FINITE ELEMENT METHOD

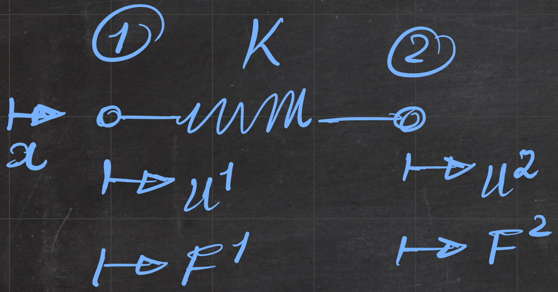
6

Ali Javili

Bilkent University

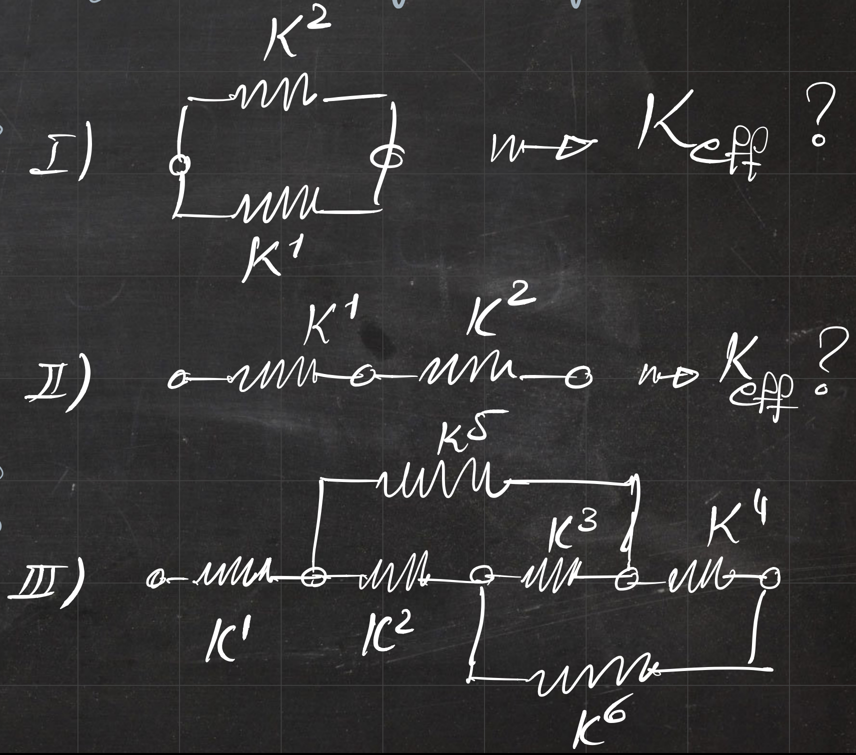
FINITE ELEMENT METHOD

Understanding key ingredients of FEM using springs:

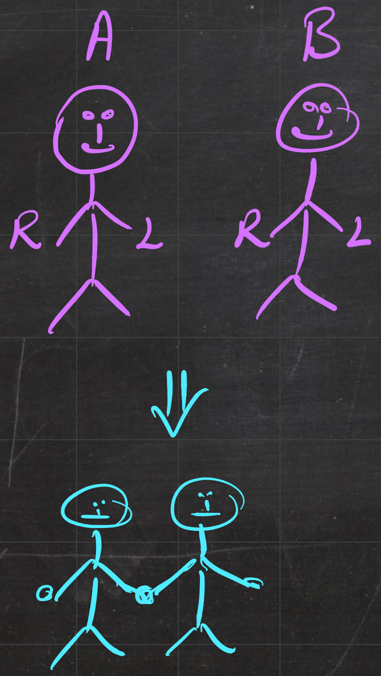


$$\begin{bmatrix} F^1 \\ F^2 \end{bmatrix} = \begin{bmatrix} K & -K \\ -K & K \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix}$$

$$\mathbb{F} = \mathbb{K} \cdot \mathbb{U}$$



TOWARDS AN ALGORITHMIC APPROACH TO ASSEMBLY:

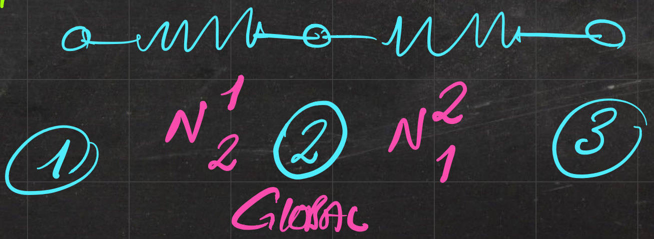


Superscript: GLOBAL

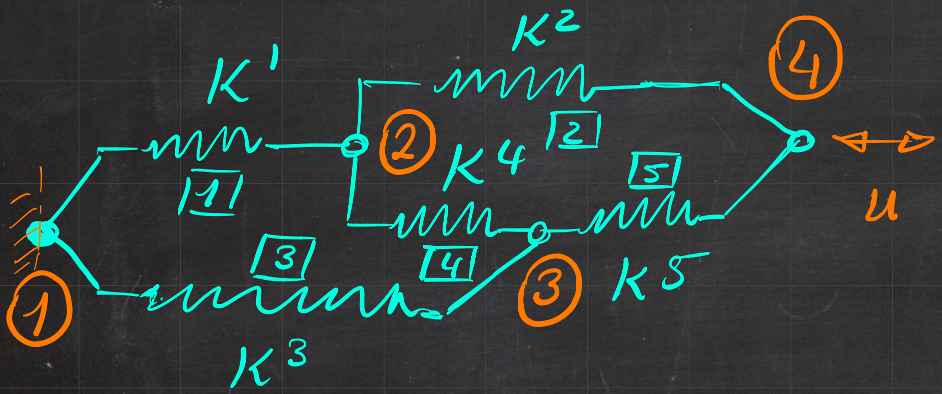
subscript: LOCAL

GLOBAL NODE

$$N^2 = N_2^1 = N_1^2$$



TOWARDS AN ALGORITHMIC APPROACH TO ASSEMBLY:



ELEMENT 5

$$\mathbb{K}^5 = \begin{bmatrix} K^5 & -K^5 \\ -K^5 & K^5 \end{bmatrix}$$

ELEMENT 1

$$\mathbb{K}^1 = \begin{bmatrix} K^1 & -K^1 \\ -K^1 & K^1 \end{bmatrix}$$

BOLD

ELEMENT 2

$$\mathbb{K}^2 = \begin{bmatrix} K^2 & -K^2 \\ -K^2 & K^2 \end{bmatrix}$$

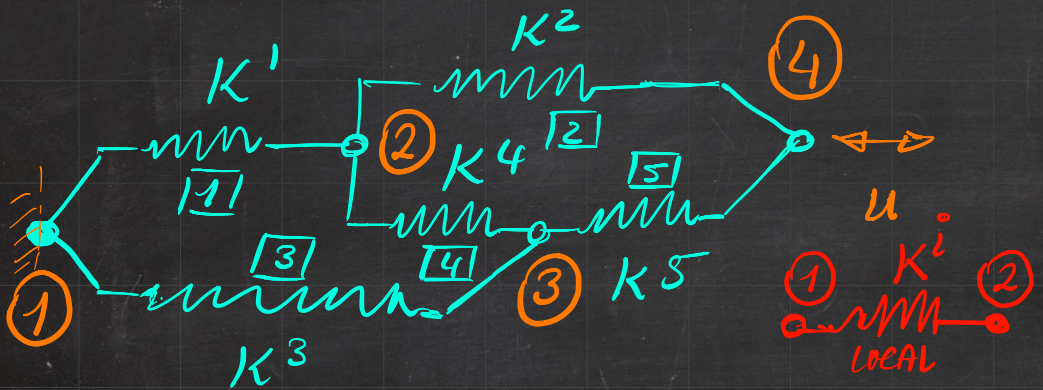
ELEMENT 3

$$\mathbb{K}^3 = \begin{bmatrix} K^3 & -K^3 \\ -K^3 & K^3 \end{bmatrix}$$

ELEMENT 4

$$\mathbb{K}^4 = \begin{bmatrix} K^4 & -K^4 \\ -K^4 & K^4 \end{bmatrix}$$

TOWARDS AN ALGORITHMIC APPROACH TO ASSEMBLY:



$$\mathbb{K}^5 = \begin{array}{cc} 3 & 4 \\ \left[\begin{array}{cc} K^5 & -K^5 \\ -K^5 & K^5 \end{array}\right] & \begin{array}{c} 3 \\ 4 \end{array} \end{array}$$

$$\mathbb{K}^1 = \begin{array}{cc} 1 \quad 2 & \\ \left[\begin{array}{cc} K^1 & -K^1 \\ -K^1 & K^1 \end{array}\right] & \begin{array}{c} 1 \\ 2 \end{array} \end{array} \quad \mathbb{K}^2 = \begin{array}{cc} 2 \quad 4 & \\ \left[\begin{array}{cc} K^2 & -K^2 \\ -K^2 & K^2 \end{array}\right] & \begin{array}{c} 2 \\ 4 \end{array} \end{array} \quad \mathbb{K}^3 = \begin{array}{cc} 1 \quad 3 & \\ \left[\begin{array}{cc} K^3 & -K^3 \\ -K^3 & K^3 \end{array}\right] & \begin{array}{c} 1 \\ 3 \end{array} \end{array} \quad \mathbb{K}^4 = \begin{array}{cc} 2 \quad 3 & \\ \left[\begin{array}{cc} K^4 & -K^4 \\ -K^4 & K^4 \end{array}\right] & \begin{array}{c} 2 \\ 3 \end{array} \end{array}$$

$$\mathbb{K}^4 = \begin{array}{c|c} & \begin{array}{cc} 2 & 3 \end{array} \\ \hline \begin{array}{c} 2 \\ 3 \end{array} & \begin{bmatrix} K^4 & -K^4 \\ -K^4 & K^4 \end{bmatrix} \end{array}$$

$$\mathbb{K}^5 = \begin{array}{c|c} & \begin{array}{cc} 3 & 4 \end{array} \\ \hline \begin{array}{c} 3 \\ 4 \end{array} & \begin{bmatrix} K^5 & -K^5 \\ -K^5 & K^5 \end{bmatrix} \end{array}$$

