MECHANICS AND MATERIALS I

20

Ali Javili

Bilkent University

MECHANICS AND MATERIALS I

Deflection of beams and shafts

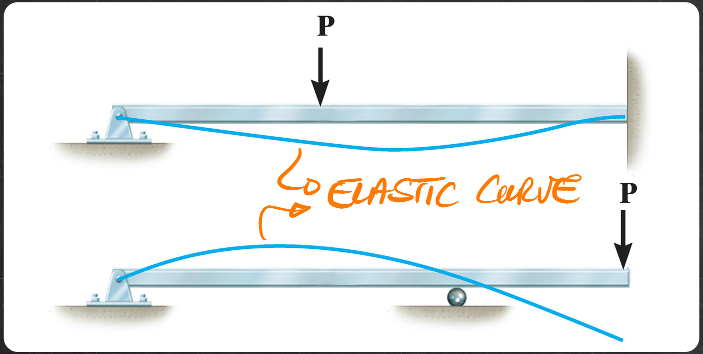
Sect. ... 12.1 – 12.3 ... 12.5 – 12.7 ... 12.9

Chap. 12

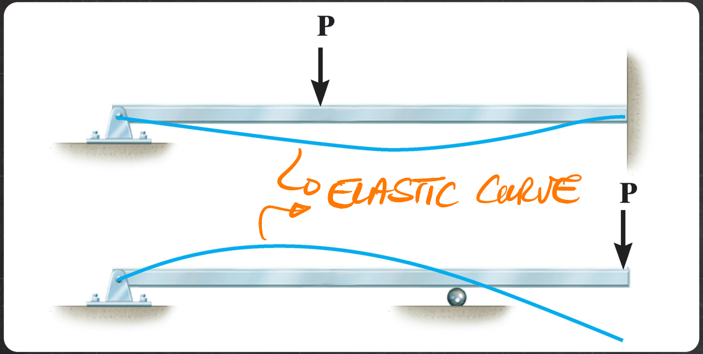
[Hibbeler 9th edition]

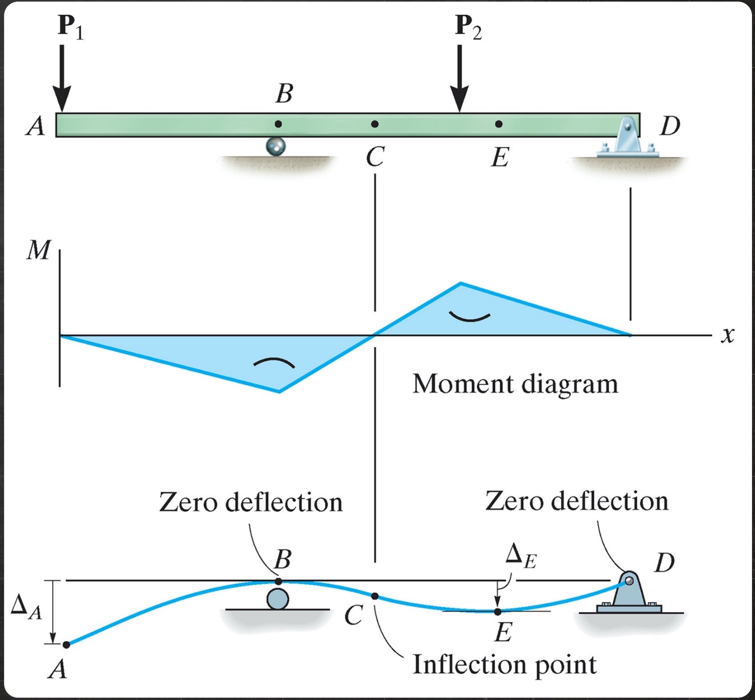
ELASTIC CURVE

ELASTIC CURVE



ELASTIC CURVE





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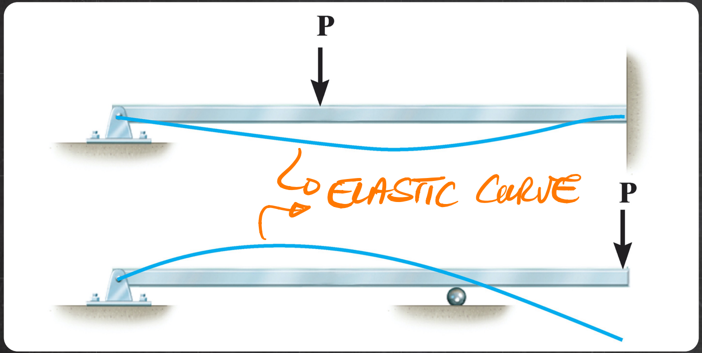

