

FINITE ELEMENT METHOD

ФИНАЛ ЕЛЕМЕНТ МЕТОД

2

Big Picture

↳ ELEMENT
↳ NODES
↳ BCs

FINITE ELEMENT METHOD

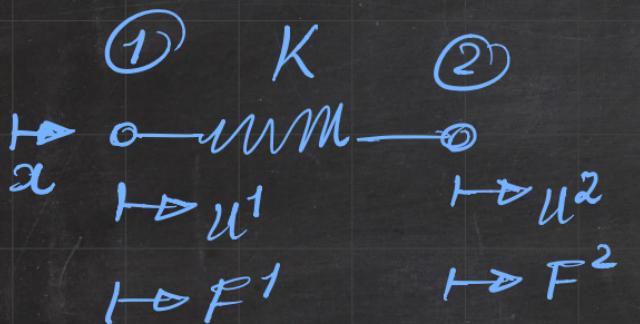
FINITE ELEMENT METHOD

ELEMENT-WISE
Assembly
↓
GLOBAL \Rightarrow

MATRIX

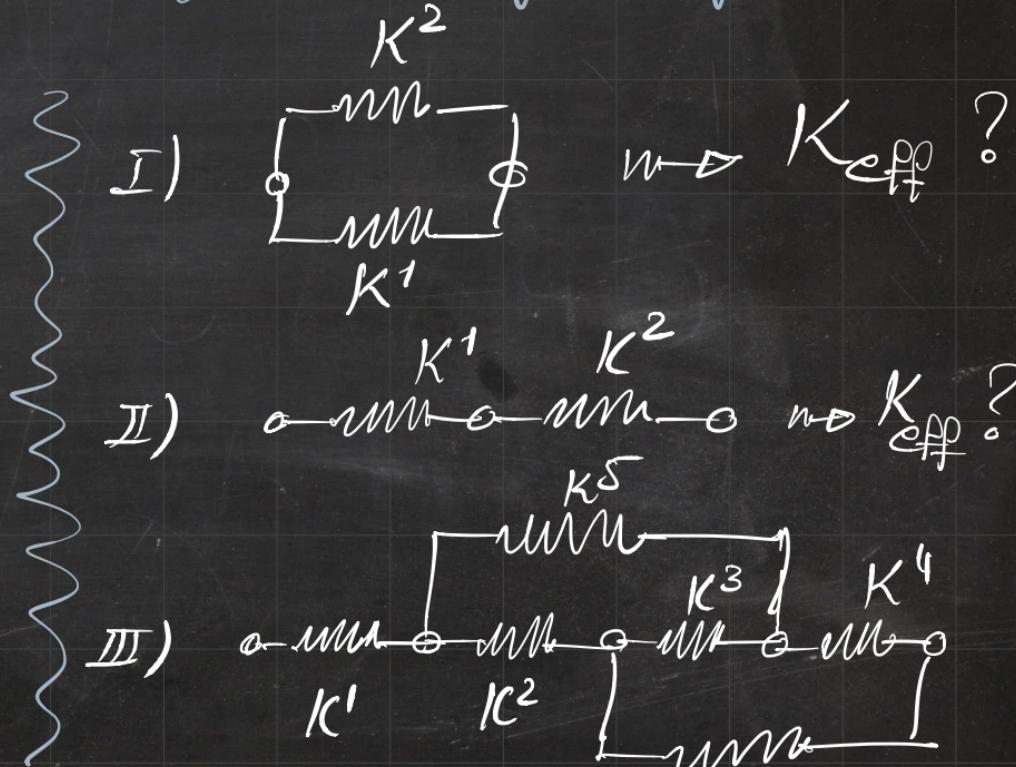
@ ELEMENTS
@ NODES
Stiffness
Displacements
Forces
 $\mathbf{K} \cdot \mathbf{U} = \mathbf{F}$
VECTOR