GLOBAL

$$\mathbb{K} = \begin{array}{c|c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \hline \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} K^1 + K^3 & -K^1 & -K^3 & 0 \\ -K^1 & K^1 + K^2 + K^4 & -K^4 & -K^2 \\ -K^3 & -K^4 & K^3 + K^4 + K^5 & -K^5 \\ 0 & -K^2 & -K^5 & K^2 + K^5 \end{bmatrix} \end{array}$$

$$\mathbb{K}^1 = \begin{array}{c|c} & \begin{array}{cc} 1 & 2 \end{array} \\ \hline \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} K^1 & -K^1 \\ -K^1 & K^1 \end{bmatrix} \end{array}$$

$$\mathbb{K}^2 = \begin{array}{c|c} & \begin{array}{cc} 2 & 4 \end{array} \\ \hline \begin{array}{c} 2 \\ 4 \end{array} & \begin{bmatrix} K^2 & -K^2 \\ -K^2 & K^2 \end{bmatrix} \end{array}$$

$$\mathbb{K}^3 = \begin{array}{c|c} & \begin{array}{cc} 1 & 3 \end{array} \\ \hline \begin{array}{c} 1 \\ 3 \end{array} & \begin{bmatrix} K^3 & -K^3 \\ -K^3 & K^3 \end{bmatrix} \end{array}$$

$\mathrm{DET}\ \mathbb{K}^{\mathrm{GLOBAL}} = 0 \qquad \mathbb{K}^{\mathrm{GLOBAL}}$: SYM

$$\begin{bmatrix} F^1 \\ F^2 \\ F^3 \\ F^4 \end{bmatrix} = \begin{bmatrix} K^{11} & K^{12} & K^{13} & K^{14} \\ K^{21} & K^{22} & K^{23} & K^{24} \\ K^{31} & K^{32} & K^{33} & K^{34} \\ K^{41} & K^{42} & K^{43} & K^{44} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \\ u^4 \end{bmatrix}$$

GLOBAL GLOBAL GLOBAL

$$\mathbb{F} = \mathbb{K} \cdot \mathbb{U}$$

NoN x 1 NoN x NoN NoN x 1

[PD x NoN] x 1

NoN ≡ NoN x PD

1D Problem Dimension

⇒ 4 EQ. & 4 Unknowns ← BCs ?

NEUMANN BCs. ⇐ FORCE BASED

DISPLACEMENT BASED ⇒ DIRICHLET BCs.

$$\begin{bmatrix} F^1 \\ F^2 \\ F^3 \\ F^4 \end{bmatrix} = \begin{bmatrix} K^{11} & K^{12} & K^{13} & K^{14} \\ K^{21} & K^{22} & K^{23} & K^{24} \\ K^{31} & K^{32} & K^{33} & K^{34} \\ K^{41} & K^{42} & K^{43} & K^{44} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \\ u^4 \end{bmatrix}$$

$u^1 = 0$

$u^4 = u$

	Homogeneous	Non Homogeneous
DIRICHLET ↳ Displacement	$= 0$	$\neq 0$
NEUMANN ↳ Force	$= 0$	$\neq 0$

$$= \begin{bmatrix} K^{11} \\ K^{21} \\ K^{31} \\ K^{41} \end{bmatrix} u^1 + \begin{bmatrix} K^{12} \\ K^{22} \\ K^{32} \\ K^{42} \end{bmatrix} u^2 + \begin{bmatrix} K^{13} \\ K^{23} \\ K^{33} \\ K^{43} \end{bmatrix} u^3 + \begin{bmatrix} K^{14} \\ K^{24} \\ K^{34} \\ K^{44} \end{bmatrix} u^4$$

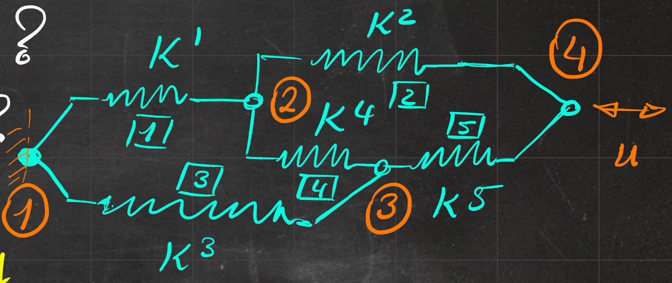
4 EQN. & 4 Unknowns

NEUMANN

NEUMANN

$$\begin{bmatrix} \cancel{F^1} \to ? \\ \cancel{F^2} \to 0 \\ \cancel{F^3} \to 0 \\ \cancel{F^4} \to ? \end{bmatrix} = \begin{bmatrix} K^{11} & K^{12} & K^{13} & K^{14} \\ K^{21} & K^{22} & K^{23} & K^{24} \\ K^{31} & K^{32} & K^{33} & K^{34} \\ K^{41} & K^{42} & K^{43} & K^{44} \end{bmatrix} \begin{bmatrix} \cancel{u^1} \to 0 \\ \cancel{u^2} \to ? \\ \cancel{u^3} \to ? \\ \cancel{u^4} \to u \end{bmatrix}$$

DIRICHLET (Prescribed Displacement)



DIRICHLET (Prescribed Displ.)

$$\overset{\checkmark}{\begin{bmatrix} \mathbb{b} \end{bmatrix}} = \begin{bmatrix} \mathbb{A} \end{bmatrix} \overset{?}{\begin{bmatrix} \mathbb{X} \end{bmatrix}}$$

$$\mathbb{A} \cdot \mathbb{X} = \mathbb{b} \Rightarrow \mathbb{X} = \mathbb{A}^{-1} \cdot \mathbb{b}$$

4 EQN. & 4 Unknowns

$$\begin{bmatrix} \cancel{F^1} \\ \cancel{F^2} \\ \cancel{F^3} \\ \cancel{F^4} \end{bmatrix} = \begin{bmatrix} K^{11} & K^{12} & K^{13} & K^{14} \\ K^{21} & K^{22} & K^{23} & K^{24} \\ K^{31} & K^{32} & K^{33} & K^{34} \\ K^{41} & K^{42} & K^{43} & K^{44} \end{bmatrix} \begin{bmatrix} \cancel{u^1} \\ \cancel{u^2} \\ \cancel{u^3} \\ \cancel{u^4} \end{bmatrix}$$

$\cancel{F^1} \to ?$

$\cancel{F^2} \to 0$ NEUMANN

$\cancel{F^3} \to 0$ NEUMANN

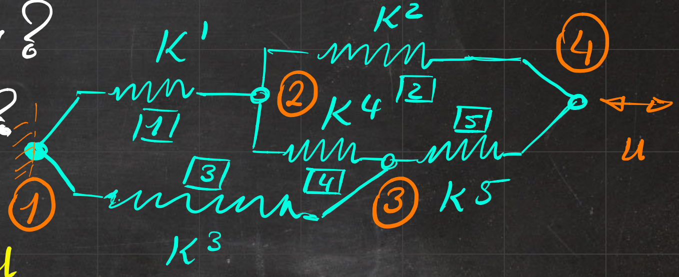
$\cancel{F^4} \to ?$

$\cancel{u^1} \to 0$ ← DIRICHLET (Prescribed Displacement)

$\cancel{u^2} \to ?$

$\cancel{u^3} \to ?$

$\cancel{u^4} \to u$ ← DIRICHLET (Prescribed Displ.)



$$\begin{bmatrix} F^P \\ \hline F^u \end{bmatrix} = \left[\begin{array}{c|c} K^{Pu} & K^{PP} \\ \hline K^{uu} & K^{uP} \end{array}\right] \begin{bmatrix} u^u \\ \hline u^P \end{bmatrix}$$

