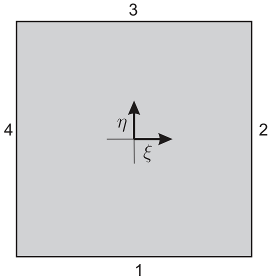
D2QU4N

bilinear quadrilateral element

Nodes



Edges



Node Number	Coordinates	
	ξ	η
1	-1	-1
2	1	-1
3	1	1
4	-1	1

$$N^1 = \frac{1}{4}\left(1-\xi\right)\left(1-\eta\right)$$

$$N^1_{,\xi} = -\frac{1}{4}\left(1-\eta\right) \qquad N^1_{,\eta} = -\frac{1}{4}\left(1-\xi\right)$$

$$N^2 = \frac{1}{4}\left(1+\xi\right)\left(1-\eta\right)$$

$$N^2_{,\xi} = +\frac{1}{4}\left(1-\eta\right) \qquad N^2_{,\eta} = -\frac{1}{4}\left(1+\xi\right)$$

$$N^3 = \frac{1}{4}\left(1+\xi\right)\left(1+\eta\right)$$

$$N^3_{,\xi} = +\frac{1}{4}\left(1+\eta\right) \qquad N^3_{,\eta} = +\frac{1}{4}\left(1+\xi\right)$$

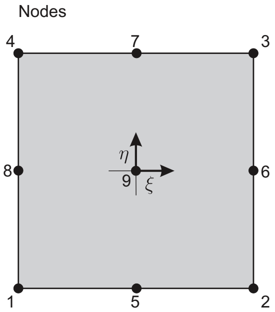
$$N^4 = \frac{1}{4}\left(1-\xi\right)\left(1+\eta\right)$$

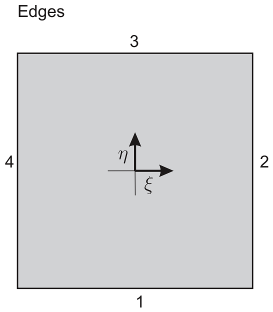
$$N^4_{,\xi} = -\frac{1}{4}\left(1+\eta\right) \qquad N^4_{,\eta} = +\frac{1}{4}\left(1-\xi\right)$$

2D Finite Element Library

D2QU9N

Lagrange biquadratic quadrilateral element





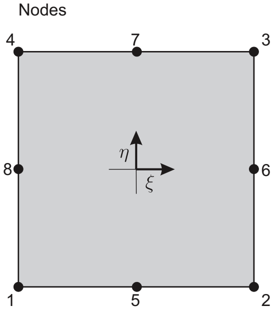
Node Number	Coordinates	
	ξ	η
1	-1	-1
2	1	-1
3	1	1
4	-1	1
5	0	-1
6	1	0
7	0	1
8	-1	0
9	0	0

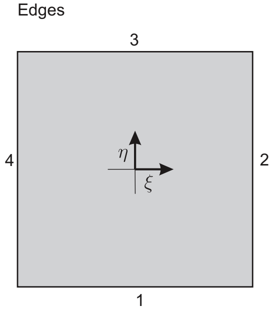
$$N^1 = +\frac{1}{4}\left(1-\xi\right)\xi\left(1-\eta\right)\eta$$
$$N^2 = -\frac{1}{4}\left(1+\xi\right)\xi\left(1-\eta\right)\eta$$
$$N^3 = +\frac{1}{4}\left(1+\xi\right)\xi\left(1+\eta\right)\eta$$
$$N^4 = -\frac{1}{4}\left(1-\xi\right)\xi\left(1+\eta\right)\eta$$
$$N^5 = -\frac{1}{2}\left(1-\xi\right)\left(1+\xi\right)\left(1-\eta\right)\eta$$
$$N^6 = +\frac{1}{2}\left(1+\xi\right)\xi\left(1-\eta\right)\left(1+\eta\right)$$
$$N^7 = +\frac{1}{2}\left(1-\xi\right)\left(1+\xi\right)\left(1+\eta\right)\eta$$
$$N^8 = -\frac{1}{2}\left(1-\xi\right)\xi\left(1-\eta\right)\left(1+\eta\right)$$
$$N^9 = \left(1-\xi\right)\left(1+\xi\right)\left(1-\eta\right)\left(1+\eta\right)$$

$$N^1_{,\xi} = +\frac{1}{4}\left(1-2\xi\right)\left(1-\eta\right)\eta$$
$$N^2_{,\xi} = -\frac{1}{4}\left(1+2\xi\right)\left(1-\eta\right)\eta$$
$$N^3_{,\xi} = +\frac{1}{4}\left(1+2\xi\right)\left(1+\eta\right)\eta$$
$$N^4_{,\xi} = -\frac{1}{4}\left(1-2\xi\right)\left(1+\eta\right)\eta$$
$$N^5_{,\xi} = \xi\eta\left(1-\eta\right)$$
$$N^6_{,\xi} = \frac{1}{2}\left(1+2\xi\right)\left(1-\eta\right)\left(1+\eta\right)$$
$$N^7_{,\xi} = -\xi\eta\left(1+\eta\right)$$
$$N^8_{,\xi} = -\frac{1}{2}\left(1-2\xi\right)\left(1-\eta\right)\left(1+\eta\right)$$
$$N^9_{,\xi} = -2\xi\left(1-\eta\right)\left(1+\eta\right)$$

$$N^1_{,\eta} = +\frac{1}{4}\left(1-\xi\right)\xi\left(1-2\eta\right)$$
$$N^2_{,\eta} = -\frac{1}{4}\left(1+\xi\right)\xi\left(1-2\eta\right)$$
$$N^3_{,\eta} = +\frac{1}{4}\left(1+\xi\right)\xi\left(1+2\eta\right)$$
$$N^4_{,\eta} = -\frac{1}{4}\left(1-\xi\right)\xi\left(1+2\eta\right)$$
$$N^5_{,\eta} = \frac{1}{2}\left(1-\xi\right)\left(1+\xi\right)\left(2\eta-1\right)$$
$$N^6_{,\eta} = -\left(1+\xi\right)\xi\eta$$
$$N^7_{,\eta} = \frac{1}{2}\left(1-\xi\right)\left(1+\xi\right)\left(1+2\eta\right)$$
$$N^8_{,\eta} = \left(1-\xi\right)\xi\eta$$
$$N^9_{,\eta} = -2\left(1-\xi\right)\left(1+\xi\right)\eta$$

D2QU8N serendipity biquadratic quadrilateral element





Node Number	Coordinates	
	ξ	η
1	-1	-1
2	1	-1
3	1	1
4	-1	1
5	0	-1
6	1	0
7	0	1
8	-1	0

$$N^1 = -\frac{1}{4}\left(1-\xi\right)\left(1-\eta\right)\left(1+\xi+\eta\right) \qquad N^1_{,\xi} = +\frac{1}{4}\left(1-\eta\right)\left(2\xi+\eta\right) \qquad N^1_{,\eta} = +\frac{1}{4}\left(1-\xi\right)\left(\xi+2\eta\right)$$

$$N^2 = -\frac{1}{4}\left(1+\xi\right)\left(1-\eta\right)\left(1-\xi+\eta\right) \qquad N^2_{,\xi} = +\frac{1}{4}\left(1-\eta\right)\left(2\xi-\eta\right) \qquad N^2_{,\eta} = +\frac{1}{4}\left(1+\xi\right)\left(-\xi+2\eta\right)$$

$$N^3 = -\frac{1}{4}\left(1+\xi\right)\left(1+\eta\right)\left(1-\xi-\eta\right) \qquad N^3_{,\xi} = +\frac{1}{4}\left(1+\eta\right)\left(2\xi+\eta\right) \qquad N^3_{,\eta} = +\frac{1}{4}\left(1+\xi\right)\left(\xi+2\eta\right)$$

$$N^4 = -\frac{1}{4}\left(1-\xi\right)\left(1+\eta\right)\left(1+\xi-\eta\right) \qquad N^4_{,\xi} = +\frac{1}{4}\left(1+\eta\right)\left(2\xi-\eta\right) \qquad N^4_{,\eta} = +\frac{1}{4}\left(1-\xi\right)\left(-\xi+2\eta\right)$$

$$N^5 = \frac{1}{2}\left(1-\xi\right)\left(1+\xi\right)\left(1-\eta\right) \qquad N^5_{,\xi} = -\xi\left(1-\eta\right) \qquad N^5_{,\eta} = -\frac{1}{2}\left(1-\xi\right)\left(1+\xi\right)$$

$$N^6 = \frac{1}{2}\left(1+\xi\right)\left(1+\eta\right)\left(1-\eta\right) \qquad N^6_{,\xi} = +\frac{1}{2}\left(1-\eta\right)\left(1+\eta\right) \qquad N^6_{,\eta} = -\left(1+\xi\right)\eta$$

$$N^7 = \frac{1}{2}\left(1-\xi\right)\left(1+\xi\right)\left(1+\eta\right) \qquad N^7_{,\xi} = -\xi\left(1+\eta\right) \qquad N^7_{,\eta} = +\frac{1}{2}\left(1-\xi\right)\left(1+\xi\right)$$

$$N^8 = \frac{1}{2}\left(1-\xi\right)\left(1+\eta\right)\left(1-\eta\right) \qquad N^8_{,\xi} = -\frac{1}{2}\left(1-\eta\right)\left(1+\eta\right) \qquad N^8_{,\eta} = -\left(1-\xi\right)\eta$$

2D Finite Element Library

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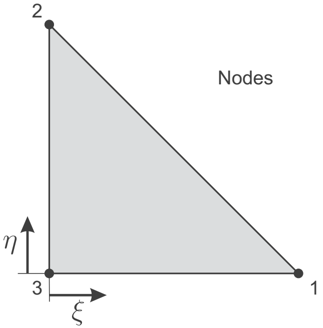
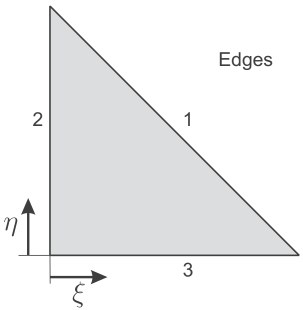
D2TR3N

constant strain triangle (CST)





Node Number	Coordinates	
	ξ	η
1	1	0
2	0	1
3	0	0

$$N^1 = \xi \qquad N^1_{,\xi} = 1 \qquad N^1_{,\eta} = 0$$

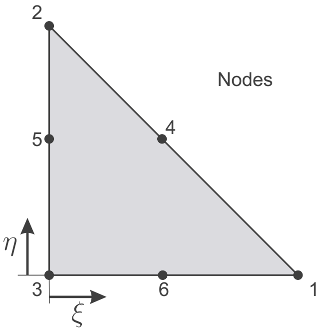
$$N^2 = \eta \qquad N^2_{,\xi} = 0 \qquad N^2_{,\eta} = 1$$

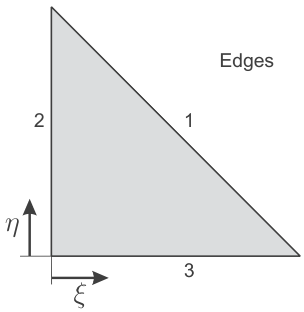
$$N^3 = (1 - \xi - \eta) \qquad N^3_{,\xi} = -1 \qquad N^3_{,\eta}(\xi, \eta) = -1$$