Understanding key ingredients of FEM using springs

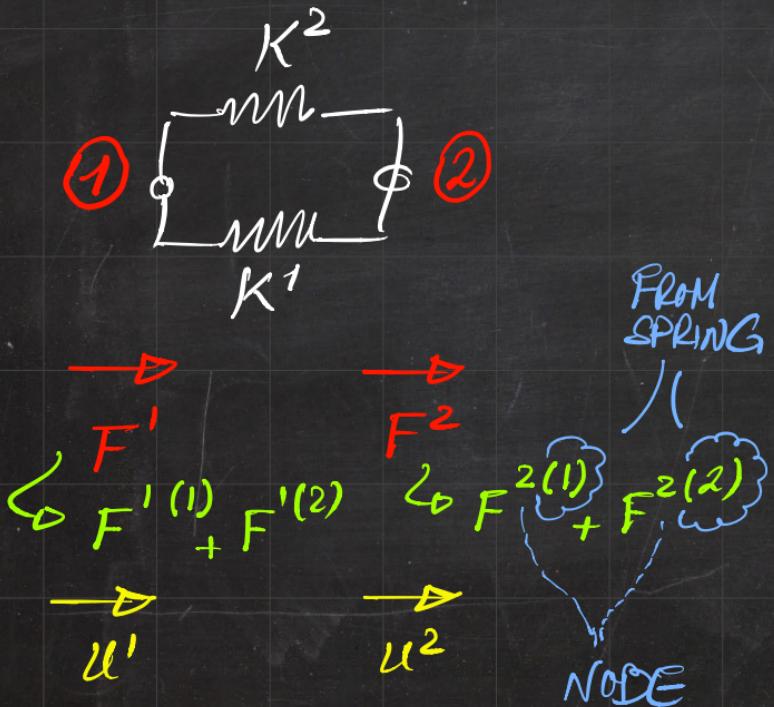


$$\begin{bmatrix} F^1 \\ F^2 \end{bmatrix} = \begin{bmatrix} K & -K \\ -K & K \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix}$$

$$[F] = [K] \cdot [u]$$



Understanding key ingredients of FEM using springs



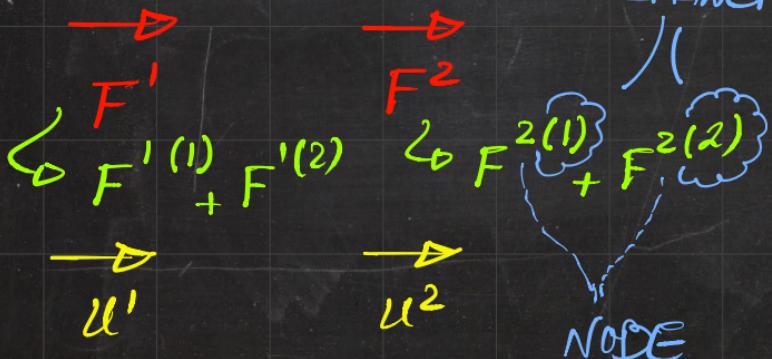
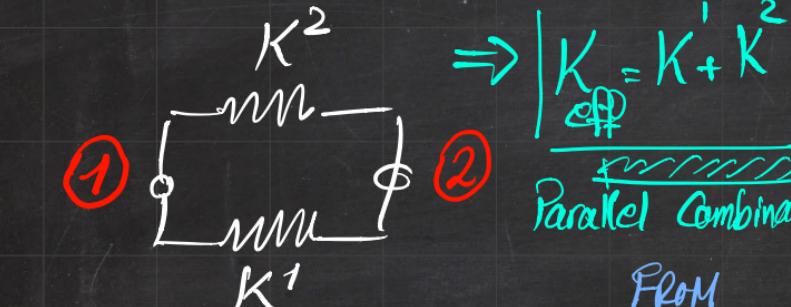
SPRING 1:

$$\begin{bmatrix} F^{1(1)} \\ F^{2(1)} \end{bmatrix} = \begin{bmatrix} K' & -K' \\ -K' & K' \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix}$$

SPRING 2:

$$\begin{bmatrix} F^{1(2)} \\ F^{2(2)} \end{bmatrix} = \begin{bmatrix} K^2 & -K^2 \\ -K^2 & K^2 \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix}$$

Understanding key ingredients of FEM using springs



SPRING 1:

$$\begin{bmatrix} F^{1(1)} \\ F^{2(1)} \end{bmatrix} = \begin{bmatrix} K^1 & -K^1 \\ -K^1 & K^1 \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix}$$

