FINITE ELEMENT METHOD

2

Ali Javili

Bilkent University

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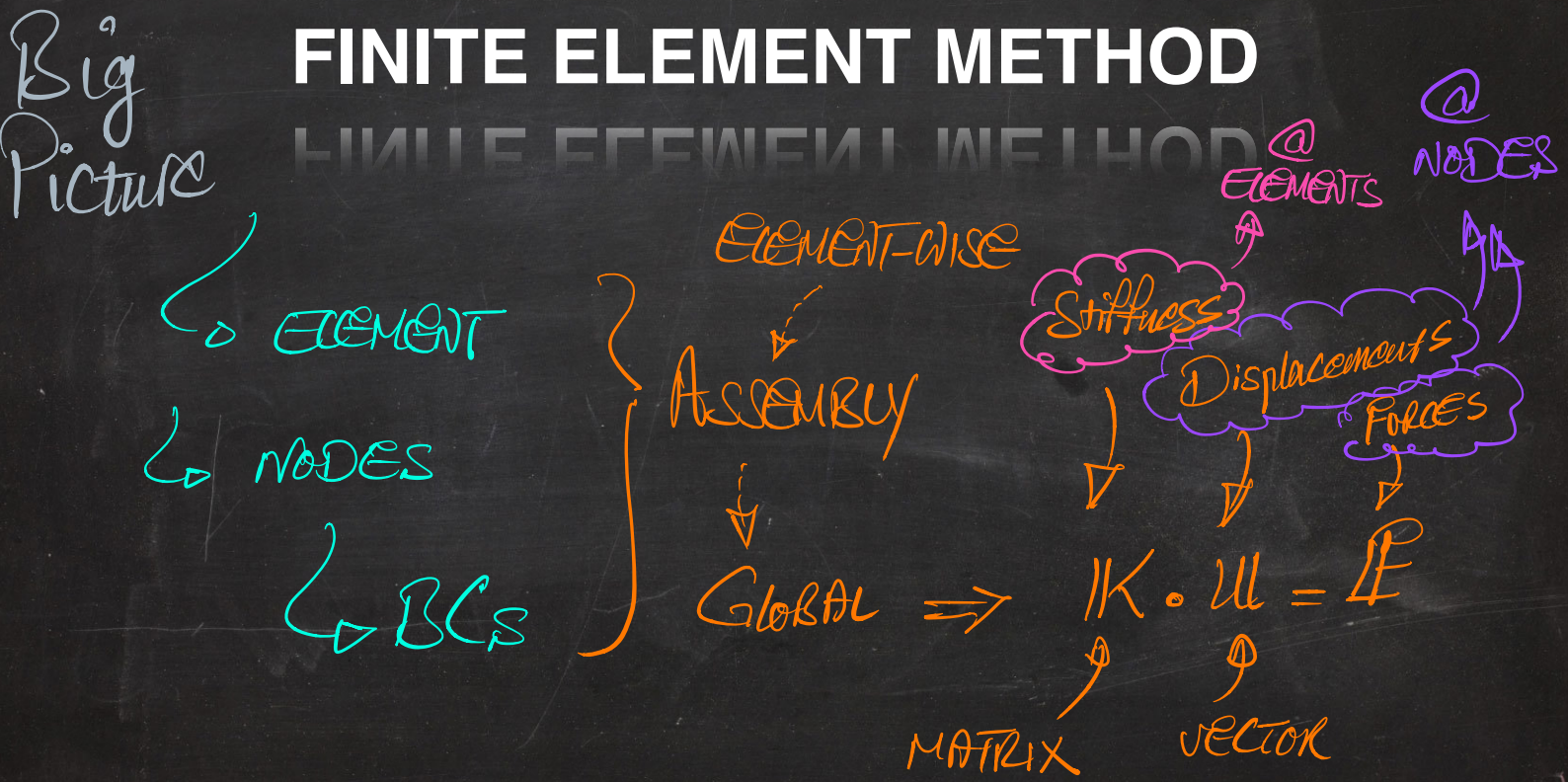
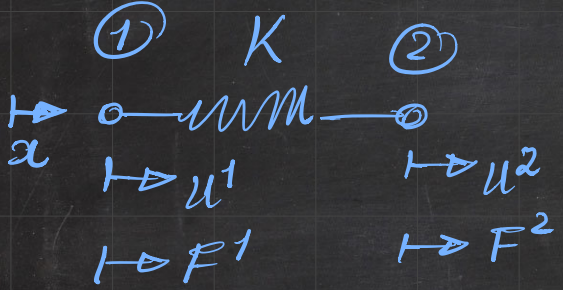