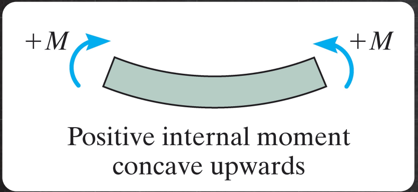

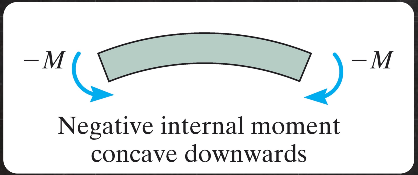

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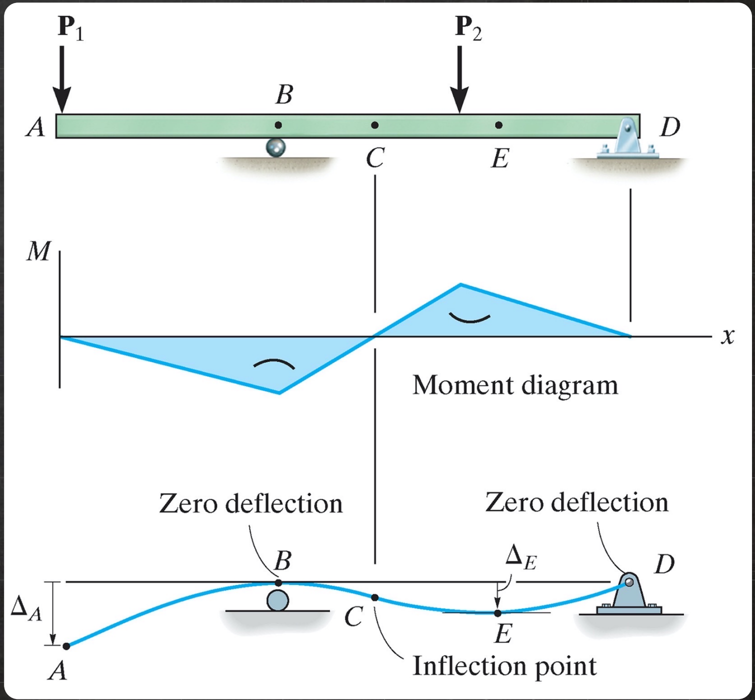
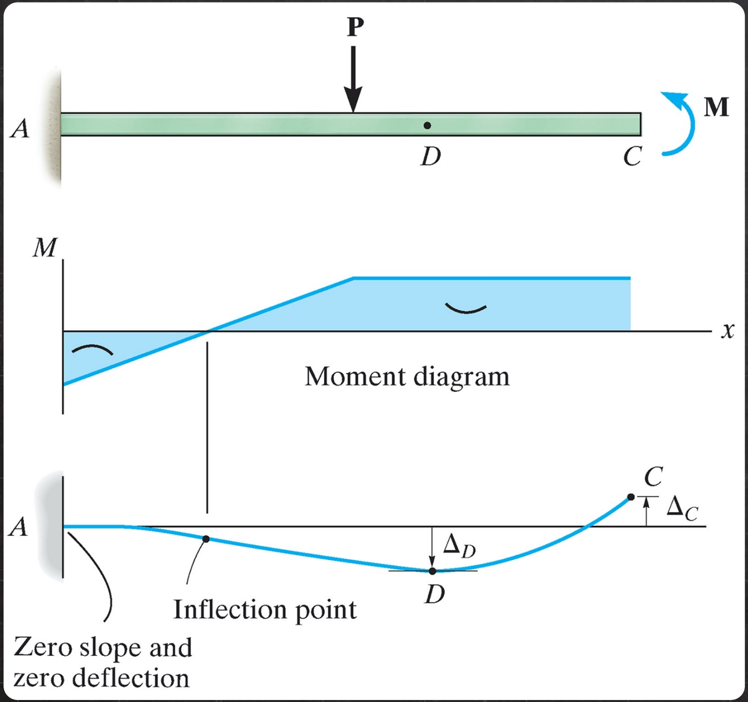
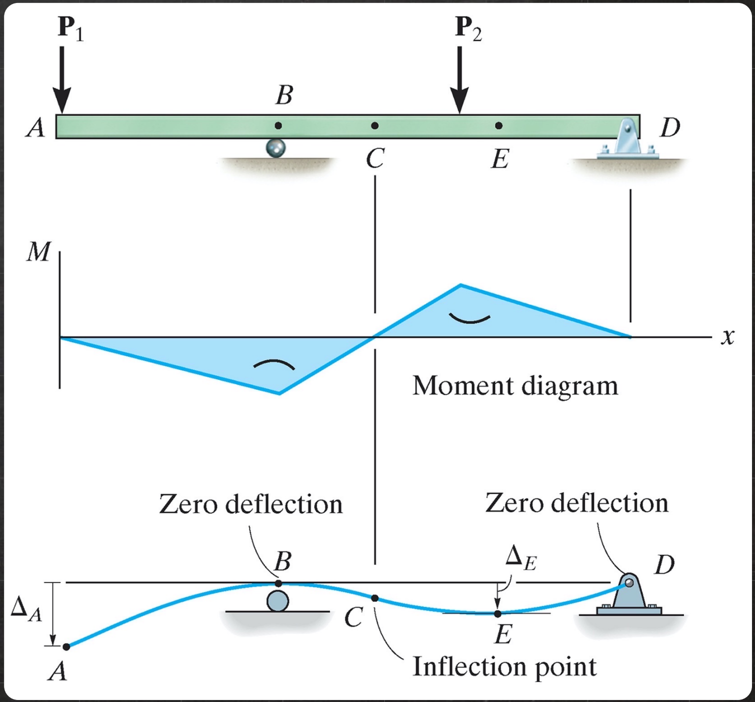
Sign Convention