FREE NODES (NEUMANN) ⇒ DoF ← DEGREES OF FREEDOM

CONSTRAINED NODES (DIRICHLET) ⇒ DoC ← DEGREES OF CONSTRAINT

4 EQN. & 4 Unknowns

$$
\begin{bmatrix} \cancel{F^1} \;?\\ \cancel{F^2} \;0 \\ \cancel{F^3} \;0 \\ \cancel{F^4} \;? \end{bmatrix}
=
\begin{bmatrix} K^{11} & K^{12} & K^{13} & K^{14} \\ K^{21} & K^{22} & K^{23} & K^{24} \\ K^{31} & K^{32} & K^{33} & K^{34} \\ K^{41} & K^{42} & K^{43} & K^{44} \end{bmatrix}
\begin{bmatrix} \cancel{u^1} \;0 \\ \cancel{u^2} \;? \\ \cancel{u^3} \;? \\ \cancel{u^4} \;u \end{bmatrix}
$$

NEUMANN

NEUMANN

$$[F^P] = [K^{Pu}][u^u] + [K^{PP}][u^P]$$

$$\Downarrow$$

$$\underbrace{[K^{Pu}]}_{A}\underbrace{[u^u]}_{x} = \underbrace{[F^P] - [K^{PP}][u^P]}_{b}$$

$$[x] = [A]^{-1}[b] \Leftarrow A \cdot x = b$$

$$
\begin{bmatrix} F^P \\ \hline F^u \end{bmatrix}
=
\left[\begin{array}{c|c} K^{Pu} & K^{PP} \\ \hline K^{uu} & K^{uP} \end{array}\right]
\begin{bmatrix} u^u \\ \hline u^P \end{bmatrix}
$$

$$
\underset{\text{FORCE}}{\begin{bmatrix} \text{DoF} \\ \hline \text{DoC} \end{bmatrix}}
=
\underset{\text{STIFFNESS}}{\left[\begin{array}{c|c} \text{DoF}\times\text{DoF} & \text{DoF}\times\text{DoC} \\ \hline \text{DoC}\times\text{DoF} & \text{DoC}\times\text{DoC} \end{array}\right]}
\underset{\text{DISPLACEMENT}}{\begin{bmatrix} \text{DoF} \\ \hline \text{DoC} \end{bmatrix}}
$$

4 EQN. & 4 Unknowns

$$\begin{bmatrix} \cancel{F^1} \\ \cancel{F^2} \\ \cancel{F^3} \\ \cancel{F^4} \end{bmatrix} = \begin{bmatrix} K^{11} & K^{12} & K^{13} & K^{14} \\ K^{21} & K^{22} & K^{23} & K^{24} \\ K^{31} & K^{32} & K^{33} & K^{34} \\ K^{41} & K^{42} & K^{43} & K^{44} \end{bmatrix} \begin{bmatrix} \cancel{u^1} \\ \cancel{u^2} \\ \cancel{u^3} \\ \cancel{u^4} \end{bmatrix}$$

F^1: ?, F^2: 0 (NEUMANN), F^3: 0 (NEUMANN), F^4: ?

u^1: 0, u^2: ?, u^3: ?, u^4: u

$$\begin{bmatrix} F^p \\ \hline F^u \end{bmatrix} = \begin{bmatrix} K^{pu} & K^{pp} \\ \hline K^{uu} & K^{up} \end{bmatrix} \begin{bmatrix} u^u \\ \hline u^p \end{bmatrix}$$

$$[F^p] = [K^{pu}][u^u] + [K^{pp}][u^p]$$

$$\Downarrow$$

$$\underbrace{[K^{pu}]}_{\text{REDUCED STIFFNESS}}[u^u] = [F^p] - [K^{pp}][u^p]$$

$$\Rightarrow \underbrace{[u^u] = [K^{pu}]^{-1} \cdot \{[F^p] - [K^{pp}][u^p]\}}_{\text{REDUCED SYSTEM}}$$

$$\underset{\text{DOF}\times\text{DOF}}{\mathbb{A}} \cdot \mathbb{x} = \mathbb{b}$$

4 EQN. & 4 Unknowns

$$\begin{bmatrix} \cancel{F^1} \\ \cancel{F^2} \\ \cancel{F^3} \\ \cancel{F^4} \end{bmatrix} \begin{matrix} ? \\ 0 \\ 0 \\ ? \end{matrix} = \begin{bmatrix} K^{11} & K^{12} & K^{13} & K^{14} \\ K^{21} & K^{22} & K^{23} & K^{24} \\ K^{31} & K^{32} & K^{33} & K^{34} \\ K^{41} & K^{42} & K^{43} & K^{44} \end{bmatrix} \begin{bmatrix} \cancel{u^1} \\ \cancel{u^2} \\ \cancel{u^3} \\ \cancel{u^4} \end{bmatrix} \begin{matrix} 0 \\ ? \\ ? \\ u \end{matrix}$$

NEUMANN

NEUMANN

$$[F^P] = [K^{Pu}][u^u] + [K^{PP}][u^P]$$

$$\Downarrow$$

$$\underbrace{[K^{Pu}][u^u] = [F^P] - [K^{PP}][u^P]}_{\checkmark}$$

REDUCED SYSTEM

$$\Rightarrow [u^u] = [K^{Pu}]^{-1} \cdot \{[F^P] - [K^{PP}][u^P]\}$$

$$\begin{bmatrix} F^P \\ \hline F^u \end{bmatrix} = \left[\begin{array}{c|c} K^{Pu} & K^{PP} \\ \hline K^{uu} & K^{uP} \end{array}\right] \begin{bmatrix} u^u \\ \hline u^P \end{bmatrix}$$

$$\Rightarrow [F^u] = [K^{uu}][u^u] + [K^{uP}][u^P]$$

STATIC CONDENSATION ✓

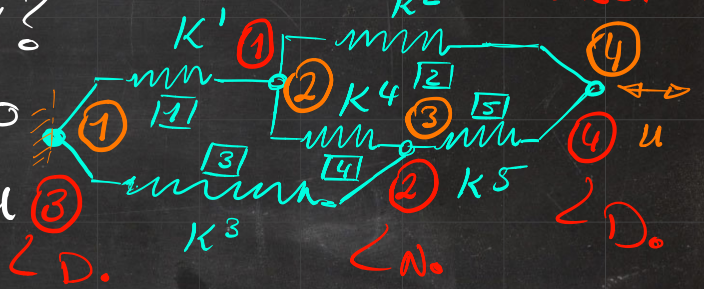
4 EQN. & 4 Unknowns

$$\begin{bmatrix} \cancel{F^1} = 0 \\ \cancel{F^2} = 0 \\ \cancel{F^3} = ? \\ \cancel{F^4} = ? \end{bmatrix} = \left[\begin{array}{cc|cc} K^{11} & K^{12} & K^{13} & K^{14} \\ K^{21} & K^{22} & K^{23} & K^{24} \\ \hline K^{31} & K^{32} & K^{33} & K^{34} \\ K^{41} & K^{42} & K^{43} & K^{44} \end{array}\right] \begin{bmatrix} \cancel{u^1} = ? \\ \cancel{u^2} = ? \\ \cancel{u^3} = 0 \\ \cancel{u^4} = u \end{bmatrix}$$

$$\Downarrow$$

$$\left[\begin{array}{c} F^P \\ \hline F^u \end{array}\right] = \left[\begin{array}{c|c} K^{Pu} & K^{PP} \\ \hline K^{uu} & K^{uP} \end{array}\right] \left[\begin{array}{c} u^u \\ \hline u^P \end{array}\right]$$

NUMBERING CAREFULLY FROM THE ONSET



NL. DEGREE ENL

{ N.	(X,Y)	1	→ 3
	(X,Y)	2	→ 1
{ D.	(X,Y)	3	→ 2
	(X,Y)	4	→ 4

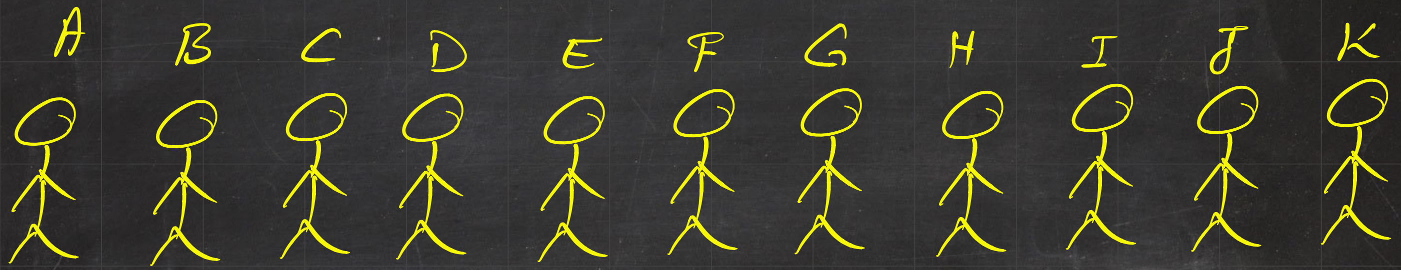
Loop over nodes

ASSIGN DEGREES TO NODES

end

EXTENDED NODE LIST

→ THE NAMING (NUMBERING) IS ARBITRARY!