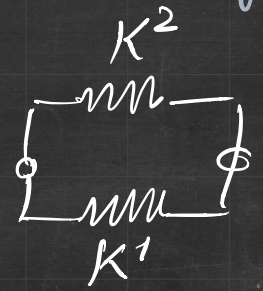
Understanding key ingredients of FEM using springs:



$$\begin{bmatrix} F^1 \\ F^2 \end{bmatrix} = \begin{bmatrix} K & -K \\ -K & K \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix}$$

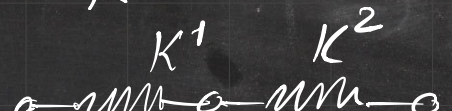
$$F = K \cdot u$$

I)



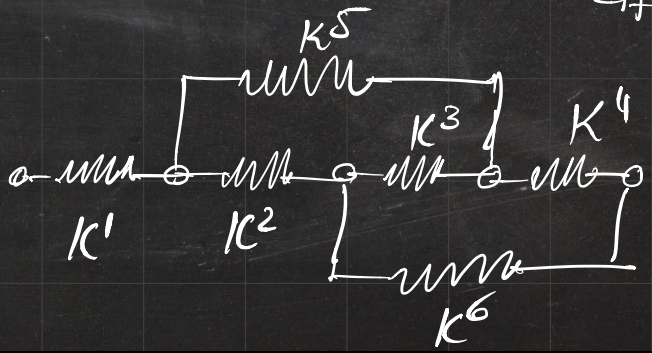
K_{eff} ?

II)

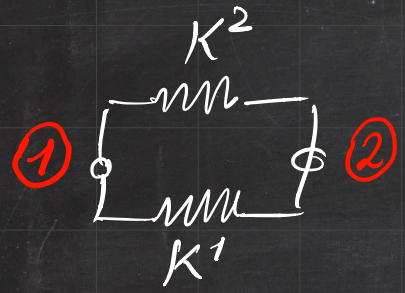


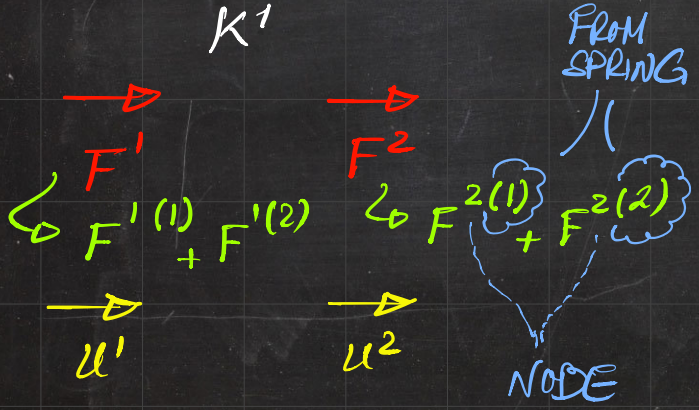
K_{eff} ?

III)



Understanding key ingredients of FEM using springs:





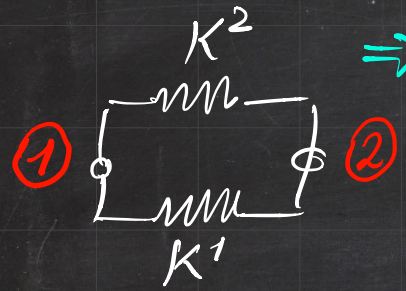
SPRING 1:

$$\begin{bmatrix} F^{1(1)} \\ F^{2(1)} \end{bmatrix} = \begin{bmatrix} K^1 & -K^1 \\ -K^1 & K^1 \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix}$$

SPRING 2:

$$\begin{bmatrix} F^{1(2)} \\ F^{2(2)} \end{bmatrix} = \begin{bmatrix} K^2 & -K^2 \\ -K^2 & K^2 \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix}$$

Understanding key ingredients of FEM using springs:



$\Rightarrow$ $K_{eff} = K^1 + K^2$

Parallel Combination

SPRING 1:

$$\begin{bmatrix} F^{1(1)} \\ F^{2(1)} \end{bmatrix} = \begin{bmatrix} K^1 & -K^1 \\ -K^1 & K^1 \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix}$$

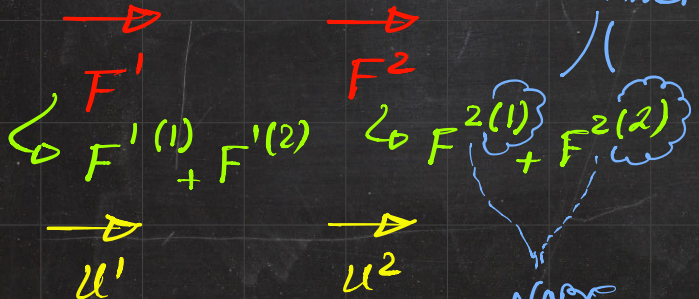
SPRING 2:

$$\begin{bmatrix} F^{1(2)} \\ F^{2(2)} \end{bmatrix} = \begin{bmatrix} K^2 & -K^2 \\ -K^2 & K^2 \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix}$$

FORCE @ NODE 1

FORCE @ NODE 2

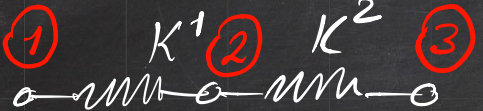
$$\begin{bmatrix} F^1 \\ F^2 \end{bmatrix} = \begin{bmatrix} F^{1(1)} + F^{1(2)} \\ F^{2(1)} + F^{2(2)} \end{bmatrix}$$

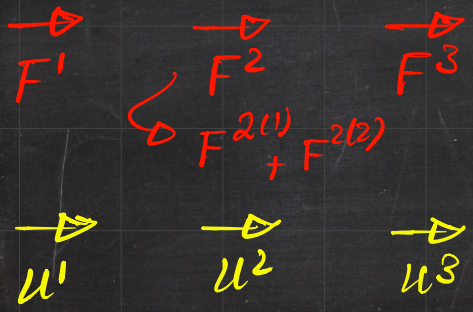


$$\Rightarrow \begin{bmatrix} F^1 \\ F^2 \end{bmatrix} = \begin{bmatrix} [K^1+K^2] & -[K^1+K^2] \\ -[K^1+K^2] & [K^1+K^2] \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix}$$

$$\Rightarrow \mathbb{F} = \mathbb{K}_{eff} \cdot \mathbb{U} \qquad \mathbb{K}_{eff} = \begin{bmatrix} K_{eff} & -K_{eff} \\ -K_{eff} & K_{eff} \end{bmatrix}$$

Understanding key ingredients of FEM using springs:





SPRING 1:

$$\begin{bmatrix} F^1 \\ F^{2(1)} \end{bmatrix} = \begin{bmatrix} K^1 & -K^1 \\ -K^1 & K^1 \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix}$$

ELEMENT-WISE STIFFNESS MATRIX

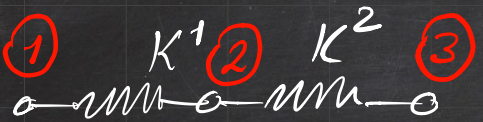
SPRING 2:

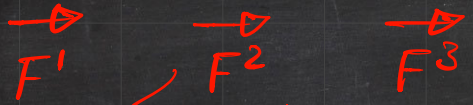
$$\begin{bmatrix} F^{2(2)} \\ F^3 \end{bmatrix} = \begin{bmatrix} K^2 & -K^2 \\ -K^2 & K^2 \end{bmatrix} \begin{bmatrix} u^2 \\ u^3 \end{bmatrix}$$

GLOBAL STIFFNESS MATRIX

$$\begin{bmatrix} F^1 \\ F^2 \\ F^3 \end{bmatrix} = \begin{bmatrix} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \end{bmatrix}$$

Understanding key ingredients of FEM using springs:





$$\frac{1}{K_{eff}} = \frac{1}{K_1} + \frac{1}{K_2}$$

SPRING 1:

$$\begin{bmatrix} F^1 \\ F^{2(1)} \end{bmatrix} = \begin{bmatrix} K^1 & -K^1 \\ -K^1 & K^1 \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix}$$