2D Finite Element Library

D2TR6N

quadratic triangle





Node Number	Coordinates	
	ξ	η
1	1	0
2	0	1
3	0	0
4	1/2	1/2
5	0	1/2
6	1/2	0

$$N^1 = \xi(2\xi - 1) \qquad N^1_{,\xi} = -1 + 4\xi \qquad N^1_{,\eta} = 0$$

$$N^2 = \eta(2\eta - 1) \qquad N^2_{,\xi} = 0 \qquad N^2_{,\eta} = -1 + 4\eta$$

$$N^3 = (1 - \xi - \eta)(1 - 2\xi - 2\eta) \qquad N^3_{,\xi} = -3 + 4\xi + 4\eta \qquad N^3_{,\eta} = -3 + 4\xi + 4\eta$$

$$N^4 = 4\xi\eta \qquad N^4_{,\xi} = 4\eta \qquad N^4_{,\eta} = 4\xi$$

$$N^5 = 4\eta(1 - \xi - \eta) \qquad N^5_{,\xi} = -4\eta \qquad N^5_{,\eta} = -4(-1 + 2\eta + \xi)$$

$$N^6 = 4\xi(1 - \xi - \eta) \qquad N^6_{,\xi} = -4(-1 + \eta + 2\xi) \qquad N^6_{,\eta} = -4\xi$$

2D Finite Element Library

two-dimensional quadrature rule i

Triangular Elements Gauss Point Rule

$$\int_0^1 \int_0^{1-\eta} \{\bullet\}\ \mathrm{d}\xi\ \mathrm{d}\eta \approx \frac{1}{2} \sum_{i=1}^{N_{\text{Gauss Points}}} \alpha_i\ \{\bullet\}\big|_{\text{Gauss Point}^i}$$

Gauss Point Number	Coordinates		Weight Factor
	ξ	η	α
1	1/3	1/3	1

Gauss Point Number	Coordinates		Weight Factor
	ξ	η	α
1	1/6	1/6	1/3
2	4/6	1/6	1/3
3	1/6	4/6	1/3

two-dimensional quadrature rule ii

Quadrilateral Elements Gauss Point Rule

$$\int_{-1}^{1}\int_{-1}^{1}\{\bullet\}\ \mathrm{d}\xi\ \mathrm{d}\eta \approx \sum_{i=1}^{N_{\text{Gauss Points}}} \alpha_i\ \{\bullet\}\big|_{\text{Gauss Point}^i}$$

Gauss Point Number	Coordinates		Weight Factor
	ξ	η	α
1	0	0	2×2

Gauss Point Number	Coordinates		Weight Factor
	ξ	η	α
1	$-1/\sqrt{3}$	$-1/\sqrt{3}$	1×1
2	$+1/\sqrt{3}$	$-1/\sqrt{3}$	1×1
3	$+1/\sqrt{3}$	$+1/\sqrt{3}$	1×1
4	$-1/\sqrt{3}$	$+1/\sqrt{3}$	1×1

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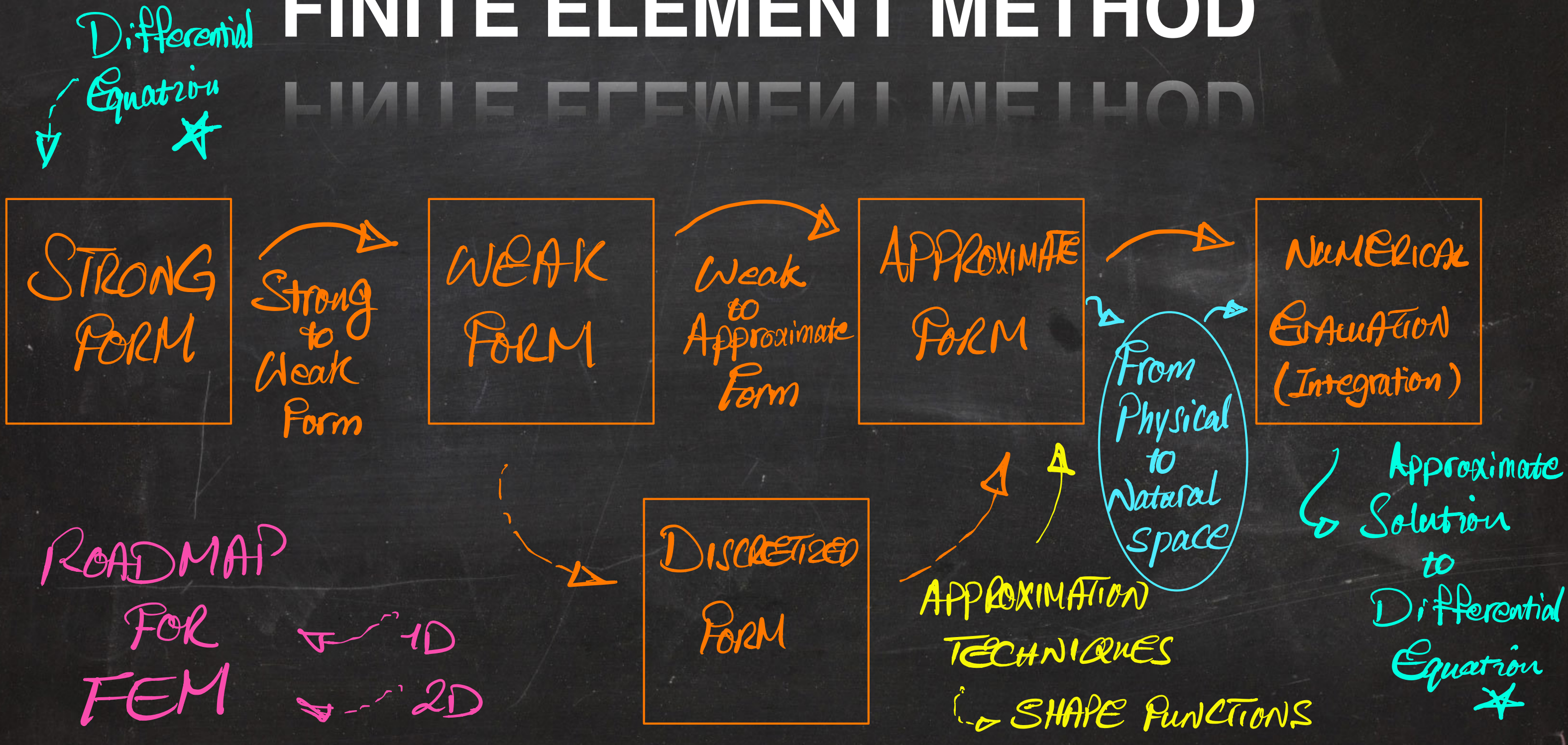
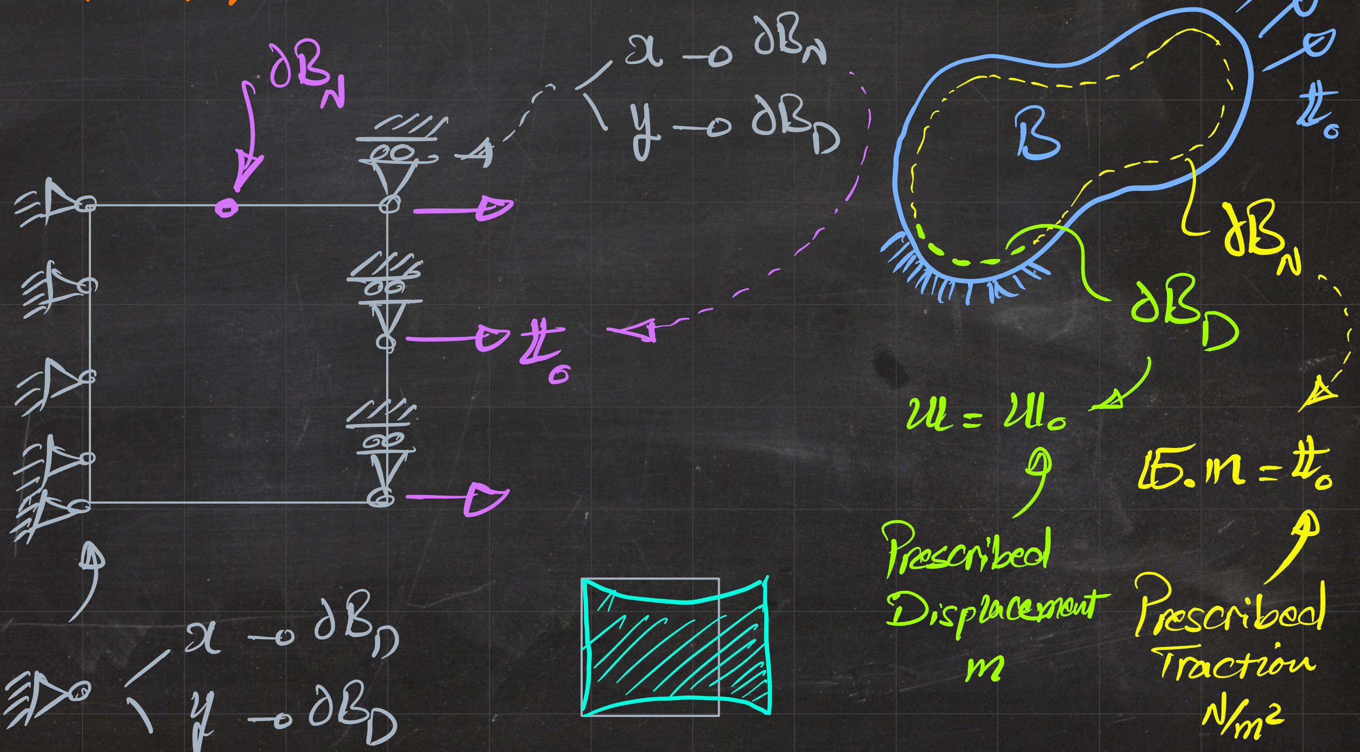
two-dimensional quadrature rule iii

Gauss Point Number	Coordinates		Weight Factor
	ξ	η	α
1	$-\sqrt{3/5}$	$-\sqrt{3/5}$	$5/9 \times 5/9$
2	$+\sqrt{3/5}$	$-\sqrt{3/5}$	$5/9 \times 5/9$
3	$\sqrt{3/5}$	$\sqrt{3/5}$	$5/9 \times 5/9$
4	$-\sqrt{3/5}$	$\sqrt{3/5}$	$5/9 \times 5/9$
5	0	$-\sqrt{3/5}$	$5/9 \times 8/9$
6	$+\sqrt{3/5}$	0	$5/9 \times 8/9$
7	0	$\sqrt{3/5}$	$5/9 \times 8/9$
8	$-\sqrt{3/5}$	0	$5/9 \times 8/9$
9	0	0	$8/9 \times 8/9$