→ Every Person Can Say One Word ← Programming : One Loop!

How many people?

How many right-handed? → Assign Degrees

How many left-handed? → Assign Degrees

EXTENDED NODE LIST → THE NAMING (NUMBERING) IS ARBITRARY!

A	B	C	D	E	F	G	H	I	J	K
1	-1	2	-2	-3	3	4	5	-4	6	-5
1	7	2	8	9	3	4	5	10	6	11

GLOBAL DEGREE

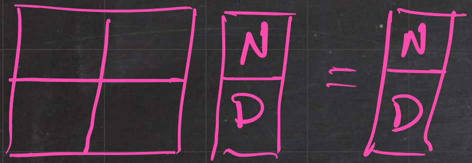
→ 6 ← # DoFs

→ 5 ← # DoCs

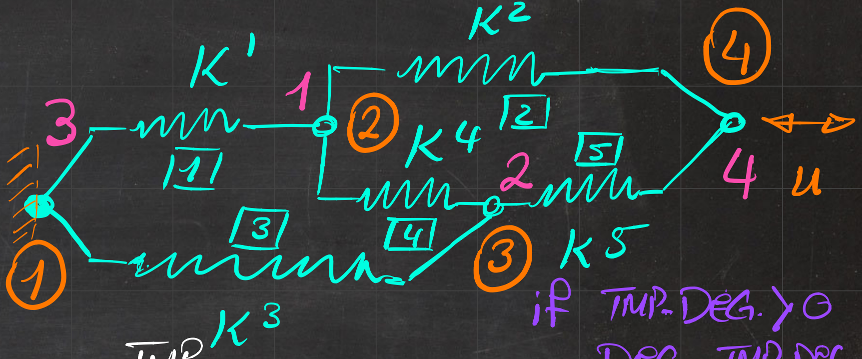
LEFT HANDED PEOPLE

RIGHT HANDED PEOPLE

EXTENDED NODE LIST



$[K][u]=[F]$



NL	COOR.	BC INFO	TMP DEGREE	DEGREE
1	0 0 0	D	-1	3
2	0 0 0	N	1	1
3	0 0 0	N	2	2
4	0 0 0	D	-2	4

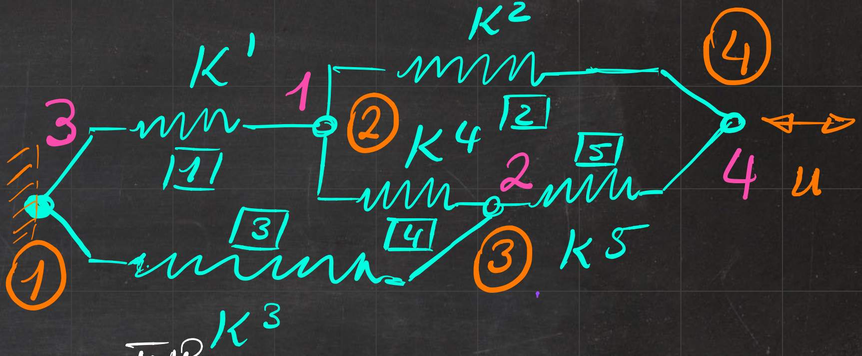
```
if TMP_DEG > 0
    DEG = TMP_DEG
else
    DEG = DoFs + |TMP_D|
```

```
DoFs=0; DoCs=0;
IF (D)
    DoCs = DoCs-1;
IF (N)
    DoFs = DoFs+1;
```

EXTENDED NODE LIST

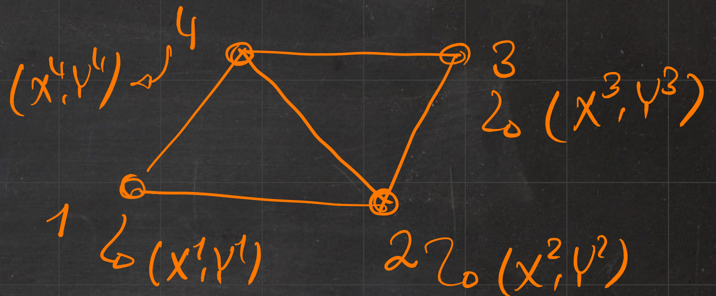
$$\begin{bmatrix} \; & \; \\ \; & \; \end{bmatrix}\begin{bmatrix} N \\ D \end{bmatrix} = \begin{bmatrix} N \\ D \end{bmatrix}$$

$\mathbb{K}\cdot\mathbb{U}=\mathbb{F}$



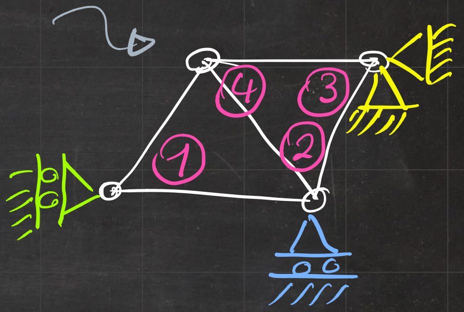
NL	COOR.	BC INFO	TMP DEGREE	DEGREE	DISP	FORCE
1	ooo	D	-1	3	0	?
2	ooo	N	1	1	?	0
3	ooo	N	2	2	?	0
4	ooo	D	-2	4	u ↳ Pres.	?

TRUSS STRUCTURES IN 2D:



NL	COOR.	BC INFO	TMP DEGREE	DEGREE	DISP	FORCE
1	X^1, Y^1	D, N	-1, 1	5, 1		
2	⋮	N, D	2, -2	2, 6		
3	⋮	D, D	-3, -4	7, 8		
4		N, N	3, 4	3, 4		

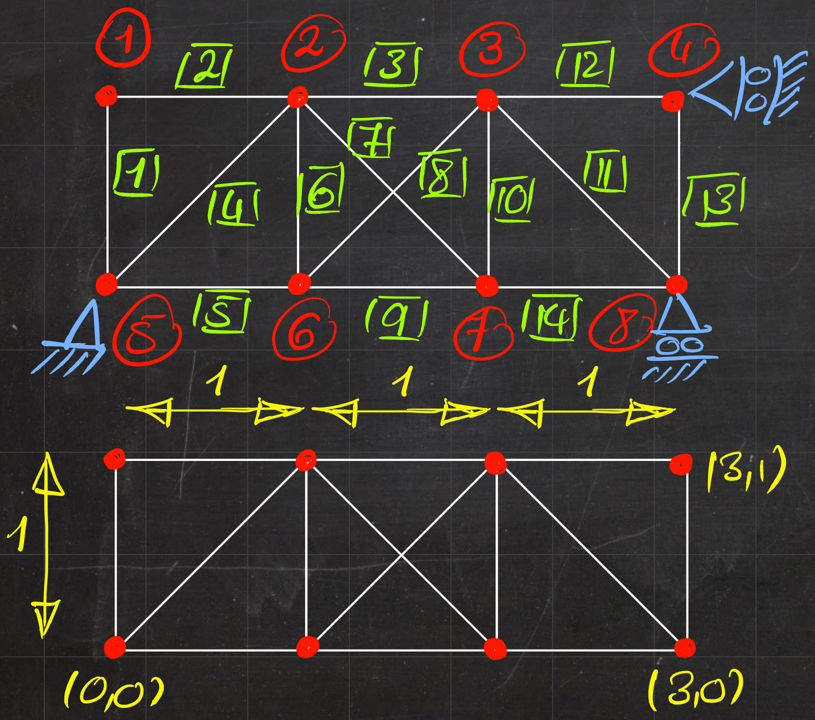
Truss Structures in 2D:



System with 4 DOFs

NL	Coor.	BC info	TMP Degree	Degree	Disp	Force
1	X^1, Y^1	D, N	-1, 1	5, 1		
2	⋮	N, D	2, -2	2, 6		
3		D, D	-3, -4	7, 8		
4		N, N	3, 4	3, 4		

NODE LIST ⟶ EXTENDED NODE LIST   CONNECTIVITY

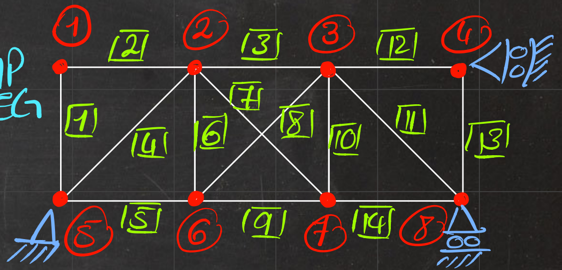


x, y

NL	COOR
1	0, 1
2	1, 1
3	2, 1
4	3, 1
5	0, 0
6	1, 0
7	2, 0
8	3, 0

EL	
1	5, 1
2	1, 2
3	2, 3
4	5, 2
5	5, 6
6	6, 2
7	2, 7
8	⋮
9	
10	
11	
⋮	

EXTENDED NODE LIST



NODE NUMBER	COOR		BC INFO		IMP DEG	
	X	Y	X	Y	X	Y
1	0	1	N	N	1	2
2	1	1	N	N	3	4
3	2	1	N	N	5	6
4	3	1	D	N	-1	7
5	0	0	D	D	-2	-3
6	1	0	N	N	8	9
7	2	0	N	N	10	11
8	3	0	N	D	12	-4