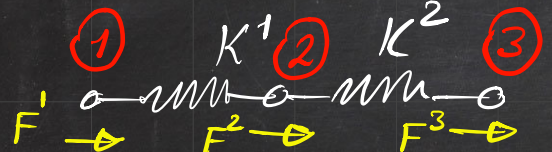
SPRING 2:

$$\begin{bmatrix} F^{2(2)} \\ F^3 \end{bmatrix} = \begin{bmatrix} K^2 & -K^2 \\ -K^2 & K^2 \end{bmatrix} \begin{bmatrix} u^2 \\ u^3 \end{bmatrix}$$

GLOBAL STIFFNESS

$$\begin{bmatrix} F^1 \\ F^2 \\ F^3 \end{bmatrix} = \begin{bmatrix} K^1 & -K^1 & 0 \\ -K^1 & K^1+K^2 & -K^2 \\ 0 & -K^2 & K^2 \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \end{bmatrix}$$

Understanding key ingredients of FEM using springs:



$$\begin{bmatrix} F^1 \\ F^2 \\ F^3 \end{bmatrix} = \begin{bmatrix} K^1 & -K^1 & 0 \\ -K^1 & K^1+K^2 & -K^2 \\ 0 & -K^2 & K^2 \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \end{bmatrix}$$

$$\begin{bmatrix} F^1 \\ 0 \\ F^3 \end{bmatrix} = \begin{bmatrix} K^1 & -K^1 & 0 \\ -K^1 & K^1+K^2 & -K^2 \\ 0 & -K^2 & K^2 \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \end{bmatrix}$$

$$-K^1u^1 + [K^1+K^2]u^2 - K^2u^3 = 0$$

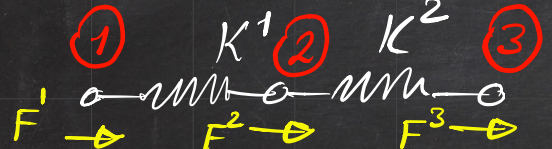
$$\Rightarrow [K^1+K^2]u^2 = K^1u^1 + K^2u^3$$

$$\Rightarrow u^2 = \frac{K^1u^1 + K^2u^3}{K^1+K^2}$$

$$F^1 = K^1u^1 - K^1u^2 = K^1u^1 - K^1\frac{K^1u^1 + K^2u^3}{K^1+K^2}$$

$$= \frac{K^1K^1u^1 + K^2K^1u^1 - K^1K^1u^1 - K^1K^2u^3}{K^1+K^2}$$

Understanding key ingredients of FEM using springs:



$$-K^1u^1 + [K^1+K^2]u^2 - K^2u^3 = 0$$

$$\Rightarrow u^2 = \frac{K^1u^1 + K^2u^3}{K^1+K^2}$$

$$\begin{bmatrix} F^1 \\ F^2 \\ F^3 \end{bmatrix} = \begin{bmatrix} K^1 & -K^1 & 0 \\ -K^1 & K^1+K^2 & -K^2 \\ 0 & -K^2 & K^2 \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \end{bmatrix}$$

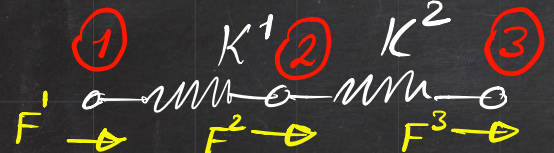
$$F^1 = K^1u^1 - K^1u^2 = K^1u^1 - K^1\frac{K^1u^1 + K^2u^3}{K^1+K^2}$$

$$= \frac{K^1K^1u^1 + K^2K^1u^1 - K^1K^1u^1 - K^1K^2u^3}{K^1+K^2}$$

$$\begin{bmatrix} F^1 \\ 0 \\ F^3 \end{bmatrix} = \begin{bmatrix} K^1 & -K^1 & 0 \\ -K^1 & K^1+K^2 & -K^2 \\ 0 & -K^2 & K^2 \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \end{bmatrix}$$

$$\Rightarrow F^1 = \frac{K^1K^2}{K^1+K^2}[u^1 - u^3]$$

Understanding key ingredients of FEM using springs:



$$\begin{bmatrix} F^1 \\ F^2 \\ F^3 \end{bmatrix} = \begin{bmatrix} K^1 & -K^1 & 0 \\ -K^1 & K^1+K^2 & -K^2 \\ 0 & -K^2 & K^2 \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \end{bmatrix}$$

$$\begin{bmatrix} F^1 \\ 0 \\ F^3 \end{bmatrix} = \begin{bmatrix} K^1 & -K^1 & 0 \\ -K^1 & K^1+K^2 & -K^2 \\ 0 & -K^2 & K^2 \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \end{bmatrix}$$

$$-K^1u^1 + [K^1+K^2]u^2 - K^2u^3 = 0 \Rightarrow u^2 = \frac{K^1u^1 + K^2u^3}{K^1+K^2}$$

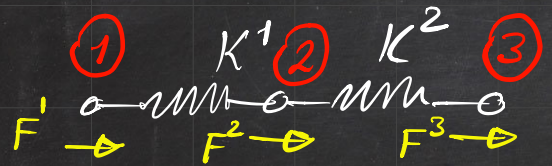
$$\Rightarrow F^1 = \frac{K^1K^2}{K^1+K^2}[u^1-u^3]$$

$$F^3 = -K^2u^2 + K^2u^3$$

$$= -K^2\frac{K^1u^1+K^2u^3}{K^1+K^2} + K^2u^3$$

$$= \frac{-K^2K^1u^1 - K^2K^2u^3 + K^1K^2u^3 + K^2K^2u^3}{K^1+K^2}$$

Understanding key ingredients of FEM using springs:



$$\begin{bmatrix} F^1 \\ F^2 \\ F^3 \end{bmatrix} = \begin{bmatrix} K^1 & -K^1 & 0 \\ -K^1 & K^1+K^2 & -K^2 \\ 0 & -K^2 & K^2 \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \end{bmatrix}$$

$$\begin{bmatrix} F^1 \\ 0 \\ F^3 \end{bmatrix} = \begin{bmatrix} K^1 & -K^1 & 0 \\ -K^1 & K^1+K^2 & -K^2 \\ 0 & -K^2 & K^2 \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \end{bmatrix}$$

$$-K^1u^1 + [K^1+K^2]u^2 - K^2u^3 = 0 \Rightarrow u^2 = \frac{K^1u^1 + K^2u^3}{K^1+K^2}$$

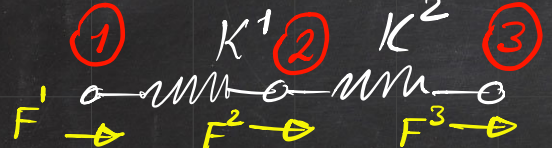
$$\Rightarrow F^1 = \frac{K^1K^2}{K^1+K^2}[u^1-u^3]$$

$$F^3 = -K^2u^2 + K^2u^3$$

$$= \frac{-K^2K^1u^1 - K^2K^2u^3 + K^1K^2u^3 + K^2K^2u^3}{K^1+K^2}$$

$$= \frac{K^1K^2}{K^1+K^2}[u^3-u^1]$$

Understanding key ingredients of FEM using springs:



$$-K^1u^1 + [K^1+K^2]u^2 - K^2u^3 = 0 \Rightarrow u^2 = \frac{K^1u^1 + K^2u^3}{K^1+K^2}$$

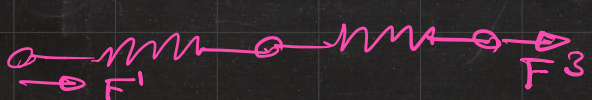
$$\begin{bmatrix} F^1 \\ F^2 \\ F^3 \end{bmatrix} = \begin{bmatrix} K^1 & -K^1 & 0 \\ -K^1 & K^1+K^2 & -K^2 \\ 0 & -K^2 & K^2 \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \end{bmatrix}$$

$$\checkmark \Rightarrow F^1 = \frac{K^1K^2}{K^1+K^2}[u^1-u^3]$$

$$0 \Leftarrow + \Rightarrow F^3 = \frac{K^1K^2}{K^1+K^2}[u^3-u^1] \rightarrow K_{eff}$$