FINITE ELEMENT METHOD

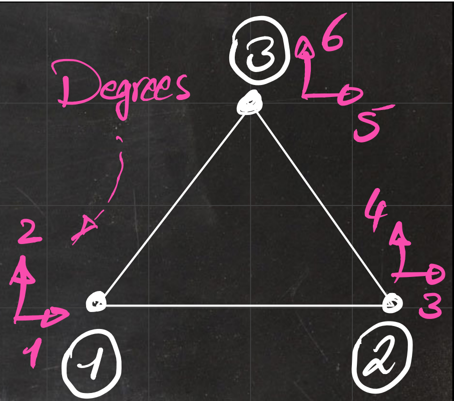


FROM STRONG FORM TO WEAK FORM



$$\underset{\Delta}{\mathbb{K}} = \begin{bmatrix} \mathbb{K}^{11} & \mathbb{K}^{12} & \mathbb{K}^{13} \\ \mathbb{K}^{21} & \mathbb{K}^{22} & \mathbb{K}^{23} \\ \mathbb{K}^{31} & \mathbb{K}^{32} & \mathbb{K}^{33} \end{bmatrix} \longleftarrow 6\times 6$$

$\hookrightarrow \underset{\uparrow 3}{\text{NoN}} \times \underset{\uparrow 2}{\text{PD}}$



$$\underset{\Delta}{\mathbb{K}} = \begin{array}{c|cc|cc|cc|c} & 1_x & 1_y & 2_x & 2_y & 3_x & 3_y & \\ \hline & K^{11}_{11} & K^{11}_{12} & K^{12}_{11} & K^{12}_{12} & K^{13}_{11} & K^{13}_{12} & 1_x \\ & K^{11}_{21} & K^{11}_{22} & K^{12}_{21} & K^{12}_{22} & K^{13}_{21} & K^{13}_{22} & 1_y \\ \hline & K^{21}_{11} & K^{21}_{12} & K^{22}_{11} & K^{22}_{12} & K^{23}_{11} & K^{23}_{12} & 2_x \\ & K^{21}_{21} & K^{21}_{22} & K^{22}_{21} & K^{22}_{22} & K^{23}_{21} & K^{23}_{22} & 2_y \\ \hline & K^{31}_{11} & K^{31}_{12} & K^{32}_{11} & K^{32}_{12} & K^{33}_{11} & K^{33}_{12} & 3_x \\ & K^{31}_{21} & K^{31}_{22} & K^{32}_{21} & K^{32}_{22} & K^{33}_{21} & K^{33}_{22} & 3_y \end{array}$$

$1 \to \text{NODE}^1\ x$

$2 \to \text{NODE}^1\ y$

$3 \to \text{NODE}^2\ x$

$4 \to \text{NODE}^2\ y$

$5 \to \text{NODE}^3\ x$

$6 \to \text{NODE}^3\ y$

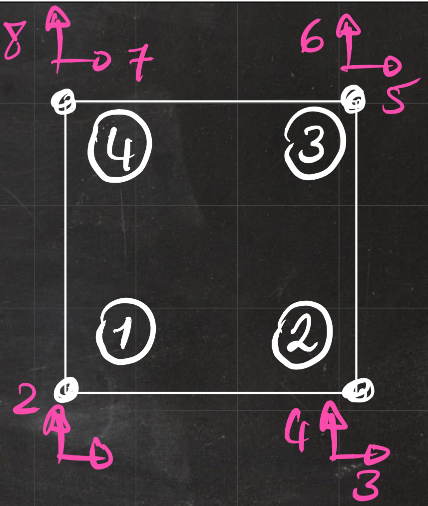
$$\mathbb{K}_{\square} = \begin{bmatrix} \mathbb{K}^{11} & \mathbb{K}^{12} & \mathbb{K}^{13} & \mathbb{K}^{14} \\ \mathbb{K}^{21} & \mathbb{K}^{22} & \mathbb{K}^{23} & \mathbb{K}^{24} \\ \mathbb{K}^{31} & \mathbb{K}^{32} & \mathbb{K}^{33} & \mathbb{K}^{34} \\ \mathbb{K}^{41} & \mathbb{K}^{42} & \mathbb{K}^{43} & \mathbb{K}^{44} \end{bmatrix} \rightarrow 8 \times 8$$

D2QU4N

$\hookrightarrow$ NoN x PD

$\hookrightarrow 4 \quad \hookrightarrow 2$

$$\mathbb{K}^{ij} = \begin{bmatrix} K^{ij}_{11} & K^{ij}_{12} \\ K^{ij}_{21} & K^{ij}_{22} \end{bmatrix} = \begin{bmatrix} K^{ij}_{xx} & K^{ij}_{xy} \\ K^{ij}_{yx} & K^{ij}_{yy} \end{bmatrix}$$



1, 2 → $\text{NODE}^1_{x,y}$

3, 4 → $\text{NODE}^2_{x,y}$

5, 6 → $\text{NODE}^3_{x,y}$

7, 8 → $\text{NODE}^4_{x,y}$

D2TR3N ← $[\mathbb{K}]_{6x6}$

D2TR6N ← $[\mathbb{K}]_{12x12}$

D2QU4N ← $[\mathbb{K}]_{8x8}$

D2QU8N ← $[\mathbb{K}]_{16x16}$

D2QU9N ← $[\mathbb{K}]_{18x18}$

PROBLEMS TO ADDRESS

- ↳ INTEGRAL ↳ GAUSS QUADRATURE RULE
- ↳ $f(x)$ ↳ $x \to \$$
- ↳ E_{abcd} ↳ ?

$$[\mathbb{K}^{ij}]_{ac} = \int_{B} [N^{i}_{,x}]_{b} \, E_{abcd} \, [N^{j}_{,x}]_{d} \, dA$$

D2TR3N ← $[\mathbb{K}]_{6\times6}$

D2TR6N ← $[\mathbb{K}]_{12\times12}$

D2QU4N ← $[\mathbb{K}]_{8\times8}$

D2QU8N ← $[\mathbb{K}]_{16\times16}$

D2QU9N ← $[\mathbb{K}]_{18\times18}$

$$\mathbb{E}_{abcd} = \frac{E}{2[1+\nu]}\left[\delta_{ad}\delta_{bc}+\delta_{ac}\delta_{bd}\right] + \frac{E\nu}{1-\nu^2}\delta_{ab}\delta_{cd}$$

CONSTITUTIVE TENSOR

4th. O.

2x2x2x2 = 16 COMPONENTS

Young's Modulus

ν: Poisson's Ratio

\nu

δ: Kronecker Delta

$$[\mathbb{K}^{ij}]_{ac} = \int_{B} [N^{i}_{,x}]_{b}\, \mathbb{E}_{abcd}\, [N^{j}_{,x}]_{d}\, dA$$

$$[\mathbb{K}^{ij}]_{ac} = \int_B [N^i_{,x}]_b \, E_{abcd} \, [N^j_{,x}]_d \, dA$$

$$E_{abcd} = \frac{E}{2[1+\nu]}\left[\delta_{ad}\delta_{bc} + \delta_{ac}\delta_{bd}\right] + \frac{E\nu}{1-\nu^2}\delta_{ab}\delta_{cd}$$

$$E_{1111} = \frac{E}{[1-\nu^2]} \qquad E_{1112} = 0 \qquad E_{1121} = 0 \qquad E_{1122} = \frac{E\nu}{[1-\nu^2]}$$

$$E_{1211} = 0 \qquad E_{1212} = \frac{E}{2[1+\nu]} \qquad E_{1221} = \frac{E}{2[1+\nu]} \qquad E_{1222} = 0$$

$$E_{2111} = 0 \qquad E_{2112} = \frac{E}{2[1+\nu]} \qquad E_{2121} = \frac{E}{2[1+\nu]} \qquad E_{2122} = 0$$

$$E_{2211} = \frac{E\nu}{[1-\nu^2]} \qquad E_{2212} = 0 \qquad E_{2221} = 0 \qquad E_{2222} = \frac{E}{[1-\nu^2]}$$

$$[\mathbb{K}^{ij}]_{ac} = \int_{\mathcal{B}} [N^{i}_{,\boldsymbol{x}}]_b \, E_{abcd} \, [N^{j}_{,\boldsymbol{x}}]_d \, dA$$

$$x = x(\xi, \eta)$$

$$y = y(\xi, \eta)$$

$$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{\xi})$$

$$N^{i}_{,\boldsymbol{x}} \rightarrow \begin{bmatrix} \dfrac{\partial N^i}{\partial x} \\[2ex] \dfrac{\partial N^i}{\partial y} \end{bmatrix}$$

$$\frac{\partial N^i}{\partial x} = \frac{\partial N^i}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial N^i}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$\frac{\partial N^i}{\partial y} = \frac{\partial N^i}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial N^i}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$\xi = \xi(x, y)$$

$$\eta = \eta(x, y)$$

$$\boldsymbol{\xi} = \boldsymbol{\xi}(\boldsymbol{x})$$

$$[\mathbb{K}^{ij}]_{ac} = \int_{\mathcal{B}} [N^{i}_{,x}]_b \, E_{abcd} \, [N^{j}_{,x}]_d \, dA$$

$$[\mathbb{K}^{ij}]_{ac} = \int_{\mathcal{B}} [\vec{\mathbb{J}}^{-t} \cdot N^{i}_{,\xi}]_b \, E_{abcd} \, [\vec{\mathbb{J}}^{-t} \cdot N^{j}_{,\xi}]_d \, dA$$

$$\mathbb{J} = \frac{\partial x}{\partial \xi} \hookrightarrow x = x(\xi) \hookleftarrow x = N^{s} x^{s}$$

$$\hookrightarrow N^{s}(\xi, \eta)$$

$$[\mathbb{K}^{ij}]_{ac} = \int_{\mathcal{B}} [\mathbf{J}^{-t} \cdot N^{i}_{,\xi}]_{b} \, \mathbb{E}_{abcd} \, [\mathbf{J}^{-t} \cdot N^{j}_{,\xi}]_{d} \, dA$$

$$J_{11} = \frac{\partial x}{\partial \xi} = x^1 \frac{\partial N^1}{\partial \xi} + x^2 \frac{\partial N^2}{\partial \xi} + \cdots + x^{NPE} \frac{\partial N^{NPE}}{\partial \xi}$$

$$J_{12} = \frac{\partial x}{\partial \eta} = x^1 \frac{\partial N^1}{\partial \eta} + x^2 \frac{\partial N^2}{\partial \eta} + \cdots + x^{NPE} \frac{\partial N^{NPE}}{\partial \eta}$$

$$J_{21} = \frac{\partial y}{\partial x} = y^1 \frac{\partial N^1}{\partial \xi} + y^2 \frac{\partial N^2}{\partial \xi} + \cdots + y^{NPE} \frac{\partial N^{NPE}}{\partial \xi}$$

$$J_{22} = \frac{\partial y}{\partial \eta} = y^1 \frac{\partial N^1}{\partial \eta} + y^2 \frac{\partial N^2}{\partial \eta} + \cdots + y^{NPE} \frac{\partial N^{NPE}}{\partial \eta}$$

$$[\mathbb{K}^{ij}]_{ac} = \int_{\mathcal{B}} [\mathbb{J}^{-t} \cdot N^{i}_{,\xi}]_{b} \, \mathbb{E}_{abcd} \, [\mathbb{J}^{-t} \cdot N^{j}_{,\xi}]_{d} \, dA$$

$$\underbrace{\begin{bmatrix} J_{11} & J_{21} \\ J_{12} & J_{22} \end{bmatrix}}_{\mathbb{J}^{t}} = \begin{bmatrix} \frac{\partial N^1}{\partial \xi} & \frac{\partial N^2}{\partial \xi} & \cdots & \frac{\partial N^{NPE}}{\partial \xi} \\ \frac{\partial N^1}{\partial \eta} & \frac{\partial N^2}{\partial \eta} & \cdots & \frac{\partial N^{NPE}}{\partial \eta} \end{bmatrix}_{2 \times NPE} \begin{bmatrix} x^1 & y^1 \\ x^2 & y^2 \\ \vdots & \vdots \\ x^{NPE} & y^{NPE} \end{bmatrix}_{NPE \times 2}$$

$$K^{ij}_{ac} = \sum_{gp=1}^{GPE} [\mathbb{J}^{-t} \cdot N^{i}_{,\xi}]_b \; E_{abcd} \; [\mathbb{J}^{-t} \cdot N^{j}_{,\xi}]_d \; \mathrm{Det}\,\mathbb{J} \times \alpha_{gp} \times \frac{1}{2}$$