EXTENDED NODE LIST

NODE NUMBER	COOR		BC INFO		TMP DEG		(GLOBAL) DEGREE		DISP		FORCE	
	X	Y	X	Y	X	Y	X	Y	X	Y	X	Y
1	0	1	N	N	1	2	1	2				
2	1	1	N	N	3	4	3	4				
3	2	1	N	N	5	6	5	6				
4	3	1	D	N	-1	7	13	7				
5	0	0	D	D	-2	-3	14	15				
6	1	0	N	N	8	9	8	9				
7	2	0	N	N	10	11	10	11				
8	3	0	N	D	12	-4	12	16				

EXTENDED NODE LIST

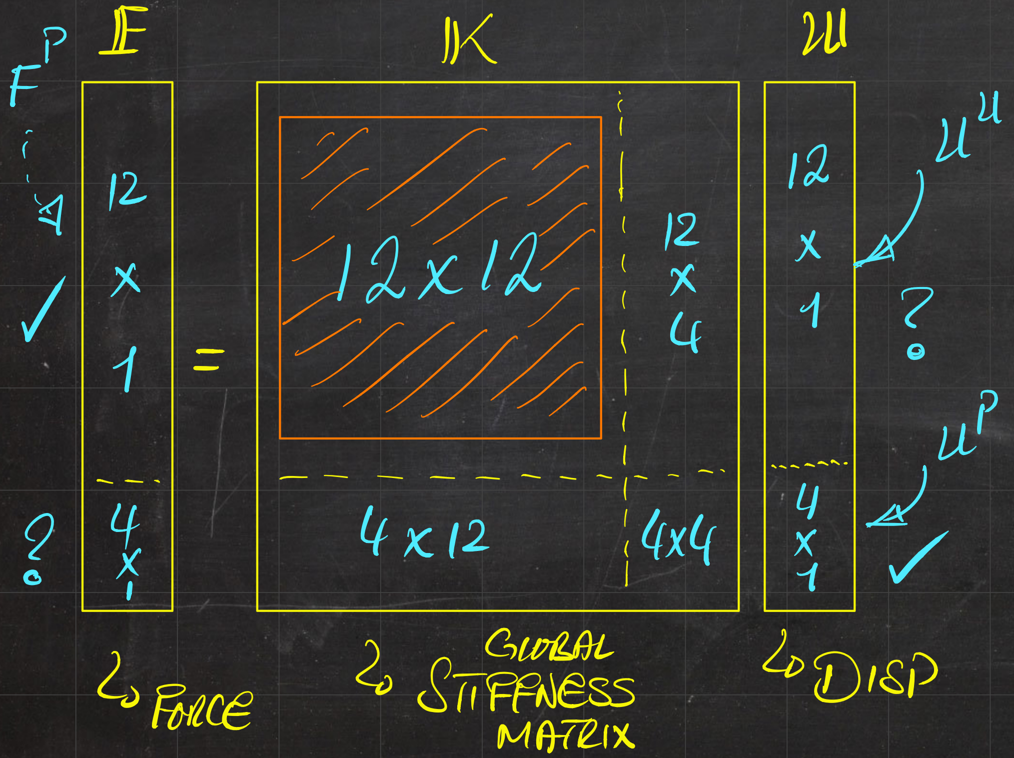
$$
\overset{F^P}{\checkmark},\ ?\qquad
\underbrace{\overset{\mathbb{F}}{\begin{bmatrix} 12\times 1 \\ \hline 4\times 1 \end{bmatrix}}}_{\text{Force}}
=
\underbrace{\overset{\mathbb{K}}{\begin{bmatrix} 12\times 12 & 12\times 4 \\ \hline 4\times 12 & 4\times 4 \end{bmatrix}}}_{\text{Global Stiffness Matrix}}
\underbrace{\overset{\mathbb{U}}{\begin{bmatrix} 12\times 1 \\ \hline 4\times 1 \end{bmatrix}}}_{\text{Disp}}
\qquad u^u\ ?,\quad u^P\ \checkmark
$$

(GLOBAL) DEGREE

X	Y
1	2
3	4
5	6
13	7
14	15
8	9
10	11
12	16

12 DoF

EXTENDED NODE LIST



→ STATIC CONDENSATION

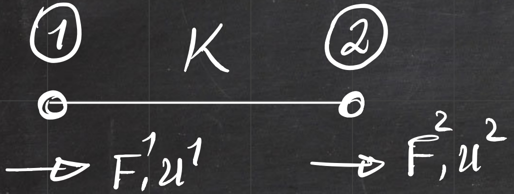
↓

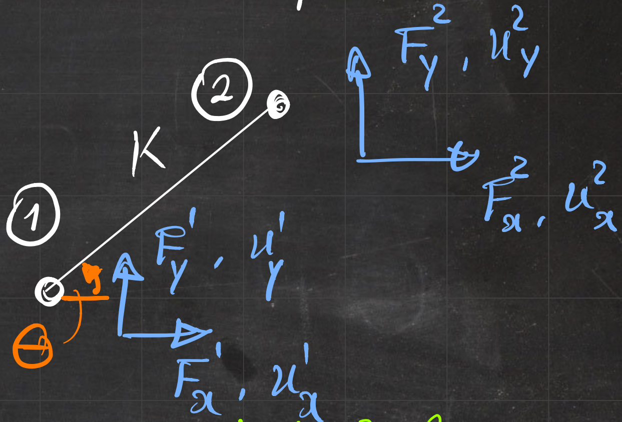
REDUCED SYSTEM

↳ $\mathbb{A} \cdot x = \mathbb{b}$

↳ $\square$ 12x12

To compute stiffness of 1D element in 2D space





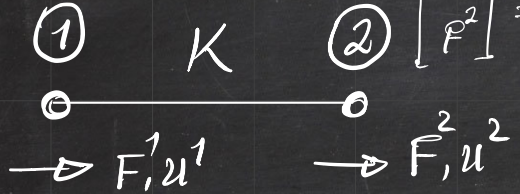
$$\begin{bmatrix} F^1 \\ F^2 \end{bmatrix} = \begin{bmatrix} K & -K \\ -K & K \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix}$$

$$\mathbb{K} = \begin{array}{cccc} 1_x & 1_y & 2_x & 2_y \end{array}$$

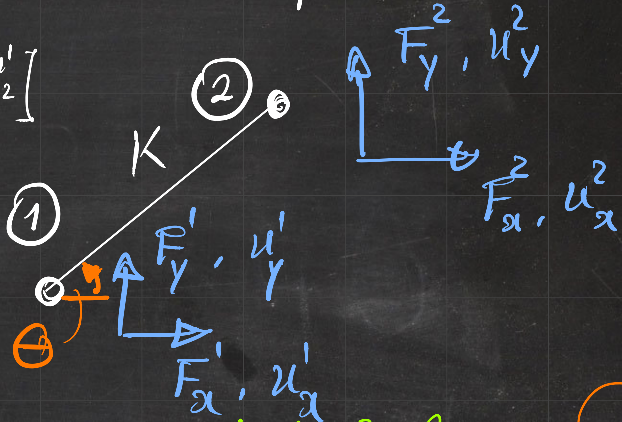
? $f(\theta)$

4x4

To compute stiffness of 1D element in 2D space



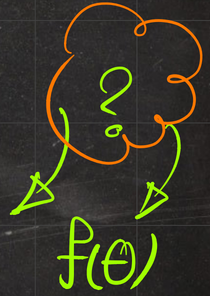
$$\begin{bmatrix} F^1 \\ F^2 \end{bmatrix} = \begin{bmatrix} K & -K \\ -K & K \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix}$$



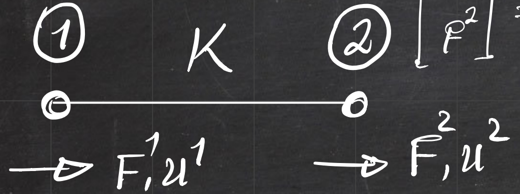
$$\begin{bmatrix} F^1_x \\ F^1_y \\ F^2_x \\ F^2_y \end{bmatrix} = \begin{array}{cccc} 1_x & 1_y & 2_x & 2_y \\ \end{array} \begin{bmatrix} \; & \; & \; & \; \\ \; & \; & \; & \; \\ \; & \; & \; & \; \\ \; & \; & \; & \; \end{bmatrix} \begin{bmatrix} u^1_x \\ u^1_y \\ u^2_x \\ u^2_y \end{bmatrix}$$

$$\mathbb{K} = \begin{array}{cccc} 1_x & 1_y & 2_x & 2_y \\ \end{array} \begin{bmatrix} \; & \; & \; & \; \\ \; & \; & \; & \; \\ \; & \; & \; & \; \\ \; & \; & \; & \; \end{bmatrix}$$