$$F^1 + F^3 = 0$$

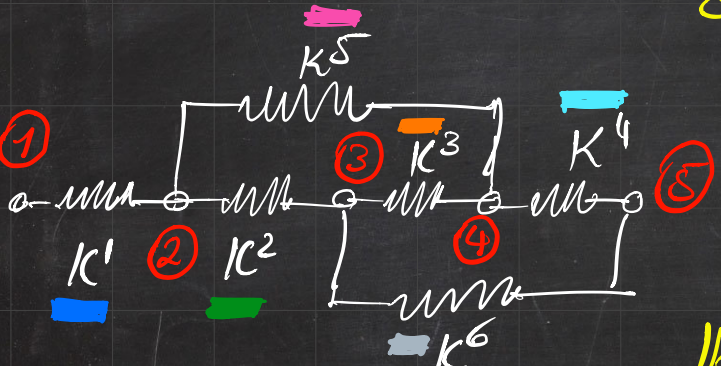
$$\begin{bmatrix} F^1 \\ 0 \\ F^3 \end{bmatrix} = \begin{bmatrix} K^1 & -K^1 & 0 \\ -K^1 & K^1+K^2 & -K^2 \\ 0 & -K^2 & K^2 \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \end{bmatrix}$$



$$\begin{bmatrix} F^1 \\ F^3 \end{bmatrix} = \begin{bmatrix} K_{eff} & -K_{eff} \\ -K_{eff} & K_{eff} \end{bmatrix} \begin{bmatrix} u^1 \\ u^3 \end{bmatrix}$$

SYM, DET=0

Understanding key ingredients of FEM using springs:



$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} \leftarrow \\ \leftarrow \\ \leftarrow \end{matrix} \begin{bmatrix} F^1 \\ F^2 \\ F^3 \\ F^4 \\ F^5 \end{bmatrix} = \begin{bmatrix} K^{11} & K^{12} & K^{13} & K^{14} & K^{15} \\ K^{21} & K^{22} & K^{23} & \cdots & \\ K^{31} & \cdots & \cdots & & \\ K^{41} & \cdots & \cdots & & \\ K^{51} & \cdots & & & K^{55} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \\ u^4 \\ u^5 \end{bmatrix} \Rightarrow F = K \cdot u$$

K

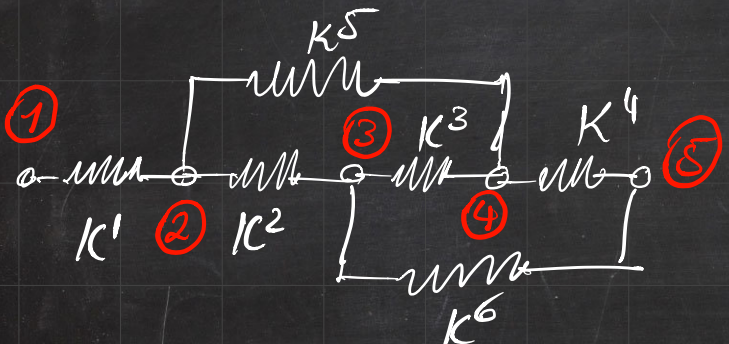
SYM. & DET=0

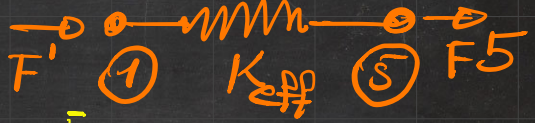
$$K = \begin{matrix} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ K^1 & -K^1 & 0 & 0 & 0 \\ -K^1 & K^1+K^2+K^5 & -K^2 & -K^5 & 0 \\ 0 & -K^2 & K^2+K^3+K^6 & -K^3 & -K^6 \\ 0 & -K^5 & -K^3 & K^3+K^4+K^5 & -K^4 \\ 0 & 0 & -K^6 & -K^4 & K^4+K^6 \end{bmatrix}$$

5x5

Understanding key ingredients of FEM using springs:

$K^1 = K^2 = K^3 = K^4 = K^5 = K^6 = 1$





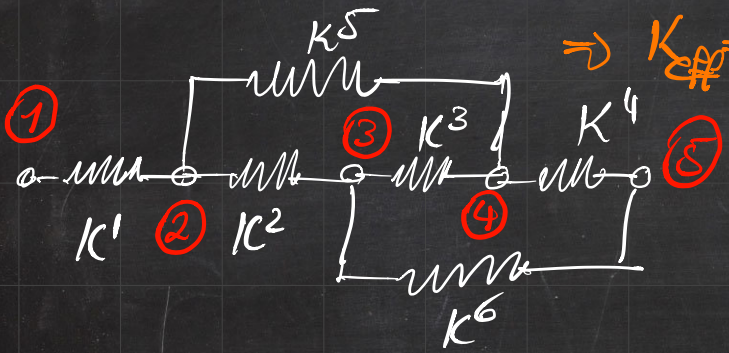
$$\begin{bmatrix} F^1 \\ F^2 \\ F^3 \\ F^4 \\ F^5 \end{bmatrix} = \begin{bmatrix} K^{11} & K^{12} & K^{13} & K^{14} & K^{15} \\ K^{21} & K^{22} & K^{23} & \cdots & \\ K^{31} & \cdots & \cdots & \cdots & \\ K^{41} & \cdots & \cdots & \cdots & \\ K^{51} & \cdots & \cdots & \cdots & K^{55} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \\ u^4 \\ u^5 \end{bmatrix} \Rightarrow F = K \cdot u$$

(0, 0, 0 mark the zero entries of F^2, F^3, F^4; K is 5×5)

$$K = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ 0 & -1 & 3 & -1 & -1 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$

Understanding key ingredients of FEM using springs:

$k^1 = k^2 = k^3 = k^4 = k^5 = k^6 = 1$



$\Rightarrow K_{eff} = 1.5$

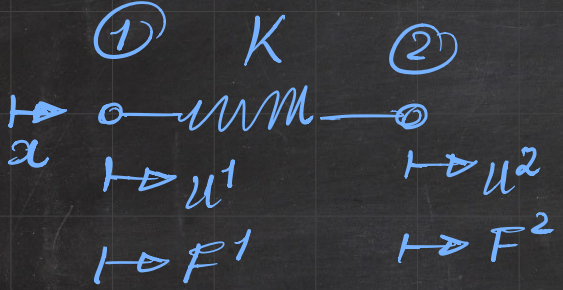
$$\begin{bmatrix} F^1 \\ F^5 \end{bmatrix} = \begin{bmatrix} 1.5 & -1.5 \\ -1.5 & 1.5 \end{bmatrix} \begin{bmatrix} u^1 \\ u^5 \end{bmatrix}$$

$\Rightarrow F^5 = 1.5\,[u^5 - u^1]$

$$K = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ 0 & -1 & 3 & -1 & -1 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}_{5\times 5}$$

$\Rightarrow F = Ku$

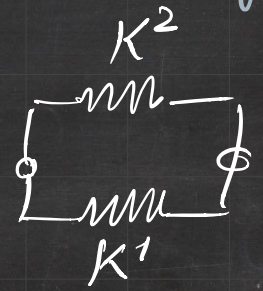
Understanding key ingredients of FEM using springs:



$$\begin{bmatrix} F^1 \\ F^2 \end{bmatrix} = \begin{bmatrix} K & -K \\ -K & K \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix}$$

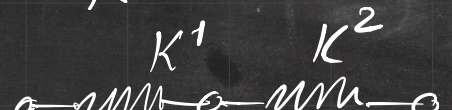
$$\mathbb{F} = \mathbb{K} \cdot \mathbb{u}$$

I)



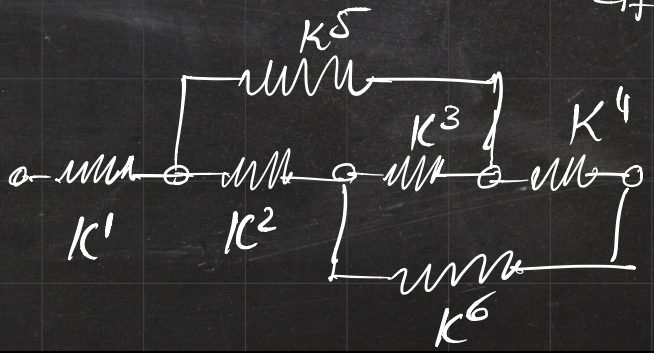
$\Rightarrow K_{eff}$?