SPRING 2:

$$\begin{bmatrix} F^{1(2)} \\ F^{2(2)} \end{bmatrix} = \begin{bmatrix} K^2 & -K^2 \\ -K^2 & K^2 \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix}$$

Force @ NODE 1

$$\begin{bmatrix} F^1 \\ F^2 \end{bmatrix} = \begin{bmatrix} F^{1(1)} + F^{1(2)} \\ F^{2(1)} + F^{2(2)} \end{bmatrix}$$

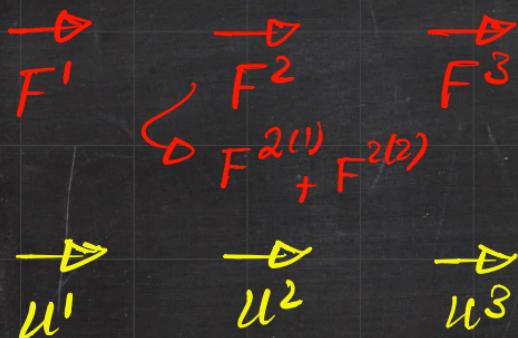
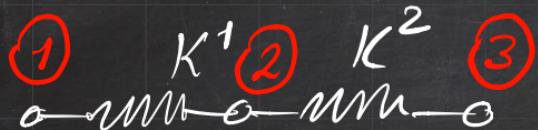
Force @ NODE 2

$$\Rightarrow \begin{bmatrix} F^1 \\ F^2 \end{bmatrix} = \begin{bmatrix} [K^1 + K^2] - [K^1 + K^2] \\ -[K^1 + K^2] [K^1 + K^2] \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix}$$

$$\Rightarrow \mathbf{F} = \mathbf{K} \cdot \mathbf{u}$$

$$\mathbf{K}_{\text{eff}} = \begin{bmatrix} K_{\text{eff}} & -K_{\text{eff}} \\ -K_{\text{eff}} & K_{\text{eff}} \end{bmatrix}$$

Understanding key ingredients of FEM using springs



SPRING 1:

$$\begin{bmatrix} F^1 \\ F^{2(1)} \end{bmatrix} = \underbrace{\begin{bmatrix} K^1 & -K^1 \\ -K^1 & K^1 \end{bmatrix}}_{\text{STIFFNESS MATRIX}} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix}$$

ELEMENT-WISE

STIFFNESS
MATRIX

SPRING 2:

$$\begin{bmatrix} F^{2(2)} \\ F^3 \end{bmatrix} = \underbrace{\begin{bmatrix} K^2 & -K^2 \\ -K^2 & K^2 \end{bmatrix}}_{\text{STIFFNESS MATRIX}} \begin{bmatrix} u^2 \\ u^3 \end{bmatrix}$$

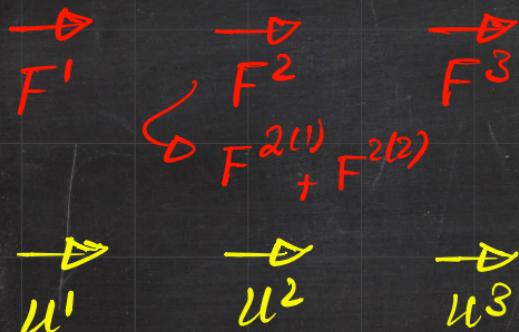
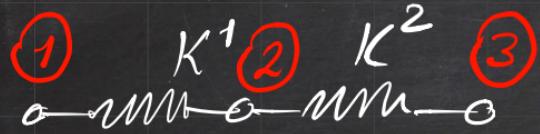
GLOBAL

STIFFNESS
MATRIX

$$\begin{bmatrix} F^1 \\ F^2 \\ F^3 \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \end{bmatrix}$$

$$\begin{bmatrix} u^1 \\ u^2 \\ u^3 \end{bmatrix}$$

Understanding key ingredients of FEM using springs



$$\frac{1}{K_{\text{eff}}} = \frac{1}{K_1} + \frac{1}{K_2}$$

SPRING 1:

$$\begin{bmatrix} F^1 \\ F^{2(1)} \end{bmatrix} = \begin{bmatrix} K^1 & -K^1 \\ -K^1 & K^1 \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix}$$