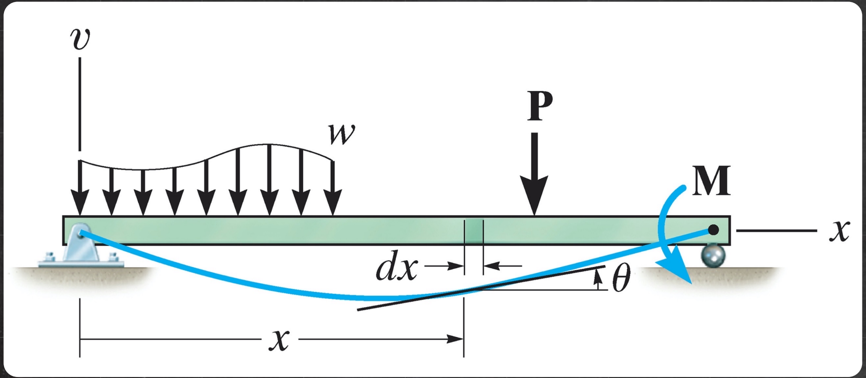
ELASTIC CURVE



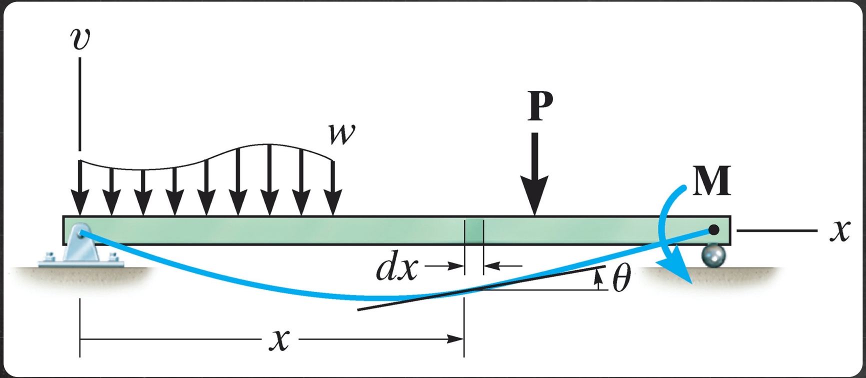


MOMENT-CURVATURE RELATIONSHIP

MOMENT-CURVATURE RELATIONSHIP

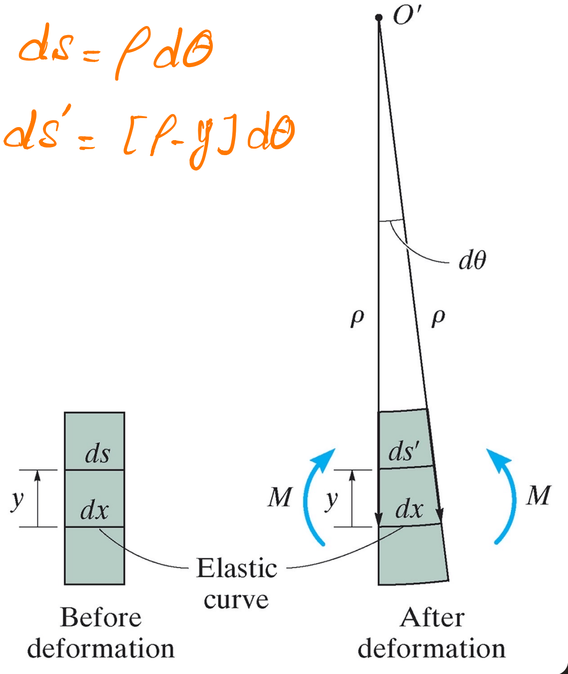


Moment-Curvature Relationship



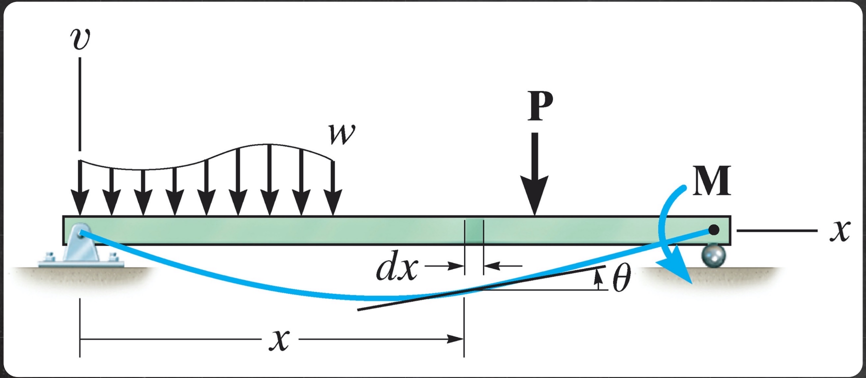
$ds = \rho \, d\theta$

$ds' = [\rho - y] \, d\theta$