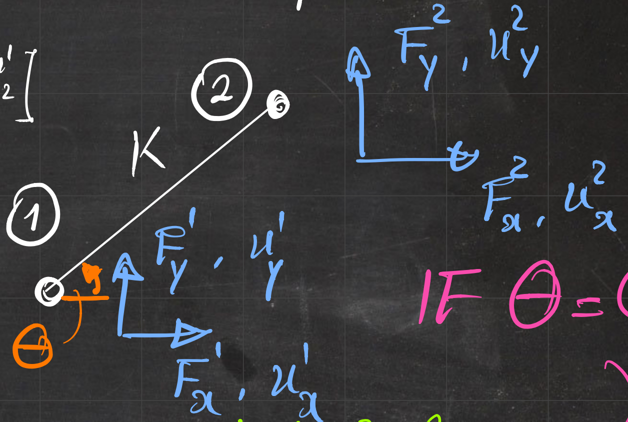
4x4



To compute stiffness of 1D element in 2D space



$$\begin{bmatrix} F^1 \\ F^2 \end{bmatrix} = \begin{bmatrix} K & -K \\ -K & K \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix}$$



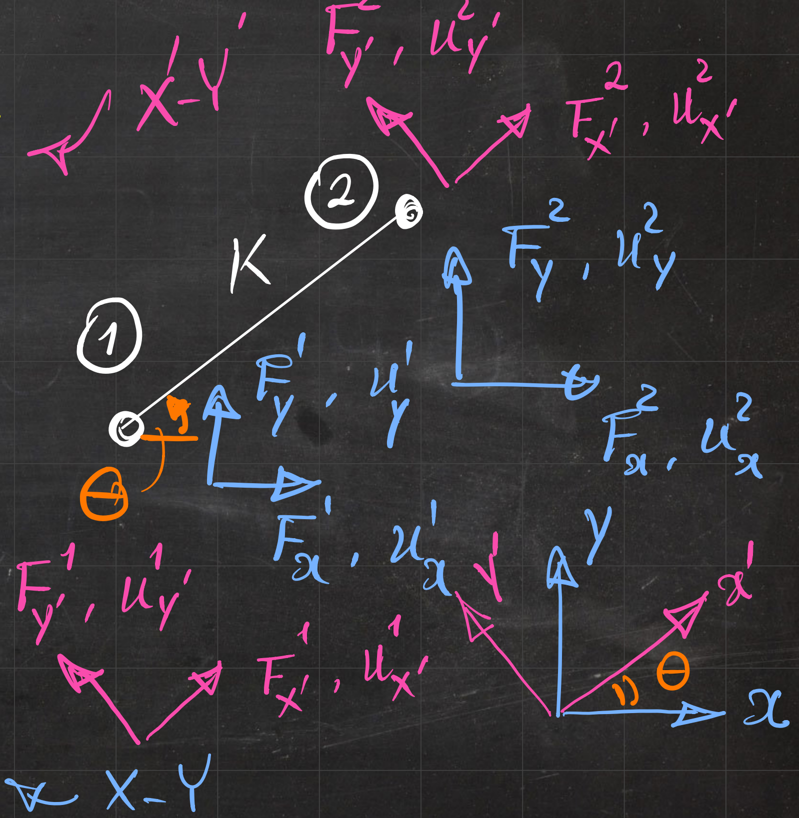
IF $\theta = 0$

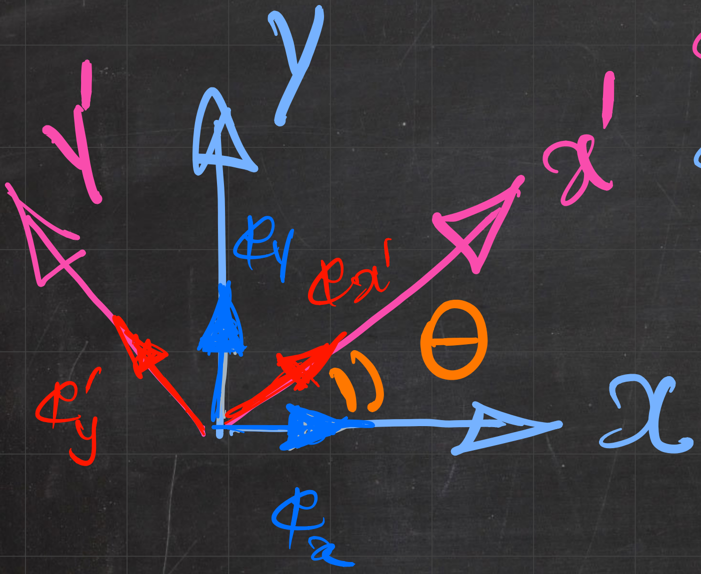
$$\begin{bmatrix} F^1_x \\ F^1_y \\ F^2_x \\ F^2_y \end{bmatrix} = \begin{array}{c} \begin{array}{cccc} 1_x & 1_y & 2_x & 2_y \end{array} \\ \begin{bmatrix} \; & \; & \; & \; \\ \; & \; & \; & \; \\ \; & \; & \; & \; \\ \; & \; & \; & \; \end{bmatrix} \end{array} \begin{bmatrix} u^1_x \\ u^1_y \\ u^2_x \\ u^2_y \end{bmatrix}$$

$$\mathbb{K} = \begin{array}{c} \begin{array}{cccc} 1_x & 1_y & 2_x & 2_y \end{array} \\ \begin{bmatrix} K & 0 & -K & 0 \\ 0 & 0 & 0 & 0 \\ -K & 0 & K & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \quad 4\times4$$

$$\begin{bmatrix}F^1_{x'}\\F^1_{y'}\\F^2_{x'}\\F^2_{y'}\end{bmatrix} = \begin{array}{cccc} 1_{x'} & 1_{y'} & 2_{x'} & 2_{y'} \\ \end{array}$$
$$\begin{bmatrix}K & 0 & -K & 0\\0 & 0 & 0 & 0\\-K & 0 & K & 0\\0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}u^1_{x'}\\u^1_{y'}\\u^2_{x'}\\u^2_{y'}\end{bmatrix}$$

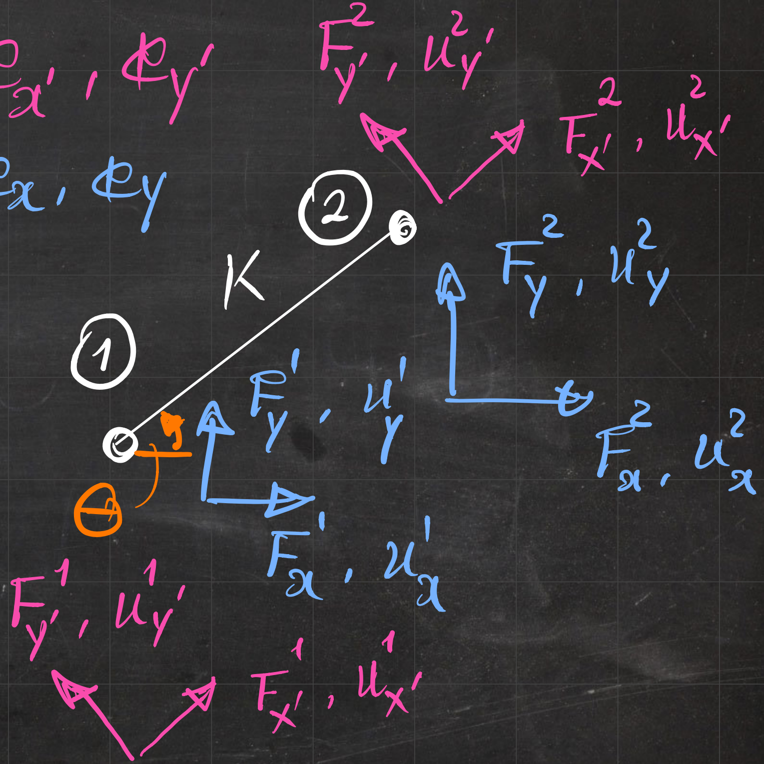
$$\begin{bmatrix}F^1_{x}\\F^1_{y}\\F^2_{x}\\F^2_{y}\end{bmatrix} = \begin{array}{cccc} 1_{x} & 1_{y} & 2_{x} & 2_{y} \\ \end{array}$$
$$\begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix}\begin{bmatrix}u^1_{x}\\u^1_{y}\\u^2_{x}\\u^2_{y}\end{bmatrix}$$