IF TRIANGLE

JACOBIAN $\frac{\partial x}{\partial \xi} \longrightarrow$

$$\mathbb{J} = \begin{bmatrix} x^1 \ldots x^{NPE} \\ y^1 \ldots y^{NPE} \end{bmatrix} \begin{bmatrix} N^1_{,\xi} & N^1_{,\eta} \\ \vdots & \vdots \\ N^{NPE}_{,\xi} & N^{NPE}_{,\eta} \end{bmatrix}$$

$$E_{abcd} = \frac{E}{2[1+\nu]} [\delta_{ad}\delta_{bc} + \delta_{ac}\delta_{bd}] + \frac{E\nu}{1-\nu^2}\delta_{ab}\delta_{cd}$$

$$\left\{ \begin{array}{llll} E_{1111} = \frac{E}{[1-\nu^2]} & E_{1112} = 0 & E_{1121} = 0 & E_{1122} = \frac{E\nu}{[1-\nu^2]} \\ E_{1211} = 0 & E_{1212} = \frac{E}{2[1+\nu]} & E_{1221} = \frac{E}{2[1+\nu]} & E_{1222} = 0 \\ E_{2111} = 0 & E_{2112} = \frac{E}{2[1+\nu]} & E_{2121} = \frac{E}{2[1+\nu]} & E_{2122} = 0 \\ E_{2211} = \frac{E\nu}{[1-\nu^2]} & E_{2212} = 0 & E_{2221} = 0 & E_{2222} = \frac{E}{[1-\nu^2]} \end{array} \right.$$

FINITE ELEMENT METHOD

2D FEM

formulation summary
& understanding via examples

$$K^{ij}_{ac}$$

stiffness between
direction "a" of node "i" &
direction "c" of node "j"

$$\equiv \frac{\delta F^i_a}{\delta u^j_c}$$

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Quadrilateral Elements

$$K^{ij}_{ac} = \sum_{\mathrm{gp}=1}^{\mathrm{GPE}} \left[\boldsymbol{J}^{-T} \cdot N^i_{,\boldsymbol{\xi}}\right]_b E_{abcd} \left[\boldsymbol{J}^{-T} \cdot N^j_{,\boldsymbol{\xi}}\right]_d \mathrm{Det}\boldsymbol{J}\, \alpha_{gp}$$

Triangular Elements

$$K^{ij}_{ac} = \sum_{\mathrm{gp}=1}^{\mathrm{GPE}} \left[\boldsymbol{J}^{-T} \cdot N^i_{,\boldsymbol{\xi}}\right]_b E_{abcd} \left[\boldsymbol{J}^{-T} \cdot N^j_{,\boldsymbol{\xi}}\right]_d \mathrm{Det}\boldsymbol{J}\, \alpha_{gp} \times \frac{1}{2}$$

Quadrilateral Elements

$$K^{\,i\,j}_{\,a\,c} = \sum_{\mathrm{gp}\,=\,1}^{\mathrm{GPE}} \left[\boldsymbol{J}^{-T} \cdot N^{i}_{,\boldsymbol{\xi}}\right]_b E_{\,a\,b\,c\,d} \left[\boldsymbol{J}^{-T} \cdot N^{j}_{,\boldsymbol{\xi}}\right]_d \,\mathrm{Det}\boldsymbol{J}\,\alpha_{gp}$$

Triangular Elements

$$K^{\,i\,j}_{\,a\,c} = \sum_{\mathrm{gp}\,=\,1}^{\mathrm{GPE}} \left[\boldsymbol{J}^{-T} \cdot N^{i}_{,\boldsymbol{\xi}}\right]_b E_{\,a\,b\,c\,d} \left[\boldsymbol{J}^{-T} \cdot N^{j}_{,\boldsymbol{\xi}}\right]_d \,\mathrm{Det}\boldsymbol{J}\,\alpha_{gp} \times \frac{1}{2}$$

$$E_{\,a\,b\,c\,d} = \frac{E}{2[1+\nu]}\left[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}\right] + \frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

2D plane strain constitutive tensor components

$$E_{abcd} = \frac{E}{2[1+\nu]}[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}] + \frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

$$\begin{aligned}
E_{1111} &= \frac{E}{1-\nu^2} & E_{1112} &= 0 & E_{1121} &= 0 & E_{1122} &= \frac{E\,\nu}{1-\nu^2} \\
E_{1211} &= 0 & E_{1212} &= \frac{E}{2[1+\nu]} & E_{1221} &= \frac{E}{2[1+\nu]} & E_{1222} &= 0 \\
E_{2111} &= 0 & E_{2112} &= \frac{E}{2[1+\nu]} & E_{2121} &= \frac{E}{2[1+\nu]} & E_{2122} &= 0 \\
E_{2211} &= \frac{E\,\nu}{1-\nu^2} & E_{2212} &= 0 & E_{2221} &= 0 & E_{2222} &= \frac{E}{1-\nu^2}
\end{aligned}$$

$$K^{\,i\,j}_{\,a\,c}$$

stiffness between
direction "a" of node "i" &
direction "c" of node "j"

Quadrilateral Elements

$$K^{\,i\,j}_{\,a\,c} = \sum_{\mathrm{gp}\,=\,1}^{\mathrm{GPE}} \left[\boldsymbol{J}^{-T}\cdot N^{i}_{,\boldsymbol{\xi}}\right]_b E_{\,a\,b\,c\,d}\,\left[\boldsymbol{J}^{-T}\cdot N^{j}_{,\boldsymbol{\xi}}\right]_d \,\mathrm{Det}\boldsymbol{J}\,\alpha_{gp}$$

Triangular Elements

$$K^{\,i\,j}_{\,a\,c} = \sum_{\mathrm{gp}\,=\,1}^{\mathrm{GPE}} \left[\boldsymbol{J}^{-T}\cdot N^{i}_{,\boldsymbol{\xi}}\right]_b E_{\,a\,b\,c\,d}\,\left[\boldsymbol{J}^{-T}\cdot N^{j}_{,\boldsymbol{\xi}}\right]_d \,\mathrm{Det}\boldsymbol{J}\,\alpha_{gp} \times \frac{1}{2}$$

$$\boldsymbol{J} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\mathrm{NPE}} \\ y^1 & y^2 & \dots & y^{\mathrm{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\mathrm{NPE}}_\xi & N^{\mathrm{NPE}}_\eta \end{bmatrix}$$

$$E_{\,a\,b\,c\,d} = \frac{E}{2[1+\nu]}\left[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}\right] + \frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

plane strain

$K^{\,i\,j}_{\,a\,c}$ *stiffness between direction "a" of node "i" & direction "c" of node "j"*

plane strain

Quadrilateral Elements

$$K^{\,i\,j}_{\,a\,c} = \sum_{\mathrm{gp}\,=\,1}^{\mathrm{GPE}} \left[\boldsymbol{J}^{-T}\cdot N^{i}_{,\boldsymbol{\xi}}\right]_{b} E_{a\,b\,c\,d} \left[\boldsymbol{J}^{-T}\cdot N^{j}_{,\boldsymbol{\xi}}\right]_{d} \,\mathrm{Det}\boldsymbol{J}\,\alpha_{gp}$$

Triangular Elements

$$K^{\,i\,j}_{\,a\,c} = \sum_{\mathrm{gp}\,=\,1}^{\mathrm{GPE}} \left[\boldsymbol{J}^{-T}\cdot N^{i}_{,\boldsymbol{\xi}}\right]_{b} E_{a\,b\,c\,d} \left[\boldsymbol{J}^{-T}\cdot N^{j}_{,\boldsymbol{\xi}}\right]_{d} \,\mathrm{Det}\boldsymbol{J}\,\alpha_{gp} \times \frac{1}{2}$$

$$E_{a\,b\,c\,d} = \frac{E}{2[1+\nu]}\left[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}\right] + \frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

$$\boldsymbol{J} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\mathrm{NPE}} \\ y^1 & y^2 & \dots & y^{\mathrm{NPE}} \end{bmatrix} \begin{bmatrix} N^1_{\xi} & N^1_{\eta} \\ N^2_{\xi} & N^2_{\eta} \\ \vdots & \vdots \\ N^{\mathrm{NPE}}_{\xi} & N^{\mathrm{NPE}}_{\eta} \end{bmatrix}$$

$E_{1\,1\,1\,1} = \dfrac{E}{1-\nu^2}$ $\quad E_{1\,1\,1\,2} = 0$ $\quad E_{1\,1\,2\,1} = 0$ $\quad E_{1\,1\,2\,2} = \dfrac{E\,\nu}{1-\nu^2}$

$E_{1\,2\,1\,1} = 0$ $\quad E_{1\,2\,1\,2} = \dfrac{E}{2[1+\nu]}$ $\quad E_{1\,2\,2\,1} = \dfrac{E}{2[1+\nu]}$ $\quad E_{1\,2\,2\,2} = 0$

$E_{2\,1\,1\,1} = 0$ $\quad E_{2\,1\,1\,2} = \dfrac{E}{2[1+\nu]}$ $\quad E_{2\,1\,2\,1} = \dfrac{E}{2[1+\nu]}$ $\quad E_{2\,1\,2\,2} = 0$

$E_{2\,2\,1\,1} = \dfrac{E\,\nu}{1-\nu^2}$ $\quad E_{2\,2\,1\,2} = 0$ $\quad E_{2\,2\,2\,1} = 0$ $\quad E_{2\,2\,2\,2} = \dfrac{E}{1-\nu^2}$

Understanding Jacobian

$$\boldsymbol{J} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\text{NPE}} \\ y^1 & y^2 & \dots & y^{\text{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\text{NPE}}_\xi & N^{\text{NPE}}_\eta \end{bmatrix}$$

linear mapping from natural space to physical space

$$\mathrm{d}\boldsymbol{x} = \boldsymbol{J}\,\mathrm{d}\boldsymbol{\xi}$$

$$J = \mathrm{Det}\boldsymbol{J} = \frac{\mathrm{d}A_{\boldsymbol{x}}}{\mathrm{d}A_{\boldsymbol{\xi}}}$$

often, the (scalar) determinant of the Jacobian matrix is also referred to as Jacobian

…

the scalar Jacobian is a linear mapping between the area elements from the natural space to the physical space

Numerical Analysis
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Example (D2TR3N) … constant strain triangle … stiffness