SPRING 2:

$$\begin{bmatrix} F^{2(2)} \\ F^3 \end{bmatrix} = \begin{bmatrix} K^2 & -K^2 \\ -K^2 & K^2 \end{bmatrix} \begin{bmatrix} u^2 \\ u^3 \end{bmatrix}$$

GLOBAL STIFFNESS

$$\begin{bmatrix} F^1 \\ F^2 \\ F^3 \end{bmatrix} = \begin{bmatrix} K^1 & -K^1 & 0 \\ -K^1 & K^1 + K^2 & -K^2 \\ 0 & -K^2 & K^2 \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \end{bmatrix}$$

Understanding key ingredients of FEM using springs



$$\begin{bmatrix} F^1 \\ F^2 \\ F^3 \end{bmatrix} = \begin{bmatrix} K^1 & -K^1 & 0 \\ -K^1 & K^1 + K^2 & -K^2 \\ 0 & -K^2 & K^2 \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \end{bmatrix}$$

$$\begin{bmatrix} F^1 \\ 0 \\ F^3 \end{bmatrix} = \begin{bmatrix} K^1 & -K^1 & 0 \\ -K^1 & K^1 + K^2 & -K^2 \\ 0 & -K^2 & K^2 \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \end{bmatrix}$$

$$-K^1 u^1 + [K^1 + K^2] u^2 - K^2 u^3 = 0$$

$$\Rightarrow [K^1 + K^2] u^2 = K^1 u^1 + K^2 u^3$$

$$\Rightarrow u^2 = \frac{K^1 u^1 + K^2 u^3}{K^1 + K^2}$$

$$\begin{aligned} F^1 &= K^1 u^1 - K^1 u^2 = K^1 u^1 - K^1 \frac{K^1 u^1 + K^2 u^3}{K^1 + K^2} \\ &= \frac{K^1 K^1 u^1 + K^1 K^2 u^3 - K^1 K^1 u^1 - K^1 K^2 u^3}{K^1 + K^2} \end{aligned}$$

Understanding key ingredients of FEM using springs



$$-K^1 u^1 + [K^1 + K^2] u^2 - K^2 u^3 = 0$$

$$\Rightarrow u^2 = \frac{K^1 u^1 + K^2 u^3}{K^1 + K^2}$$

$$F^1 = K^1 u^1 - K^1 u^2 = K^1 u^1 - K^1 \frac{K^1 u^1 + K^2 u^3}{K^1 + K^2}$$

$$= \frac{K^1 K^1 u^1 + K^2 K^1 u^1 - K^1 K^1 u^1 - K^1 K^2 u^3}{K^1 + K^2}$$

$$\Rightarrow F^1 = \frac{K^1 K^2}{K^1 + K^2} [u^1 - u^3]$$

$$\begin{bmatrix} F^1 \\ 0 \\ F^3 \end{bmatrix} = \begin{bmatrix} K^1 & -K^1 & 0 \\ -K^1 & K^1 + K^2 & -K^2 \\ 0 & -K^2 & K^2 \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \end{bmatrix}$$

Understanding key ingredients of FEM using springs



$$\begin{bmatrix} F^1 \\ F^2 \\ F^3 \end{bmatrix} = \begin{bmatrix} K^1 & -K^1 & 0 \\ -K^1 & K^1 + K^2 & -K^2 \\ 0 & -K^2 & K^2 \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \end{bmatrix}$$

$$\begin{bmatrix} F^1 \\ 0 \\ F^3 \end{bmatrix} = \begin{bmatrix} K^1 & -K^1 & 0 \\ -K^1 & K^1 + K^2 & -K^2 \\ 0 & -K^2 & K^2 \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \end{bmatrix}$$

$$-K^1 u^1 + [K^1 + K^2] u^2 - K^2 u^3 = 0 \Rightarrow u^2 = \frac{K^1 u^1 + K^2 u^3}{K^1 + K^2}$$

$$\Rightarrow F^1 = \frac{K^1 K^2}{K^1 + K^2} [u^1 - u^3]$$

$$F^3 = -K^2 u^2 + K^2 u^3$$

$$= -K^2 \frac{K^1 u^1 + K^2 u^3}{K^1 + K^2} + K^2 u^3$$

$$= \frac{-K^2 K^1 u^1 - K^2 K^2 u^3 + K^1 K^2 u^3 + K^2 K^2 u^3}{K^1 + K^2}$$

Understanding key ingredients of FEM using springs



$$\begin{bmatrix} F' \\ F^2 \\ F^3 \end{bmatrix} = \begin{bmatrix} K' & -K' & 0 \\ -K' & K + K^2 & -K^2 \\ 0 & -K^2 & K^2 \end{bmatrix} \begin{bmatrix} u' \\ u^2 \\ u^3 \end{bmatrix}$$