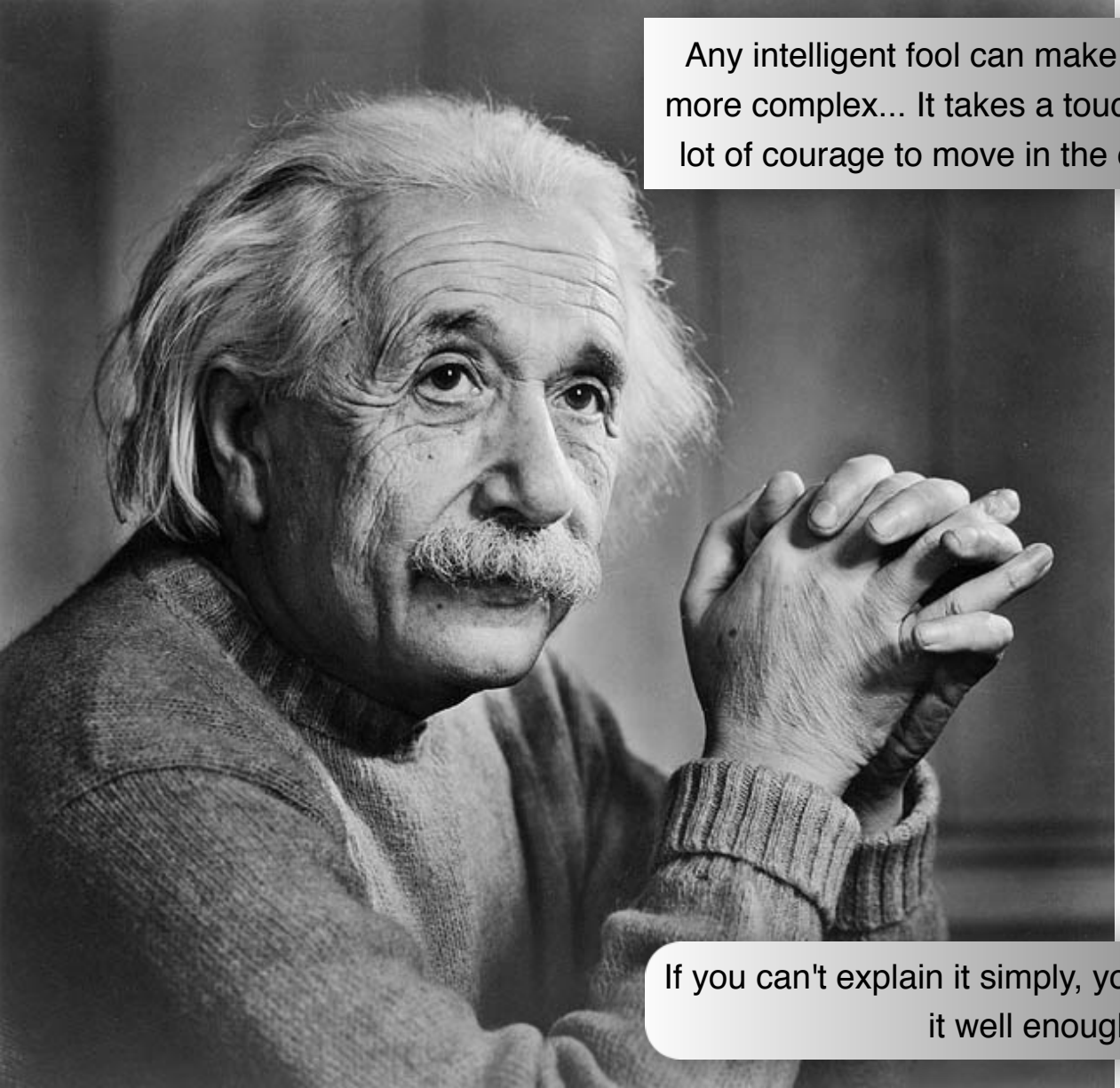
II)



$\Rightarrow K_{eff}$?

III)





Any intelligent fool can make things bigger and more complex... It takes a touch of genius - and a lot of courage to move in the opposite direction.

If you can't explain it simply, you don't understand it well enough.