Triangular Elements

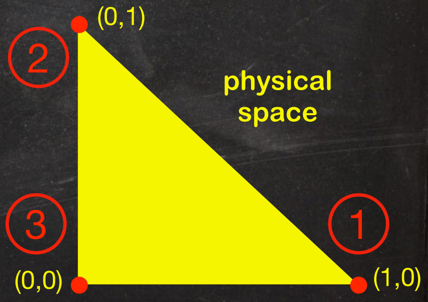
$$K^{i\,j}_{a\,c} = \sum_{\text{gp}\,=\,1}^{\text{GPE}} \left[\boldsymbol{J}^{-T}\cdot N^{i}_{,\boldsymbol{\xi}}\right]_b E_{a\,b\,c\,d}\left[\boldsymbol{J}^{-T}\cdot N^{j}_{,\boldsymbol{\xi}}\right]_d \,\text{Det}\boldsymbol{J}\,\alpha_{gp}\times\frac{1}{2}$$

$$E_{a\,b\,c\,d} = \frac{E}{2[1+\nu]}[\delta_{ad}\,\delta_{bc}+\delta_{ac}\,\delta_{bd}] + \frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

plane strain

$$\boldsymbol{K} = \begin{bmatrix} \boldsymbol{K}^{11} & \boldsymbol{K}^{12} & \boldsymbol{K}^{13} \\ \boldsymbol{K}^{21} & \boldsymbol{K}^{22} & \boldsymbol{K}^{23} \\ \boldsymbol{K}^{31} & \boldsymbol{K}^{32} & \boldsymbol{K}^{33} \end{bmatrix} = \begin{bmatrix} K^{11}_{11} & K^{11}_{12} & K^{12}_{11} & K^{12}_{12} & K^{13}_{11} & K^{13}_{12} \\ K^{11}_{21} & K^{11}_{22} & K^{12}_{21} & K^{12}_{22} & K^{13}_{21} & K^{13}_{22} \\ K^{21}_{11} & K^{21}_{12} & K^{22}_{11} & K^{22}_{12} & K^{23}_{11} & K^{23}_{12} \\ K^{21}_{21} & K^{21}_{22} & K^{22}_{21} & K^{22}_{22} & K^{23}_{21} & K^{23}_{22} \\ K^{31}_{11} & K^{31}_{12} & K^{32}_{11} & K^{32}_{12} & K^{33}_{11} & K^{33}_{12} \\ K^{31}_{21} & K^{31}_{22} & K^{32}_{21} & K^{32}_{22} & K^{33}_{21} & K^{33}_{22} \end{bmatrix}$$

$$\boldsymbol{J} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\text{NPE}} \\ y^1 & y^2 & \dots & y^{\text{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\text{NPE}}_\xi & N^{\text{NPE}}_\eta \end{bmatrix}$$



Example (D2TR3N) … constant strain triangle … stiffness

Triangular Elements

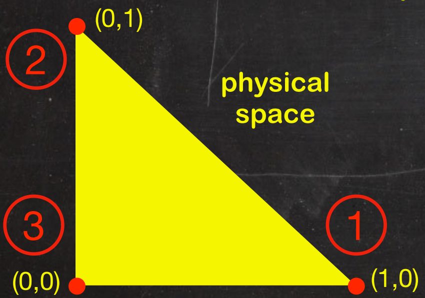
$$K^{\,i\,j}_{\,a\,c} = \sum_{\mathrm{gp}\,=\,1}^{\mathrm{GPE}} \left[\boldsymbol{J}^{-T}\cdot N^{i}_{,\boldsymbol{\xi}}\right]_b E_{a\,b\,c\,d} \left[\boldsymbol{J}^{-T}\cdot N^{j}_{,\boldsymbol{\xi}}\right]_d \,\mathrm{Det}\boldsymbol{J}\,\alpha_{gp} \times \frac{1}{2}$$

$$E_{a\,b\,c\,d} = \frac{E}{2[1+\nu]}\left[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}\right] + \frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

plane strain

$$\boldsymbol{J} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\mathrm{NPE}} \\ y^1 & y^2 & \dots & y^{\mathrm{NPE}} \end{bmatrix} \begin{bmatrix} N^1_{\xi} & N^1_{\eta} \\ N^2_{\xi} & N^2_{\eta} \\ \vdots & \vdots \\ N^{\mathrm{NPE}}_{\xi} & N^{\mathrm{NPE}}_{\eta} \end{bmatrix}$$

$$\begin{array}{llll}
E_{1111} = \dfrac{E}{1-\nu^2} & E_{1112} = 0 & E_{1121} = 0 & E_{1122} = \dfrac{E\,\nu}{1-\nu^2} \\
E_{1211} = 0 & E_{1212} = \dfrac{E}{2[1+\nu]} & E_{1221} = \dfrac{E}{2[1+\nu]} & E_{1222} = 0 \\
E_{2111} = 0 & E_{2112} = \dfrac{E}{2[1+\nu]} & E_{2121} = \dfrac{E}{2[1+\nu]} & E_{2122} = 0 \\
E_{2211} = \dfrac{E\,\nu}{1-\nu^2} & E_{2212} = 0 & E_{2221} = 0 & E_{2222} = \dfrac{E}{1-\nu^2}
\end{array}$$



natural space

$N^1 = \xi$

$N^2 = \eta$

$N^3 = (1-\xi-\eta)$

$N^1_{,\xi} = 1$ $\quad N^1_{,\eta} = 0$

$N^2_{,\xi} = 0$ $\quad N^2_{,\eta} = 1$

$N^3_{,\xi} = -1$ $\quad N^3_{,\eta}(\xi,\eta) = -1$

$$\boldsymbol{K} = \begin{bmatrix} \boldsymbol{K}^{11} & \boldsymbol{K}^{12} & \boldsymbol{K}^{13} \\ \boldsymbol{K}^{21} & \boldsymbol{K}^{22} & \boldsymbol{K}^{23} \\ \boldsymbol{K}^{31} & \boldsymbol{K}^{32} & \boldsymbol{K}^{33} \end{bmatrix}$$

Example (D2TR3N) … linear triangular element

Triangular Elements

$$K^{i\,j}_{a\,c} = \sum_{\mathrm{gp}\,=\,1}^{\mathrm{GPE}} \left[\boldsymbol{J}^{-T}\cdot N^{i}_{,\boldsymbol{\xi}}\right]_b E_{a\,b\,c\,d} \left[\boldsymbol{J}^{-T}\cdot N^{j}_{,\boldsymbol{\xi}}\right]_d \,\mathrm{Det}\boldsymbol{J}\,\alpha_{gp} \times \frac{1}{2}$$

$$E_{a\,b\,c\,d} = \frac{E}{2[1+\nu]}[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}] + \frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

plane strain

$$\boldsymbol{J} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\mathrm{NPE}} \\ y^1 & y^2 & \dots & y^{\mathrm{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\mathrm{NPE}}_\xi & N^{\mathrm{NPE}}_\eta \end{bmatrix}$$

$$\boldsymbol{K} = \begin{bmatrix} \frac{E}{2\,[1-\nu^2]} & 0 & 0 & \frac{E\nu}{2\,[1-\nu^2]} & -\frac{E}{2\,[1-\nu^2]} & -\frac{E\nu}{2\,[1-\nu^2]} \\ 0 & \frac{E}{4\,[\nu+1]} & \frac{E}{4\,[\nu+1]} & 0 & -\frac{E}{4\,[\nu+1]} & -\frac{E}{4\,[\nu+1]} \\ 0 & \frac{E}{4\,[\nu+1]} & \frac{E}{4\,[\nu+1]} & 0 & -\frac{E}{4\,[\nu+1]} & -\frac{E}{4\,[\nu+1]} \\ \frac{E\nu}{2\,[1-\nu^2]} & 0 & 0 & \frac{E}{2\,[1-\nu^2]} & -\frac{E\nu}{2\,[1-\nu^2]} & -\frac{E}{2\,[1-\nu^2]} \\ -\frac{E}{2\,[1-\nu^2]} & -\frac{E}{4\,[\nu+1]} & -\frac{E}{4\,[\nu+1]} & -\frac{E\nu}{2\,[1-\nu^2]} & \frac{E\,[3-\nu]}{4\,[1-\nu^2]} & \frac{E}{4\,[1-\nu]} \\ -\frac{E\nu}{2\,[1-\nu^2]} & -\frac{E}{4\,[\nu+1]} & -\frac{E}{4\,[\nu+1]} & -\frac{E}{2\,[1-\nu^2]} & \frac{E}{4\,[1-\nu]} & \frac{E\,[3-\nu]}{4\,[1-\nu^2]} \end{bmatrix}$$

$$\boldsymbol{x}^1 = [\,1\,,\,0\,]$$

$$\boldsymbol{x}^2 = [\,0\,,\,1\,]$$

$$\boldsymbol{x}^3 = [\,0\,,\,0\,]$$

Example (D2TR3N) … linear triangular element

… using one Gauss point …

Triangular Elements

$$K^{i\,j}_{a\,c} = \sum_{\mathrm{gp}\,=\,1}^{\mathrm{GPE}} \left[\boldsymbol{J}^{-T} \cdot N^{i}_{,\boldsymbol{\xi}}\right]_b E_{a\,b\,c\,d} \left[\boldsymbol{J}^{-T} \cdot N^{j}_{,\boldsymbol{\xi}}\right]_d \mathrm{Det}\boldsymbol{J}\,\alpha_{gp} \times \frac{1}{2}$$

$$E_{a\,b\,c\,d} = \frac{E}{2[1+\nu]}[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}] + \frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

plane strain

$$\boldsymbol{J} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\mathrm{NPE}} \\ y^1 & y^2 & \dots & y^{\mathrm{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\mathrm{NPE}}_\xi & N^{\mathrm{NPE}}_\eta \end{bmatrix}$$

$$\boldsymbol{K} = \begin{bmatrix} 5.000 & 0.000 & 0.000 & 0.000 & -5.000 & 0.000 \\ 0.000 & 2.500 & 2.500 & 0.000 & -2.500 & -2.500 \\ 0.000 & 2.500 & 2.500 & 0.000 & -2.500 & -2.500 \\ 0.000 & 0.000 & 0.000 & 5.000 & 0.000 & -5.000 \\ -5.000 & -2.500 & -2.500 & 0.000 & 7.500 & 2.500 \\ 0.000 & -2.500 & -2.500 & -5.000 & 2.500 & 7.500 \end{bmatrix}$$

$$\boldsymbol{x}^1 = [\,1\,,\,0\,]$$

$$\boldsymbol{x}^2 = [\,0\,,\,1\,]$$

$$\boldsymbol{x}^3 = [\,0\,,\,0\,]$$

$$E = 10\,,\ \nu = 0$$

Example (D2TR3N) … linear triangular element

… using one Gauss point …

Triangular Elements

$$K^{\,i\,j}_{\,a\,c} = \sum_{\mathrm{gp}\,=\,1}^{\mathrm{GPE}} \left[\boldsymbol{J}^{-T}\cdot N^{i}_{,\boldsymbol{\xi}}\right]_b E_{a\,b\,c\,d} \left[\boldsymbol{J}^{-T}\cdot N^{j}_{,\boldsymbol{\xi}}\right]_d \, \mathrm{Det}\boldsymbol{J}\,\alpha_{gp} \times \frac{1}{2}$$

$$E_{a\,b\,c\,d} = \frac{E}{2[1+\nu]}[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}] + \frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