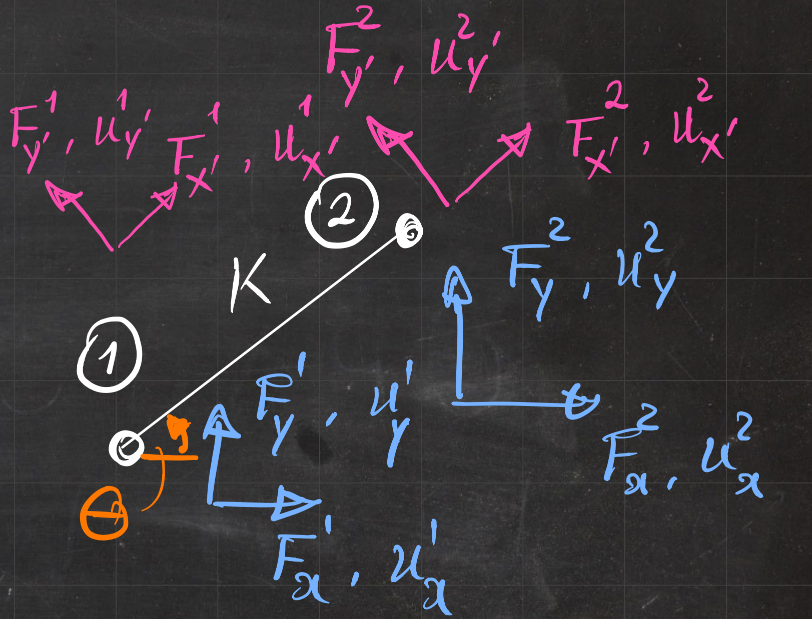
$e_{x'}$, $e_{y'}$

e_x, e_y



$$\begin{cases} e_{x'} = \cos\theta \, e_x + \sin\theta \, e_y \\ e_{y'} = -\sin\theta \, e_x + \cos\theta \, e_y \end{cases}$$

$$\begin{cases} \mathbf{e}_{x'} = \cos\theta\, \mathbf{e}_x + \sin\theta\, \mathbf{e}_y \\ \mathbf{e}_{y'} = -\sin\theta\, \mathbf{e}_x + \cos\theta\, \mathbf{e}_y \end{cases}$$

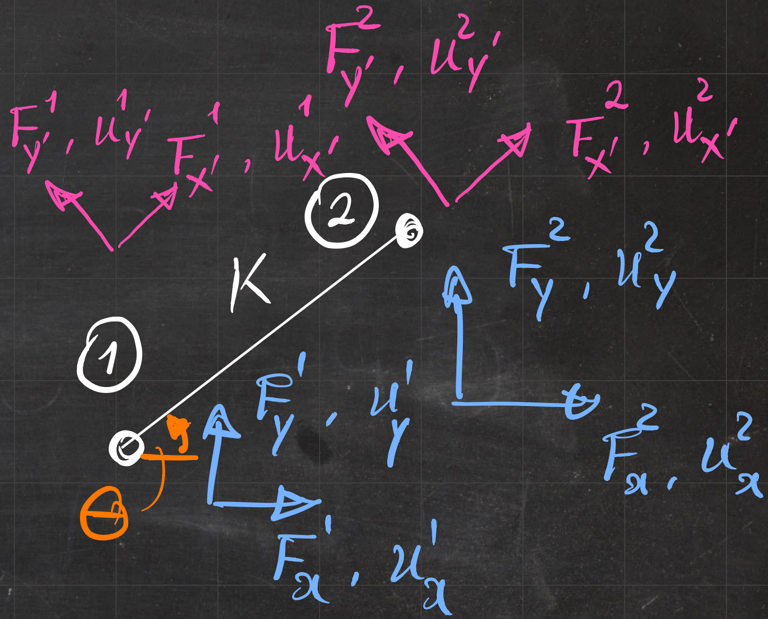


$$\mathbf{F} = F_{x'}\, \mathbf{e}_{x'} + F_{y'}\, \mathbf{e}_{y'}$$

$$= F_x\, \mathbf{e}_x + F_y\, \mathbf{e}_y \quad \dots$$

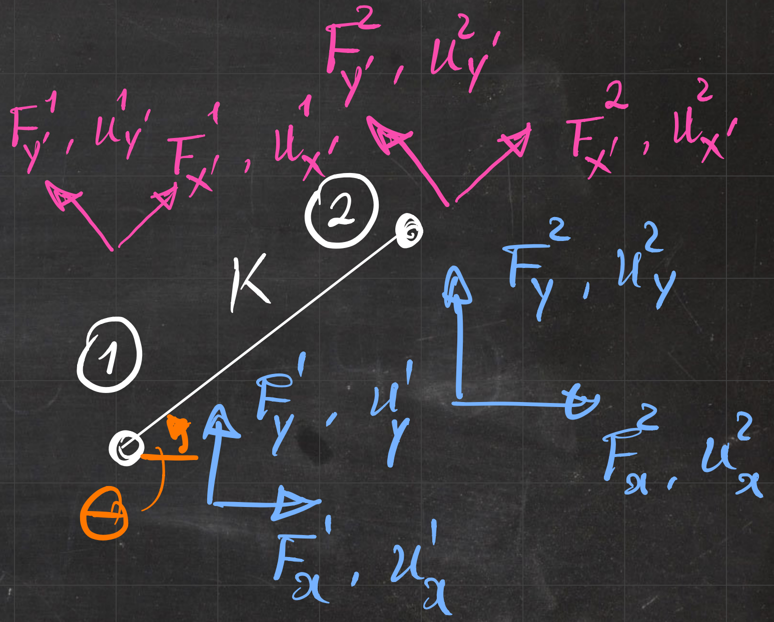
$$\begin{bmatrix} F_x \\ F_y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ +\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} F_{x'} \\ F_{y'} \end{bmatrix} \Longleftrightarrow \begin{bmatrix} F_{x'} \\ F_{y'} \end{bmatrix} = \begin{bmatrix} \cos\theta & +\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix}$$

$$\begin{bmatrix} F^1_{x'} \\ F^1_{y'} \\ F^2_{x'} \\ F^2_{y'} \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix}}_{\mathbb{R}} \begin{bmatrix} F^1_{x} \\ F^1_{y} \\ F^2_{x} \\ F^2_{y} \end{bmatrix}$$



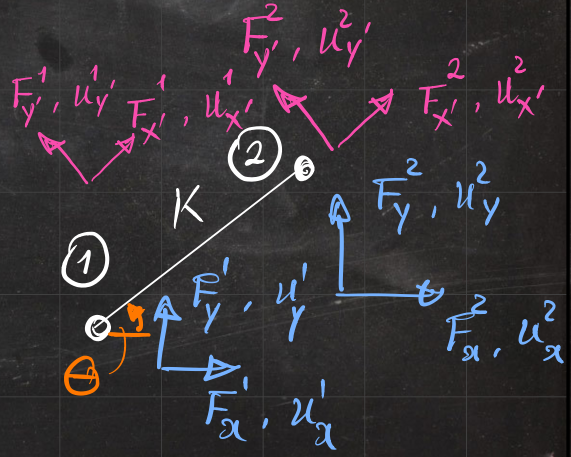
$$\begin{bmatrix} F_x \\ F_y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ +\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} F_{x'} \\ F_{y'} \end{bmatrix} \Longleftrightarrow \begin{bmatrix} F_{x'} \\ F_{y'} \end{bmatrix} = \begin{bmatrix} \cos\theta & +\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} F_{x} \\ F_{y} \end{bmatrix}$$

$$
\begin{bmatrix} F_{x'}^{1} \\ F_{y'}^{1} \\ F_{x'}^{2} \\ F_{y'}^{2} \end{bmatrix}
=
\underbrace{\begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix}}_{\mathbb{R}(\theta)}
\begin{bmatrix} F_{x}^{1} \\ F_{y}^{1} \\ F_{x}^{2} \\ F_{y}^{2} \end{bmatrix}
$$



$$\begin{bmatrix} F^1_{x'} \\ F^1_{y'} \\ F^2_{x'} \\ F^2_{y'} \end{bmatrix} = \mathbb{R}_\theta \begin{bmatrix} F^1_x \\ F^1_y \\ F^2_x \\ F^2_y \end{bmatrix} , \quad \begin{bmatrix} u^1_{x'} \\ u^1_{y'} \\ u^2_{x'} \\ u^2_{y'} \end{bmatrix} = \mathbb{R}_\theta \begin{bmatrix} u^1_x \\ u^1_y \\ u^2_x \\ u^2_y \end{bmatrix}$$

$$\mathbb{R}_\theta^{-1} = \mathbb{R}_{-\theta}$$



$$
\begin{bmatrix} F^1_{x'} \\ F^1_{y'} \\ F^2_{x'} \\ F^2_{y'} \end{bmatrix}
=
\begin{array}{c}
\begin{array}{cccc} 1_{x'} & 1_{y'} & 2_{x'} & 2_{y'} \end{array} \\
\underbrace{\begin{bmatrix} K & 0 & -K & 0 \\ 0 & 0 & 0 & 0 \\ -K & 0 & K & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathbb{K}_0}
\end{array}
\begin{bmatrix} u^1_{x'} \\ u^1_{y'} \\ u^2_{x'} \\ u^2_{y'} \end{bmatrix}
$$