$$\begin{bmatrix} F' \\ 0 \\ F^3 \end{bmatrix} = \begin{bmatrix} K' & -K' & 0 \\ -K' & K + K^2 & -K^2 \\ 0 & -K^2 & K^2 \end{bmatrix} \begin{bmatrix} u' \\ u^2 \\ u^3 \end{bmatrix}$$

$$-K'u' + [K' + K^2]u^2 - K^2u^3 = 0 \Rightarrow u^2 = \frac{K'u' + K^2u^3}{K' + K^2}$$

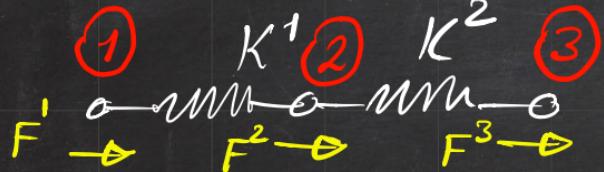
$$\Rightarrow F' = \frac{K'K^2}{K' + K^2} [u' - u^3]$$

$$F^3 = -K^2u^2 + K^2u^3$$

$$= \frac{-K^2K'u' - K^2K^2u^3 + K'K^2u^3 + K^2K^2u^3}{K' + K^2}$$

$$= \frac{K'K^2}{K' + K^2} [u^3 - u']$$

Understanding key ingredients of FEM using springs



$$\begin{bmatrix} F^1 \\ F^2 \\ F^3 \end{bmatrix} = \begin{bmatrix} K' & -K' & 0 \\ -K' & K'+K^2-K^2 & 0 \\ 0 & -K^2 & K \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \end{bmatrix}$$

$$\begin{bmatrix} F^1 \\ 0 \\ F^3 \end{bmatrix} = \begin{bmatrix} K' & -K' & 0 \\ -K' & K'+K^2-K^2 & 0 \\ 0 & -K^2 & K \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \end{bmatrix}$$



$$-K'u^1 + [K'+K^2]u^2 - K^2u^3 = 0 \Rightarrow u^2 = \frac{K'u^1 + K^2u^3}{K'+K^2}$$