plane strain

$$\boldsymbol{K} = \begin{bmatrix} 6.667 & 0.000 & 0.000 & 3.333 & -6.667 & -3.333 \\ 0.000 & 1.667 & 1.667 & 0.000 & -1.667 & -1.667 \\ 0.000 & 1.667 & 1.667 & 0.000 & -1.667 & -1.667 \\ 3.333 & 0.000 & 0.000 & 6.667 & -3.333 & -6.667 \\ -6.667 & -1.667 & -1.667 & -3.333 & 8.333 & 5.000 \\ -3.333 & -1.667 & -1.667 & -6.667 & 5.000 & 8.333 \end{bmatrix}$$

$$\boldsymbol{J} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\mathrm{NPE}} \\ y^1 & y^2 & \dots & y^{\mathrm{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\mathrm{NPE}}_\xi & N^{\mathrm{NPE}}_\eta \end{bmatrix}$$

$$\boldsymbol{x}^1 = [\,1\,,\,0\,]$$

$$\boldsymbol{x}^2 = [\,0\,,\,1\,]$$

$$\boldsymbol{x}^3 = [\,0\,,\,0\,]$$

$$E = 10\,,\;\nu = 0.5$$

Example (D2QU4N) … bilinear quadrilateral element

Quadrilateral Elements

$$K^{ij}_{ac} = \sum_{\text{gp}\,=\,1}^{\text{GPE}} \left[\boldsymbol{J}^{-T} \cdot N^{i}_{,\boldsymbol{\xi}}\right]_b E_{abcd} \left[\boldsymbol{J}^{-T} \cdot N^{j}_{,\boldsymbol{\xi}}\right]_d \, \text{Det}\boldsymbol{J} \, \alpha_{gp}$$

$$E_{abcd} = \frac{E}{2[1+\nu]}[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}] + \frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

plane strain

$$\boldsymbol{K} = \begin{bmatrix} \boldsymbol{K}^{11} & \boldsymbol{K}^{12} & \boldsymbol{K}^{13} & \boldsymbol{K}^{14} \\ \boldsymbol{K}^{21} & \boldsymbol{K}^{22} & \boldsymbol{K}^{23} & \boldsymbol{K}^{24} \\ \boldsymbol{K}^{31} & \boldsymbol{K}^{32} & \boldsymbol{K}^{33} & \boldsymbol{K}^{34} \\ \boldsymbol{K}^{41} & \boldsymbol{K}^{42} & \boldsymbol{K}^{43} & \boldsymbol{K}^{44} \end{bmatrix} = \begin{bmatrix} K^{11}_{11} & K^{11}_{12} & K^{12}_{11} & K^{12}_{12} & K^{13}_{11} & K^{13}_{12} & K^{14}_{11} & K^{14}_{12} \\ K^{11}_{21} & K^{11}_{22} & K^{12}_{21} & K^{12}_{22} & K^{13}_{21} & K^{13}_{22} & K^{14}_{21} & K^{14}_{22} \\ K^{21}_{11} & K^{21}_{12} & K^{22}_{11} & K^{22}_{12} & K^{23}_{11} & K^{23}_{12} & K^{24}_{11} & K^{24}_{12} \\ K^{21}_{21} & K^{21}_{22} & K^{22}_{21} & K^{22}_{22} & K^{23}_{21} & K^{23}_{22} & K^{24}_{21} & K^{24}_{22} \\ K^{31}_{11} & K^{31}_{12} & K^{32}_{11} & K^{32}_{12} & K^{33}_{11} & K^{33}_{12} & K^{34}_{11} & K^{34}_{12} \\ K^{31}_{21} & K^{31}_{22} & K^{32}_{21} & K^{32}_{22} & K^{33}_{21} & K^{33}_{22} & K^{34}_{21} & K^{34}_{22} \\ K^{41}_{11} & K^{41}_{12} & K^{42}_{11} & K^{42}_{12} & K^{43}_{11} & K^{43}_{12} & K^{44}_{11} & K^{44}_{12} \\ K^{41}_{21} & K^{41}_{22} & K^{42}_{21} & K^{42}_{22} & K^{43}_{21} & K^{43}_{22} & K^{44}_{21} & K^{44}_{22} \end{bmatrix}$$

$$\boldsymbol{J} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\text{NPE}} \\ y^1 & y^2 & \dots & y^{\text{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\text{NPE}}_\xi & N^{\text{NPE}}_\eta \end{bmatrix}$$

! ATTENTION !

element stiffness matrix is symmetric and its determinant is zero

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Example (D2QU4N) … bilinear quadrilateral element

… using one Gauss point …

Quadrilateral Elements

$$K^{ij}_{ac} = \sum_{\text{gp}=1}^{\text{GPE}} \left[\boldsymbol{J}^{-T} \cdot N^i_{,\boldsymbol{\xi}}\right]_b E_{abcd} \left[\boldsymbol{J}^{-T} \cdot N^j_{,\boldsymbol{\xi}}\right]_d \operatorname{Det}\boldsymbol{J}\, \alpha_{gp}$$

$$E_{abcd} = \frac{E}{2[1+\nu]}[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}]$$

plane strain

$$+\frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

$$\boldsymbol{K} = \begin{bmatrix}
\frac{E[\nu-3]}{8[\nu^2-1]} & -\frac{E}{8[\nu-1]} & \frac{E}{8[\nu-1]} & \frac{E[1-3\nu]}{8[\nu^2-1]} & -\frac{E[\nu-3]}{8[\nu^2-1]} & \frac{E}{8[\nu-1]} & -\frac{E}{8[\nu-1]} & -\frac{E[1-3\nu]}{8[\nu^2-1]} \\
-\frac{E}{8[\nu-1]} & \frac{E[\nu-3]}{8[\nu^2-1]} & -\frac{E[1-3\nu]}{8[\nu^2-1]} & -\frac{E}{8[\nu-1]} & \frac{E}{8[\nu-1]} & -\frac{E[\nu-3]}{8[\nu^2-1]} & \frac{E[1-3\nu]}{8[\nu^2-1]} & \frac{E}{8[\nu-1]} \\
\frac{E}{8[\nu-1]} & -\frac{E[1-3\nu]}{8[\nu^2-1]} & \frac{E[\nu-3]}{8[\nu^2-1]} & \frac{E}{8[\nu-1]} & -\frac{E}{8[\nu-1]} & \frac{E[1-3\nu]}{8[\nu^2-1]} & -\frac{E[\nu-3]}{8[\nu^2-1]} & -\frac{E}{8[\nu-1]} \\
\frac{E[1-3\nu]}{8[\nu^2-1]} & -\frac{E}{8[\nu-1]} & \frac{E}{8[\nu-1]} & \frac{E[\nu-3]}{8[\nu^2-1]} & -\frac{E[1-3\nu]}{8[\nu^2-1]} & \frac{E}{8[\nu-1]} & -\frac{E}{8[\nu-1]} & -\frac{E[\nu-3]}{8[\nu^2-1]} \\
-\frac{E[\nu-3]}{8[\nu^2-1]} & \frac{E}{8[\nu-1]} & -\frac{E}{8[\nu-1]} & -\frac{E[1-3\nu]}{8[\nu^2-1]} & \frac{E[\nu-3]}{8[\nu^2-1]} & -\frac{E}{8[\nu-1]} & \frac{E}{8[\nu-1]} & \frac{E[1-3\nu]}{8[\nu^2-1]} \\
\frac{E}{8[\nu-1]} & -\frac{E[\nu-3]}{8[\nu^2-1]} & \frac{E[1-3\nu]}{8[\nu^2-1]} & \frac{E}{8[\nu-1]} & -\frac{E}{8[\nu-1]} & \frac{E[\nu-3]}{8[\nu^2-1]} & -\frac{E[1-3\nu]}{8[\nu^2-1]} & -\frac{E}{8[\nu-1]} \\
-\frac{E}{8[\nu-1]} & \frac{E[1-3\nu]}{8[\nu^2-1]} & -\frac{E[\nu-3]}{8[\nu^2-1]} & -\frac{E}{8[\nu-1]} & \frac{E}{8[\nu-1]} & -\frac{E[1-3\nu]}{8[\nu^2-1]} & \frac{E[\nu-3]}{8[\nu^2-1]} & \frac{E}{8[\nu-1]} \\
-\frac{E[1-3\nu]}{8[\nu^2-1]} & \frac{E}{8[\nu-1]} & -\frac{E}{8[\nu-1]} & -\frac{E[\nu-3]}{8[\nu^2-1]} & \frac{E[1-3\nu]}{8[\nu^2-1]} & -\frac{E}{8[\nu-1]} & \frac{E}{8[\nu-1]} & \frac{E[\nu-3]}{8[\nu^2-1]}
\end{bmatrix}$$

$$\boldsymbol{J} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\text{NPE}} \\ y^1 & y^2 & \dots & y^{\text{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\text{NPE}}_\xi & N^{\text{NPE}}_\eta \end{bmatrix}$$

$$\boldsymbol{x}^1 = [-1\,,\,-1]$$

$$\boldsymbol{x}^2 = [1\,,\,-1]$$

$$\boldsymbol{x}^3 = [1\,,\,1]$$

$$\boldsymbol{x}^4 = [-1\,,\,1]$$

Example (D2QU4N) … bilinear quadrilateral element

… using one Gauss point …

Quadrilateral Elements

$$K^{\,i\,j}_{a\,c} = \sum_{\text{gp}\,=\,1}^{\text{GPE}} \left[\boldsymbol{J}^{-T}\cdot N^{i}_{,\boldsymbol{\xi}}\right]_b E_{a\,b\,c\,d}\left[\boldsymbol{J}^{-T}\cdot N^{j}_{,\boldsymbol{\xi}}\right]_d \,\text{Det}\boldsymbol{J}\,\alpha_{gp}$$

$$E_{a\,b\,c\,d} = \frac{E}{2[1+\nu]}[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}] + \frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

plane strain

$$\boldsymbol{J} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\text{NPE}} \\ y^1 & y^2 & \dots & y^{\text{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\text{NPE}}_\xi & N^{\text{NPE}}_\eta \end{bmatrix}$$

$$\boldsymbol{K} = \begin{bmatrix} 3.750 & 1.250 & -1.250 & -1.250 & -3.750 & -1.250 & 1.250 & 1.250 \\ 1.250 & 3.750 & 1.250 & 1.250 & -1.250 & -3.750 & -1.250 & -1.250 \\ -1.250 & 1.250 & 3.750 & -1.250 & 1.250 & -1.250 & -3.750 & 1.250 \\ -1.250 & 1.250 & -1.250 & 3.750 & 1.250 & -1.250 & 1.250 & -3.750 \\ -3.750 & -1.250 & 1.250 & 1.250 & 3.750 & 1.250 & -1.250 & -1.250 \\ -1.250 & -3.750 & -1.250 & -1.250 & 1.250 & 3.750 & 1.250 & 1.250 \\ 1.250 & -1.250 & -3.750 & 1.250 & -1.250 & 1.250 & 3.750 & -1.250 \\ 1.250 & -1.250 & 1.250 & -3.750 & -1.250 & 1.250 & -1.250 & 3.750 \end{bmatrix}$$