$$
\begin{bmatrix} F^1_{x'} \\ F^1_{y'} \\ F^2_{x'} \\ F^2_{y'} \end{bmatrix} = \mathbb{R}_\theta \begin{bmatrix} F^1_{x} \\ F^1_{y} \\ F^2_{x} \\ F^2_{y} \end{bmatrix}, \quad
\begin{bmatrix} u^1_{x'} \\ u^1_{y'} \\ u^2_{x'} \\ u^2_{y'} \end{bmatrix} = \mathbb{R}_\theta \begin{bmatrix} u^1_{x} \\ u^1_{y} \\ u^2_{x} \\ u^2_{y} \end{bmatrix}
$$

$$\mathbb{F}' = \mathbb{K}_0 \, \mathbb{u}'$$

$$
\begin{bmatrix} F^1_{x} \\ F^1_{y} \\ F^2_{x} \\ F^2_{y} \end{bmatrix}
=
\begin{array}{c}
\begin{array}{cccc} 1_{x} & 1_{y} & 2_{x} & 2_{y} \end{array} \\
\begin{bmatrix} \; & \; & \; & \; \\ \; & \; & \; & \; \\ \; & \; & \; & \; \\ \; & \; & \; & \; \end{bmatrix}
\end{array}
\begin{bmatrix} u^1_{x} \\ u^1_{y} \\ u^2_{x} \\ u^2_{y} \end{bmatrix}
$$

$$\mathbb{R}_\theta \, \mathbb{F} = \mathbb{K}_0 \, \mathbb{R}_\theta \, \mathbb{u}$$

$$\mathbb{F} = \mathbb{R}_\theta^T \, \mathbb{K}_0 \, \mathbb{R}_\theta \, \mathbb{u}$$

$$
\begin{bmatrix} F_x^{1'} \\ F_y^{1'} \\ F_x^{2'} \\ F_y^{2'} \end{bmatrix}
=
\underbrace{
\begin{array}{cccc}
1_x' & 1_y' & 2_x' & 2_y' \\
\end{array}
\begin{bmatrix} K & 0 & -K & 0 \\ 0 & 0 & 0 & 0 \\ -K & 0 & K & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
}_{\mathbb{K}_0}
\begin{bmatrix} u_x^{1'} \\ u_y^{1'} \\ u_x^{2'} \\ u_y^{2'} \end{bmatrix}
$$

$$
\begin{bmatrix} F_x^{1'} \\ F_y^{1'} \\ F_x^{2'} \\ F_y^{2'} \end{bmatrix}
= \mathbb{R}_\theta
\begin{bmatrix} F_x^{1} \\ F_y^{1} \\ F_x^{2} \\ F_y^{2} \end{bmatrix}
,\qquad
\begin{bmatrix} u_x^{1'} \\ u_y^{1'} \\ u_x^{2'} \\ u_y^{2'} \end{bmatrix}
= \mathbb{R}_\theta
\begin{bmatrix} u_x^{1} \\ u_y^{1} \\ u_x^{2} \\ u_y^{2} \end{bmatrix}
$$

$$
\mathbb{F}' = \mathbb{K}_0 \, \mathbb{U}'
$$

$$
\begin{bmatrix} F_x^{1} \\ F_y^{1} \\ F_x^{2} \\ F_y^{2} \end{bmatrix}
=
\begin{array}{cccc}
1_x & 1_y & 2_x & 2_y \\
\end{array}
\begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \\ & & & \end{bmatrix}
\begin{bmatrix} u_x^{1} \\ u_y^{1} \\ u_x^{2} \\ u_y^{2} \end{bmatrix}
$$

$$
\mathbb{F} = \underbrace{\mathbb{R}_\theta^T \, \mathbb{K}_0 \, \mathbb{R}_\theta}_{\mathbb{K}} \, \mathbb{U}
$$

$$
\mathbb{F} = \mathbb{K} \, \mathbb{U}
$$

$$
\begin{bmatrix} F^1_{x'} \\ F^1_{y'} \\ F^2_{x'} \\ F^2_{y'} \end{bmatrix}
=
\underbrace{
\begin{array}{c}
\begin{array}{cccc} 1_{x'} & 1_{y'} & 2_{x'} & 2_{y'} \end{array} \\
\left[\begin{array}{c|c|c|c} K & 0 & -K & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline -K & 0 & K & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]
\end{array}
}_{\mathbb{K}_0}
\begin{bmatrix} u^1_{x'} \\ u^1_{y'} \\ u^2_{x'} \\ u^2_{y'} \end{bmatrix}
$$

$$
\begin{bmatrix} F^1_{x'} \\ F^1_{y'} \\ F^2_{x'} \\ F^2_{y'} \end{bmatrix} = \mathbb{R}_\theta \begin{bmatrix} F^1_{x} \\ F^1_{y} \\ F^2_{x} \\ F^2_{y} \end{bmatrix}, \quad
\begin{bmatrix} u^1_{x'} \\ u^1_{y'} \\ u^2_{x'} \\ u^2_{y'} \end{bmatrix} = \mathbb{R}_\theta \begin{bmatrix} u^1_{x} \\ u^1_{y} \\ u^2_{x} \\ u^2_{y} \end{bmatrix}
$$

$$
\mathbb{F}' = \mathbb{K}_0 \, \mathbb{u}'
$$

$$
\begin{bmatrix} F^1_{x} \\ F^1_{y} \\ F^2_{x} \\ F^2_{y} \end{bmatrix}
=
\begin{array}{c}
\begin{array}{cccc} 1_{x} & 1_{y} & 2_{x} & 2_{y} \end{array} \\
\left[\begin{array}{c|c|c|c} \; & \; & \; & \; \\ \hline \; & \; & \; & \; \\ \hline \; & \; & \; & \; \\ \hline \; & \; & \; & \; \end{array}\right]
\end{array}
\begin{bmatrix} u^1_{x} \\ u^1_{y} \\ u^2_{x} \\ u^2_{y} \end{bmatrix}
$$

$$
\mathbb{F} = \mathbb{K} \, \mathbb{u}
$$

$$
\mathbb{K} = \mathbb{R}_\theta^T \, \mathbb{K}_0 \, \mathbb{R}_\theta
$$

$$
\begin{bmatrix} F^1_x \\ F^1_y \\ F^2_x \\ F^2_y \end{bmatrix}
=
\begin{array}{cccc} 1_x & 1_y & 2_x & 2_y \\ \hline \end{array}
\begin{bmatrix} \; & \; & \; & \; \\ \; & \; & \; & \; \\ \; & \; & \; & \; \\ \; & \; & \; & \; \end{bmatrix}
\begin{bmatrix} u^1_x \\ u^1_y \\ u^2_x \\ u^2_y \end{bmatrix}
$$

$$\mathbb{F} = \mathbb{K}\,\mathbb{u}$$

$$\mathbb{K} = \mathbb{R}_\theta^T\,\mathbb{K}_0\,\mathbb{R}_\theta$$

$$
\boxed{\begin{matrix} F^1_x \\ F^1_y \\ F^2_x \\ F^2_y \end{matrix}}
= K \; \frac{EA}{L}
\boxed{\begin{matrix}
\cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\
\sin\theta\cos\theta & \sin^2\theta & -\sin\theta\cos\theta & -\sin^2\theta \\
-\cos^2\theta & -\cos\theta\sin\theta & \cos^2\theta & \cos\theta\sin\theta \\
-\sin\theta\cos\theta & -\sin^2\theta & \sin\theta\cos\theta & \sin^2\theta
\end{matrix}}
\boxed{\begin{matrix} u^1_x \\ u^1_y \\ u^2_x \\ u^2_y \end{matrix}}
$$

$$
\underbrace{\begin{bmatrix}
\cos\theta & \sin\theta & 0 & 0 \\
-\sin\theta & \cos\theta & 0 & 0 \\
0 & 0 & \cos\theta & \sin\theta \\
0 & 0 & -\sin\theta & \cos\theta
\end{bmatrix}}_{\mathbb{R}(\theta)}
$$

$$
\begin{bmatrix} F^1_x \\ F^1_y \\ F^2_x \\ F^2_y \end{bmatrix} =
\begin{array}{cccc} 1_x & 1_y & 2_x & 2_y \\ \end{array}
\begin{bmatrix} \; & \; & \; & \; \\ \; & \; & \; & \; \\ \; & \; & \; & \; \\ \; & \; & \; & \; \end{bmatrix}
\begin{bmatrix} u^1_x \\ u^1_y \\ u^2_x \\ u^2_y \end{bmatrix}
$$

$$\theta = 0$$

$$\mathbb{F} = \mathbb{K}\,\mathbb{u}$$

$$\mathbb{K} = \mathbb{R}_\theta^T \,\mathbb{K}_0\, \mathbb{R}_\theta$$

$$
\boxed{\begin{matrix} F^1_x \\ F^1_y \\ F^2_x \\ F^2_y \end{matrix}} = K \boxed{\begin{matrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}} \boxed{\begin{matrix} u^1_x \\ u^1_y \\ u^2_x \\ u^2_y \end{matrix}}
$$

$$K = \frac{EA}{L}$$

$$
\underbrace{\begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix}}_{\mathbb{R}(\theta)}
$$