✓ $\Rightarrow F^1 = \frac{K'K^2}{K'+K^2} [u^1 - u^3]$

$0 \Leftarrow +$

$\Rightarrow F^3 = \frac{K'K^2}{K'+K^2} [u^3 - u^1]$

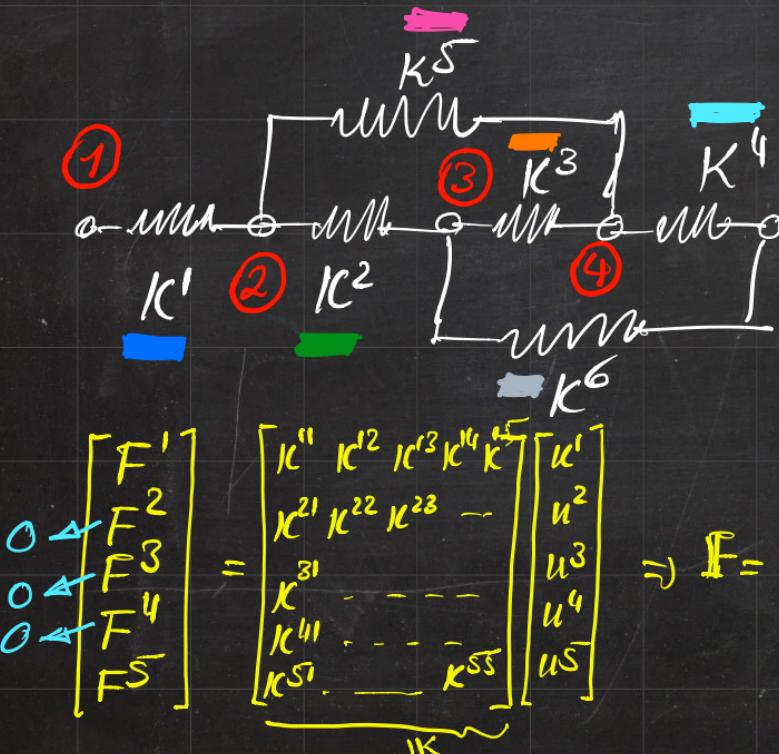
K_{eff}

$$F^1 + F^3 = 0$$

$$\begin{bmatrix} F^1 \\ F^3 \end{bmatrix} = \begin{bmatrix} K_{eff} & -K_{eff} \\ -K_{eff} & K_{eff} \end{bmatrix} \begin{bmatrix} u^1 \\ u^3 \end{bmatrix}$$

sym, $DET=0$

Understanding key ingredients of FEM using springs



$$\begin{bmatrix} F^1 \\ F^2 \\ F^3 \\ F^4 \\ F^5 \end{bmatrix} = \underbrace{\begin{bmatrix} K^{11} & K^{12} & K^{13} & K^{14} & K^{15} \\ K^{21} & K^{22} & K^{23} & - & - \\ K^{31} & - & - & - & - \\ K^{41} & - & - & - & - \\ K^{51} & - & - & - & K^{55} \end{bmatrix}}_{\mathbf{K}} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \\ u^4 \\ u^5 \end{bmatrix}$$

$$\Rightarrow \mathbf{F} = \mathbf{K} \cdot \mathbf{u}$$

SYM.

&
DET=0

(1) (2) (3) (4) (5)

$$\mathbf{K} = (3)$$

(4)
(5)

(1) (2) (3) (4) (5)

$$\begin{bmatrix} K^1 & -K^1 & 0 & 0 & 0 \\ -K^1 & K^1 + K^5 + K^6 & -K^2 & -K^5 & 0 \\ 0 & -K^2 + K^3 + K^6 & K^2 - K^3 - K^6 & -K^6 & -K^6 \\ 0 & -K^5 - K^3 + K^4 + K^5 & -K^3 - K^4 & K^3 + K^4 - K^6 & -K^4 \\ 0 & 0 & -K^6 - K^4 + K^4 + K^6 & K^4 + K^6 & -K^6 \end{bmatrix}_{5 \times 5}$$

Understanding key ingredients of FEM using springs

$$K^1 = K^2 = K^3 = K^4 = K^5 = K^6 = 1$$

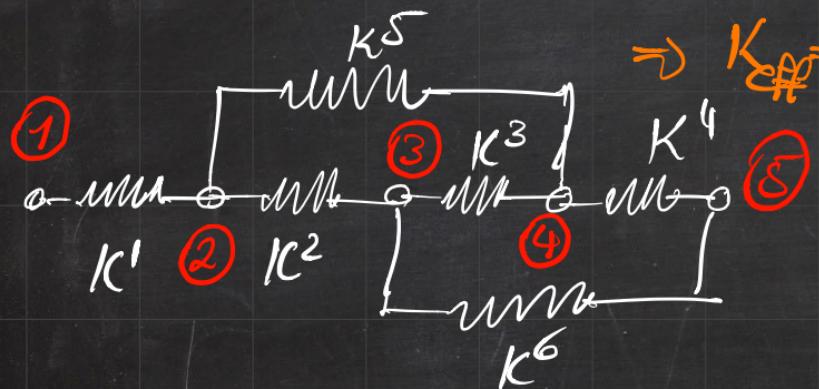


$$\begin{bmatrix} F^1 \\ F^2 \\ F^3 \\ F^4 \\ F^5 \end{bmatrix} = \underbrace{\begin{bmatrix} K^{11} & K^{12} & K^{13} & K^{14} & K^{15} \\ K^{21} & K^{22} & K^{23} & - & - \\ K^{31} & - & - & - & - \\ K^{41} & - & - & - & - \\ K^{51} & - & - & - & K^{55} \end{bmatrix}}_{\mathbf{K}} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \\ u^4 \\ u^5 \end{bmatrix} \Rightarrow \mathbf{F} = \mathbf{K} \cdot \mathbf{u}$$

$$\mathbf{K} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ 0 & -1 & 3 & -1 & -1 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}_{5 \times 5}$$