$$\boldsymbol{x}^1 = [-1\,,\,-1]$$

$$\boldsymbol{x}^2 = [1\,,\,-1]$$

$$\boldsymbol{x}^3 = [1\,,\,1]$$

$$\boldsymbol{x}^4 = [-1\,,\,1]$$

$$E = 10\,,\;\nu = 0$$

Example (D2QU4N) … bilinear quadrilateral element

… using four Gauss points …

Quadrilateral Elements

$$K^{ij}_{ac} = \sum_{\text{gp}=1}^{\text{GPE}} \left[\boldsymbol{J}^{-T} \cdot N^{i}_{,\boldsymbol{\xi}}\right]_b E_{abcd} \left[\boldsymbol{J}^{-T} \cdot N^{j}_{,\boldsymbol{\xi}}\right]_d \operatorname{Det}\boldsymbol{J}\, \alpha_{gp}$$

$$E_{abcd} = \frac{E}{2[1+\nu]}[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}]$$

plane strain

$$+\frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

$$\boldsymbol{K} = \begin{bmatrix} 5.000 & 1.250 & -2.500 & -1.250 & -2.500 & -1.250 & 0.000 & 1.250 \\ 1.250 & 5.000 & 1.250 & -0.000 & -1.250 & -2.500 & -1.250 & -2.500 \\ -2.500 & 1.250 & 5.000 & -1.250 & 0.000 & -1.250 & -2.500 & 1.250 \\ -1.250 & -0.000 & -1.250 & 5.000 & 1.250 & -2.500 & 1.250 & -2.500 \\ -2.500 & -1.250 & 0.000 & 1.250 & 5.000 & 1.250 & -2.500 & -1.250 \\ -1.250 & -2.500 & -1.250 & -2.500 & 1.250 & 5.000 & 1.250 & -0.000 \\ 0.000 & -1.250 & -2.500 & 1.250 & -2.500 & 1.250 & 5.000 & -1.250 \\ 1.250 & -2.500 & 1.250 & -2.500 & -1.250 & -0.000 & -1.250 & 5.000 \end{bmatrix}$$

$$\boldsymbol{J} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\text{NPE}} \\ y^1 & y^2 & \dots & y^{\text{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\text{NPE}}_\xi & N^{\text{NPE}}_\eta \end{bmatrix}$$

$$\boldsymbol{x}^1 = [\,-1\,,\,-1\,]$$

$$\boldsymbol{x}^2 = [\,1\,,\,-1\,]$$

$$\boldsymbol{x}^3 = [\,1\,,\,1\,]$$

$$\boldsymbol{x}^4 = [\,-1\,,\,1\,]$$

$$E = 10\,,\ \nu = 0$$

Example (D2QU4N) … bilinear quadrilateral element

… using nine Gauss points …

Quadrilateral Elements

$$K^{\,i\,j}_{\,a\,c} = \sum_{\text{gp}\,=\,1}^{\text{GPE}} \left[\boldsymbol{J}^{-T} \cdot N^i_{,\boldsymbol{\xi}}\right]_b E_{a\,b\,c\,d} \left[\boldsymbol{J}^{-T} \cdot N^j_{,\boldsymbol{\xi}}\right]_d \operatorname{Det}\boldsymbol{J}\,\alpha_{gp}$$

$$E_{a\,b\,c\,d} = \frac{E}{2[1+\nu]}\left[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}\right] + \frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

plane strain

$$\boldsymbol{J} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\text{NPE}} \\ y^1 & y^2 & \dots & y^{\text{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\text{NPE}}_\xi & N^{\text{NPE}}_\eta \end{bmatrix}$$

$$\boldsymbol{K} = \begin{bmatrix} 5.000 & 1.250 & -2.500 & -1.250 & -2.500 & -1.250 & -0.000 & 1.250 \\ 1.250 & 5.000 & 1.250 & -0.000 & -1.250 & -2.500 & -1.250 & -2.500 \\ -2.500 & 1.250 & 5.000 & -1.250 & -0.000 & -1.250 & -2.500 & 1.250 \\ -1.250 & -0.000 & -1.250 & 5.000 & 1.250 & -2.500 & 1.250 & -2.500 \\ -2.500 & -1.250 & -0.000 & 1.250 & 5.000 & 1.250 & -2.500 & -1.250 \\ -1.250 & -2.500 & -1.250 & -2.500 & 1.250 & 5.000 & 1.250 & -0.000 \\ -0.000 & -1.250 & -2.500 & 1.250 & -2.500 & 1.250 & 5.000 & -1.250 \\ 1.250 & -2.500 & 1.250 & -2.500 & -1.250 & -0.000 & -1.250 & 5.000 \end{bmatrix}$$

$$\boldsymbol{x}^1 = [-1\,,\,-1]$$
$$\boldsymbol{x}^2 = [1\,,\,-1]$$
$$\boldsymbol{x}^3 = [1\,,\,1]$$
$$\boldsymbol{x}^4 = [-1\,,\,1]$$

$$E = 10\,,\ \nu = 0$$

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Example (D2QU4N) … bilinear quadrilateral element

… using one Gauss point …

Quadrilateral Elements

$$K^{ij}_{ac} = \sum_{\text{gp}=1}^{\text{GPE}} \left[\boldsymbol{J}^{-T}\cdot N^{i}_{,\boldsymbol{\xi}}\right]_b E_{abcd} \left[\boldsymbol{J}^{-T}\cdot N^{j}_{,\boldsymbol{\xi}}\right]_d \text{Det}\boldsymbol{J}\,\alpha_{gp}$$

$$E_{abcd} = \frac{E}{2[1+\nu]}\left[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}\right] + \frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

plane strain

$$\boldsymbol{J} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\text{NPE}} \\ y^1 & y^2 & \dots & y^{\text{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\text{NPE}}_\xi & N^{\text{NPE}}_\eta \end{bmatrix}$$

$$\boldsymbol{K} = \begin{bmatrix}
4.167 & 2.500 & -2.500 & 0.833 & -4.167 & -2.500 & 2.500 & -0.833 \\
2.500 & 4.167 & -0.833 & 2.500 & -2.500 & -4.167 & 0.833 & -2.500 \\
-2.500 & -0.833 & 4.167 & -2.500 & 2.500 & 0.833 & -4.167 & 2.500 \\
0.833 & 2.500 & -2.500 & 4.167 & -0.833 & -2.500 & 2.500 & -4.167 \\
-4.167 & -2.500 & 2.500 & -0.833 & 4.167 & 2.500 & -2.500 & 0.833 \\
-2.500 & -4.167 & 0.833 & -2.500 & 2.500 & 4.167 & -0.833 & 2.500 \\
2.500 & 0.833 & -4.167 & 2.500 & -2.500 & -0.833 & 4.167 & -2.500 \\
-0.833 & -2.500 & 2.500 & -4.167 & 0.833 & 2.500 & -2.500 & 4.167
\end{bmatrix}$$

$$\boldsymbol{x}^1 = [-1\,,\,-1]$$
$$\boldsymbol{x}^2 = [1\,,\,-1]$$
$$\boldsymbol{x}^3 = [1\,,\,1]$$
$$\boldsymbol{x}^4 = [-1\,,\,1]$$

$$E = 10\,,\ \nu = 0.5$$

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Example (D2QU4N) … bilinear quadrilateral element

… using four Gauss points …

Quadrilateral Elements

$$K^{ij}_{ac} = \sum_{\text{gp}=1}^{\text{GPE}} \left[\boldsymbol{J}^{-T} \cdot N^i_{,\boldsymbol{\xi}}\right]_b E_{abcd} \left[\boldsymbol{J}^{-T} \cdot N^j_{,\boldsymbol{\xi}}\right]_d \text{Det}\boldsymbol{J}\,\alpha_{gp}$$

$$E_{abcd} = \frac{E}{2[1+\nu]}[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}]$$

plane strain

$$+\frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

$$\boldsymbol{K} = \begin{bmatrix} 5.556 & 2.500 & -3.889 & 0.833 & -2.778 & -2.500 & 1.111 & -0.833 \\ 2.500 & 5.556 & -0.833 & 1.111 & -2.500 & -2.778 & 0.833 & -3.889 \\ -3.889 & -0.833 & 5.556 & -2.500 & 1.111 & 0.833 & -2.778 & 2.500 \\ 0.833 & 1.111 & -2.500 & 5.556 & -0.833 & -3.889 & 2.500 & -2.778 \\ -2.778 & -2.500 & 1.111 & -0.833 & 5.556 & 2.500 & -3.889 & 0.833 \\ -2.500 & -2.778 & 0.833 & -3.889 & 2.500 & 5.556 & -0.833 & 1.111 \\ 1.111 & 0.833 & -2.778 & 2.500 & -3.889 & -0.833 & 5.556 & -2.500 \\ -0.833 & -3.889 & 2.500 & -2.778 & 0.833 & 1.111 & -2.500 & 5.556 \end{bmatrix}$$

$$\boldsymbol{J} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\text{NPE}} \\ y^1 & y^2 & \dots & y^{\text{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\text{NPE}}_\xi & N^{\text{NPE}}_\eta \end{bmatrix}$$

$$\boldsymbol{x}^1 = [-1\,,\,-1]$$

$$\boldsymbol{x}^2 = [1\,,\,-1]$$

$$\boldsymbol{x}^3 = [1\,,\,1]$$

$$\boldsymbol{x}^4 = [-1\,,\,1]$$

$$E = 10\,,\ \nu = 0.5$$

Example (D2QU4N) … bilinear quadrilateral element

… using nine Gauss points …

Quadrilateral Elements

$$K^{\,i\,j}_{\,a\,c} = \sum_{\text{gp}\,=\,1}^{\text{GPE}} \left[\boldsymbol{J}^{-T}\cdot N^{i}_{,\boldsymbol{\xi}}\right]_b E_{a\,b\,c\,d}\left[\boldsymbol{J}^{-T}\cdot N^{j}_{,\boldsymbol{\xi}}\right]_d \,\text{Det}\boldsymbol{J}\,\alpha_{gp}$$

$$E_{a\,b\,c\,d} = \frac{E}{2[1+\nu]}\left[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}\right] + \frac{E\,\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

plane strain

$$\boldsymbol{J} = \begin{bmatrix} x^1 & x^2 & \dots & x^{\text{NPE}} \\ y^1 & y^2 & \dots & y^{\text{NPE}} \end{bmatrix} \begin{bmatrix} N^1_\xi & N^1_\eta \\ N^2_\xi & N^2_\eta \\ \vdots & \vdots \\ N^{\text{NPE}}_\xi & N^{\text{NPE}}_\eta \end{bmatrix}$$

$$\boldsymbol{K} = \begin{bmatrix}
5.556 & 2.500 & -3.889 & 0.833 & -2.778 & -2.500 & 1.111 & -0.833 \\
2.500 & 5.556 & -0.833 & 1.111 & -2.500 & -2.778 & 0.833 & -3.889 \\
-3.889 & -0.833 & 5.556 & -2.500 & 1.111 & 0.833 & -2.778 & 2.500 \\
0.833 & 1.111 & -2.500 & 5.556 & -0.833 & -3.889 & 2.500 & -2.778 \\
-2.778 & -2.500 & 1.111 & -0.833 & 5.556 & 2.500 & -3.889 & 0.833 \\
-2.500 & -2.778 & 0.833 & -3.889 & 2.500 & 5.556 & -0.833 & 1.111 \\
1.111 & 0.833 & -2.778 & 2.500 & -3.889 & -0.833 & 5.556 & -2.500 \\
-0.833 & -3.889 & 2.500 & -2.778 & 0.833 & 1.111 & -2.500 & 5.556
\end{bmatrix}$$