Understanding key ingredients of FEM using springs

$$K^1 = K^2 = K^3 = K^4 = K^5 = K^6 = 1$$



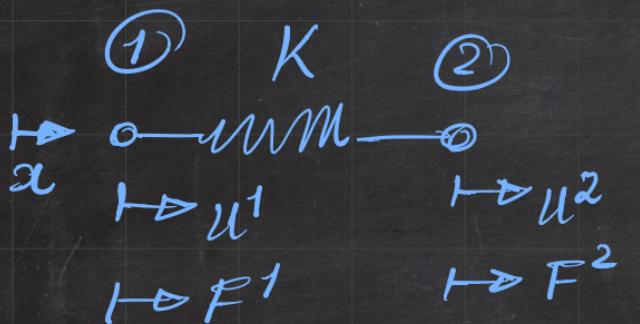
$$\begin{bmatrix} F^1 \\ F^5 \end{bmatrix} = \begin{bmatrix} 1.5 & -1.5 \\ -1.5 & 1.5 \end{bmatrix} \begin{bmatrix} u^1 \\ u^5 \end{bmatrix}$$

$$\Rightarrow F^5 = 1.5 [u^5 - u^1]$$

$$K = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ 0 & -1 & 3 & -1 & -1 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$

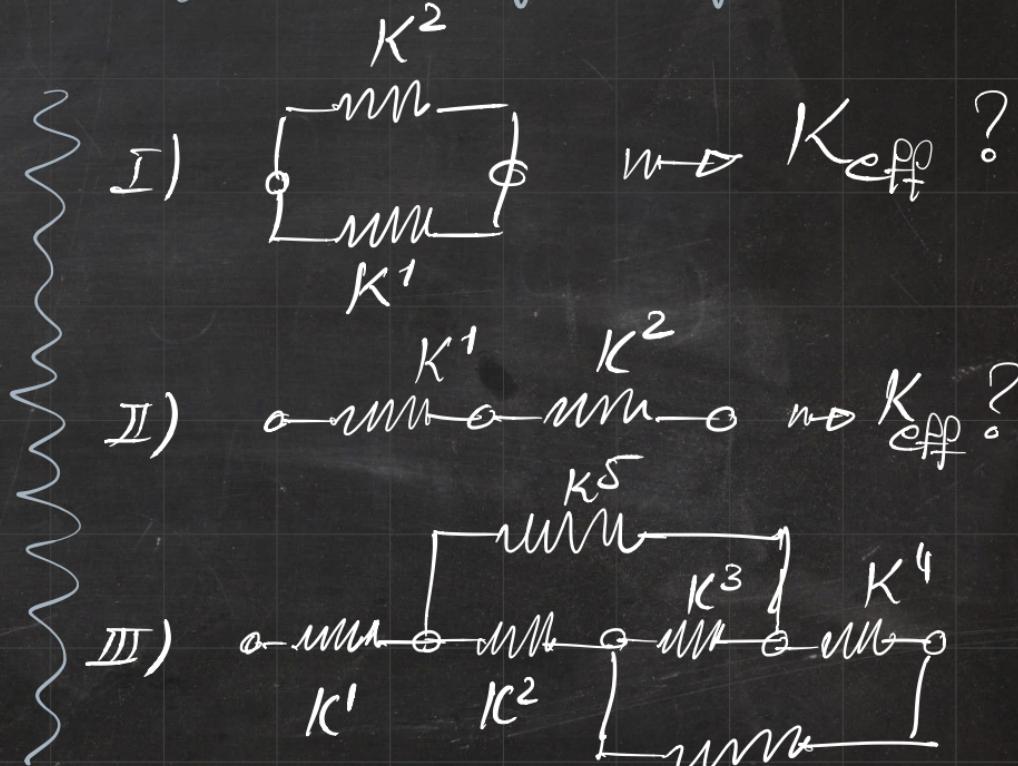
$$\Rightarrow F = Ku$$

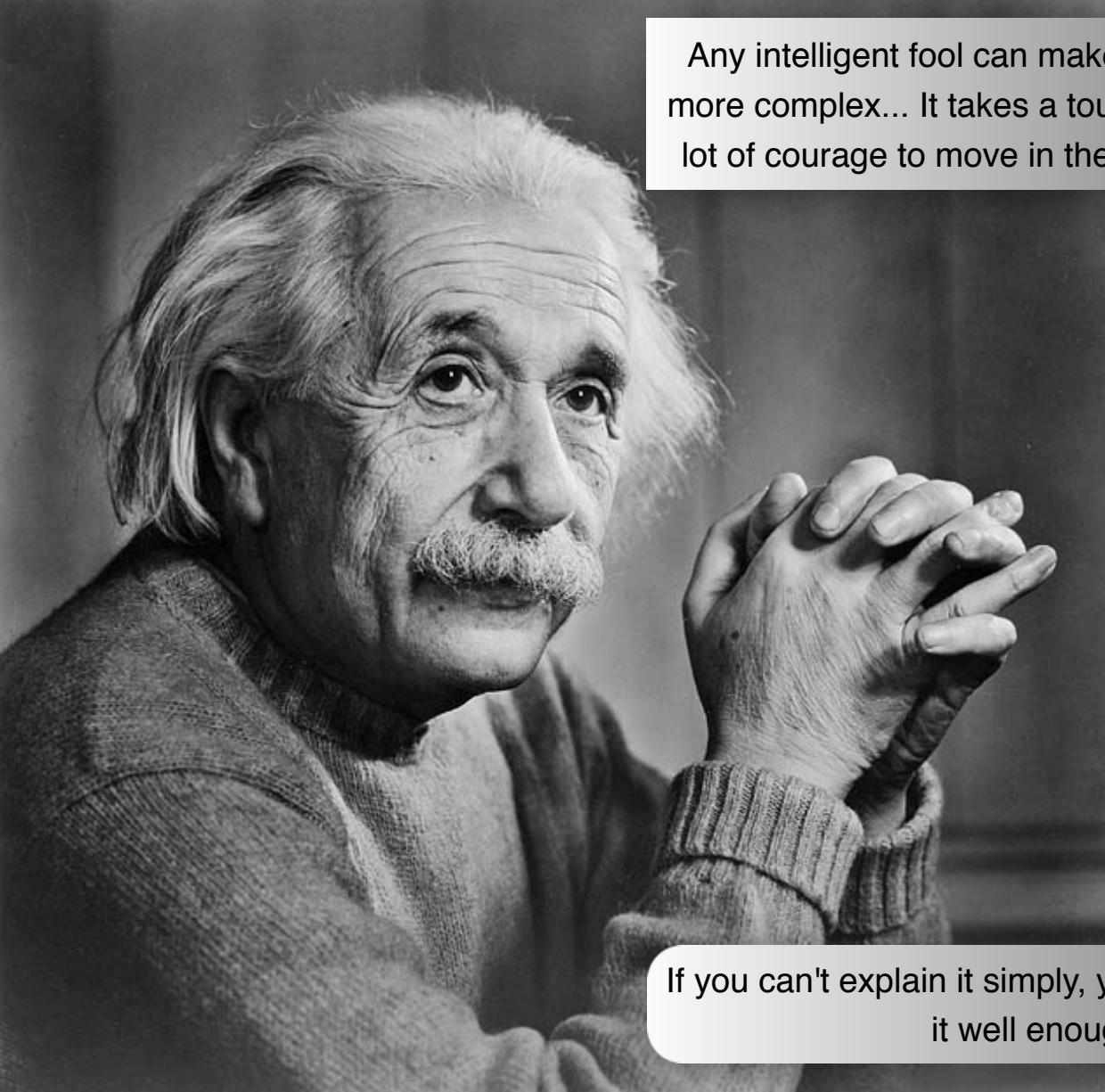
Understanding key ingredients of FEM using springs



$$\begin{bmatrix} F^1 \\ F^2 \end{bmatrix} = \begin{bmatrix} K & -K \\ -K & K \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix}$$

$$[F] = [K] \cdot [u]$$



A black and white photograph of Albert Einstein. He is shown from the chest up, wearing a light-colored, ribbed sweater. His characteristic wild, white hair is visible. He has a thoughtful expression, looking slightly to the right of the camera. His hands are clasped together in front of him, resting near his chin. The background is dark and out of focus.

Any intelligent fool can make things bigger and more complex... It takes a touch of genius - and a lot of courage to move in the opposite direction.

If you can't explain it simply, you don't understand it well enough.