$$\boldsymbol{x}^1 = [-1\,,\,-1]$$
$$\boldsymbol{x}^2 = [1\,,\,-1]$$
$$\boldsymbol{x}^3 = [1\,,\,1]$$
$$\boldsymbol{x}^4 = [-1\,,\,1]$$

$$E = 10\,,\ \nu = 0.5$$

finite elements

Examples

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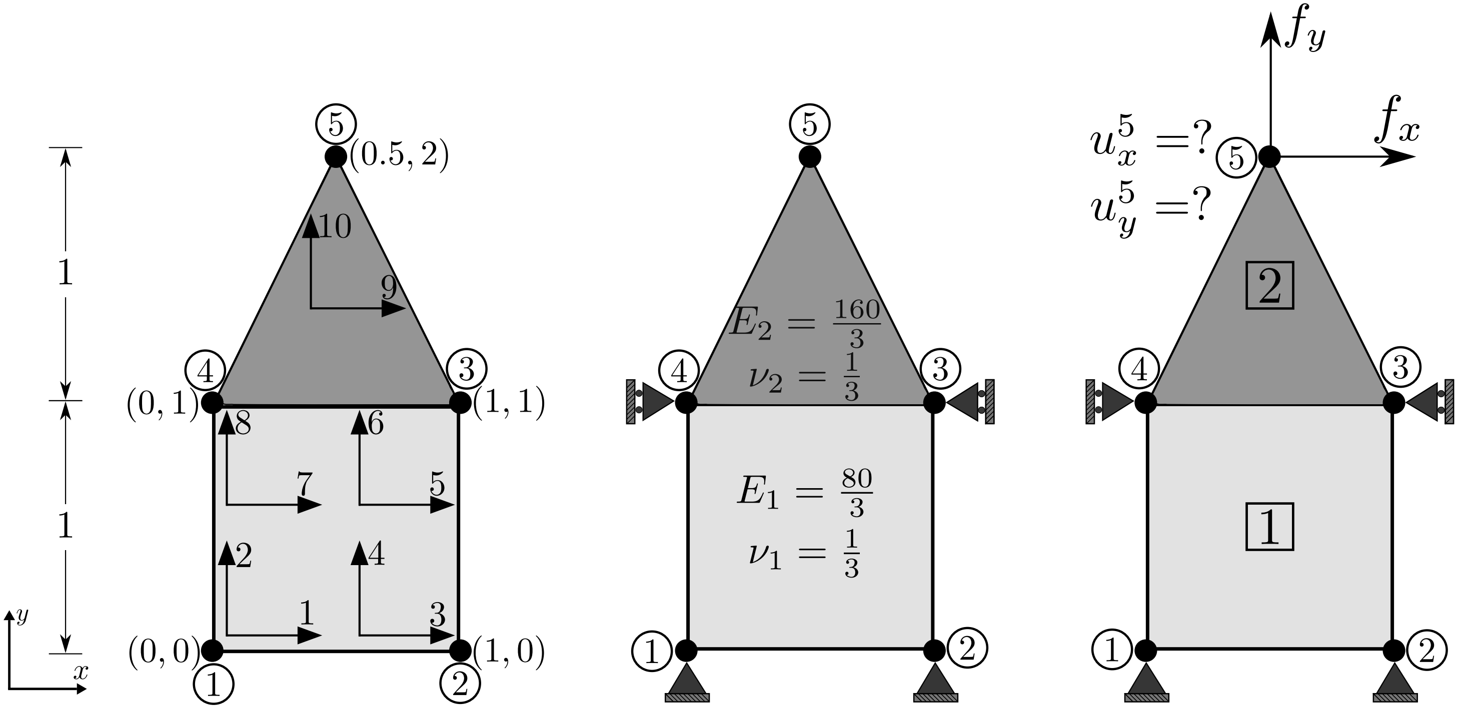
Example 1

Consider the given structure consisting of a square and a triangle element. The element 1 is bilinear quadrilateral, also referred to as D2QU4N, and element 2 is linear triangular, also referred to as D2TR3N. The material parameters of each element are shown in the figure. The material properties for the square element and the triangle element are $E_1 = 80/3\,,\ \nu_1 = 1/3$ and $E_2 = 160/3\,,\ \nu_2 = 1/3$, respectively. The numbering of the nodes, elements and the global degrees are given in the figure. The nodes are numbered with a circle around them and elements are distinguished by square around them. **You must use exactly the same numbering for the degrees and elements given in the figure**. Homogeneous Dirichlet boundary conditions are prescribed on nodes 1 and 2 in both directions. Nodes 3 and 4 are fixed only in horizontal direction, but can freely move in vertical dircetion. The forces f_x and f_y are applied on node 5 in the horizontal and vertical directions, respectively.

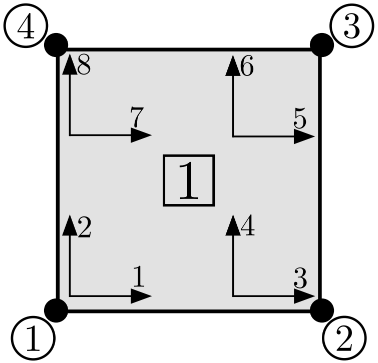
Example 1 ii



- Compute the 8×8 stiffness of the element 1 using one Gauss Point quadrature rule.
- Compute the 6×6 stiffness of the element 2 using one Gauss Point quadrature rule.
- Assemble the **reduced** stiffness for the entire system.
- Compute the displacement of node 5 in x and y directions for the three cases:

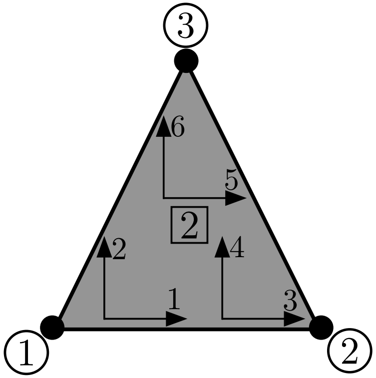
$(1)\ f_x = 1\ ,\ f_y = 0$, $(2)\ f_x = 0\ ,\ f_y = 1$, $(3)\ f_x = 1\ ,\ f_y = 1.$

Example 1



$$\mathbb{K}_1 = \begin{array}{cccccccc|c} 1x & 1y & 2x & 2y & 3x & 3y & 4x & 4y & \\ \hline 10 & 5 & -5 & 0 & -10 & -5 & 5 & 0 & 1x \\ 5 & 10 & 0 & 5 & -5 & -10 & 0 & -5 & 1y \\ -5 & 0 & 10 & -5 & 5 & 0 & -10 & 5 & 2x \\ 0 & 5 & -5 & 10 & 0 & -5 & 5 & -10 & 2y \\ -10 & -5 & 5 & 0 & 10 & 5 & -5 & 0 & 3x \\ -5 & -10 & 0 & -5 & 5 & 10 & 0 & 5 & 3y \\ 5 & 0 & -10 & 5 & -5 & 0 & 10 & -5 & 4x \\ 0 & -5 & 5 & -10 & 0 & 5 & -5 & 10 & 4y \end{array}$$

Example 1



$$\mathbb{K}_2 = \begin{array}{c} \begin{array}{cccccc} \color{red}{4x} & \color{red}{4y} & \color{red}{5x} & \color{red}{5y} & \color{red}{3x} & \color{red}{3y} \end{array} \\ \left[\begin{array}{cccccc} 32.05 & 10 & -27.5 & 0 & -5 & -10 \\ 10 & 17.5 & 0 & -2.5 & -10 & -15 \\ -27.5 & 0 & 32.5 & -10 & -5 & 10 \\ 0 & -2.5 & -10 & 17.5 & 10 & -15 \\ -5 & -10 & -5 & 10 & 10 & 0 \\ -10 & -15 & 10 & -15 & 0 & 30 \end{array}\right] \end{array} \begin{array}{c} \\ \color{red}{4x} \\ \color{red}{4y} \\ \color{red}{5x} \\ \color{red}{5y} \\ \color{red}{3x} \\ \color{red}{3y} \end{array}$$

Example 1

$$\mathbb{K}_{\mathsf{tot}} = \begin{bmatrix} 10 & 5 & -5 & 0 & -10 & -5 & 5 & 0 & 0 & 0 \\ 5 & 10 & 0 & 5 & -5 & -10 & 0 & -5 & 0 & 0 \\ -5 & 0 & 10 & -5 & 5 & 0 & -10 & 5 & 0 & 0 \\ 0 & 5 & -5 & 10 & 0 & -5 & 5 & -10 & 0 & 0 \\ -10 & -5 & 5 & 0 & 42.5 & -5 & -32.5 & 0 & -5 & 10 \\ -5 & -10 & 0 & -5 & -5 & 27.5 & 0 & 2.5 & 10 & -15 \\ 5 & 0 & -10 & 5 & -23.5 & 0 & 42.5 & 5 & -5 & -10 \\ 0 & -5 & 5 & -10 & 0 & 2.5 & 5 & 27.5 & -10 & -15 \\ 0 & 0 & 0 & 0 & -5 & 10 & -5 & -10 & 10 & 0 \\ 0 & 0 & 0 & 0 & 10 & -15 & -10 & -15 & 0 & 30 \end{bmatrix}$$

$$\mathbb{K}_{\text{red}} = \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{cccc|c} 4y & 3y & 5x & 5y & \\ \hline 27.5 & 2.5 & -10 & -15 & 4y \\ 2.5 & 27.5 & 10 & -15 & 3y \\ -10 & 10 & 10 & 0 & 5x \\ -15 & -15 & 0 & 30 & 5y \end{array}$$

$$\begin{bmatrix} 27.5 & 2.5 & -10 & -15 \\ 2.5 & 27.5 & 10 & -15 \\ -10 & 10 & 10 & 0 \\ -15 & -15 & 0 & 30 \end{bmatrix} \begin{bmatrix} u_y^4 \\ u_y^3 \\ u_x^5 \\ u_y^5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ f_x \\ f_y \end{bmatrix}$$

Example 1

$$\begin{cases} 27.5u_y^4 + 2.5u_y^3 - 10u_x^5 - 15u_y^5 = 0 \\ 2.5u_y^4 + 27.5u_y^3 + 10u_x^5 - 15u_y^5 = 0 \\ -10u_y^4 + 10u_y^3 + 10u_x^5 = f_x \\ -15u_y^4 - 15u_y^3 + 30u_y^5 = f_y \end{cases}$$

$f_x = 1\ ,\ f_y = 0$	$f_x = 0\ ,\ f_y = 1$	$f_x = 1\ ,\ f_y = 1$
$u_y^4 = 0.2$	$u_y^4 = \frac{0.1}{3}$	$u_y^4 = \frac{0.7}{3}$
$u_y^3 = -0.2$	$u_y^3 = \frac{0.1}{3}$	$u_y^3 = -\frac{0.5}{3}$
$u_x^5 = 0.5$	$u_x^5 = 0$	$u_x^5 = \frac{1.5}{3}$
$u_y^5 = 0$	$u_y^5 = \frac{0.2}{3}$	$u_y^5 = \frac{0.2}{3}$