MOMENT-CURVATURE RELATIONSHIP

$$\varepsilon = \frac{ds' - ds}{ds} \Rightarrow \varepsilon = -\frac{y}{\rho}$$



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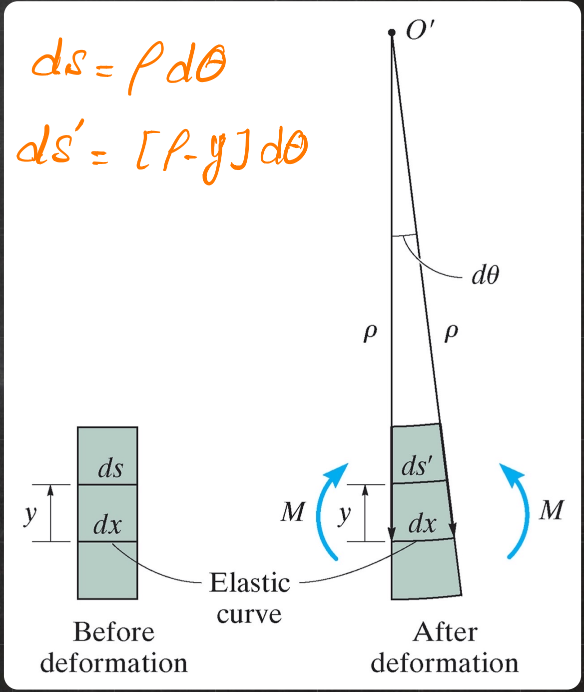

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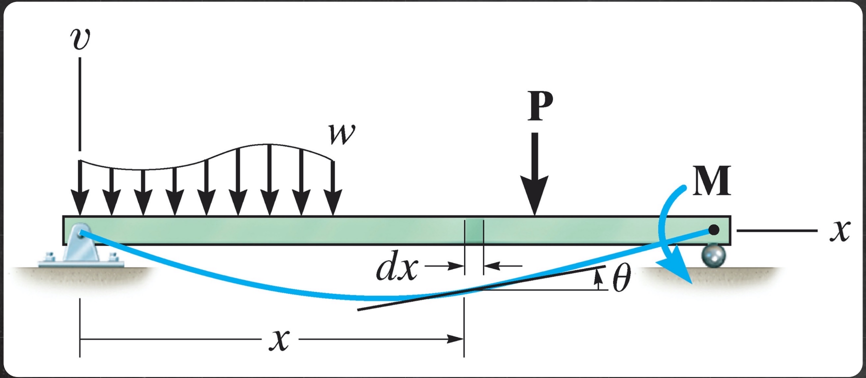

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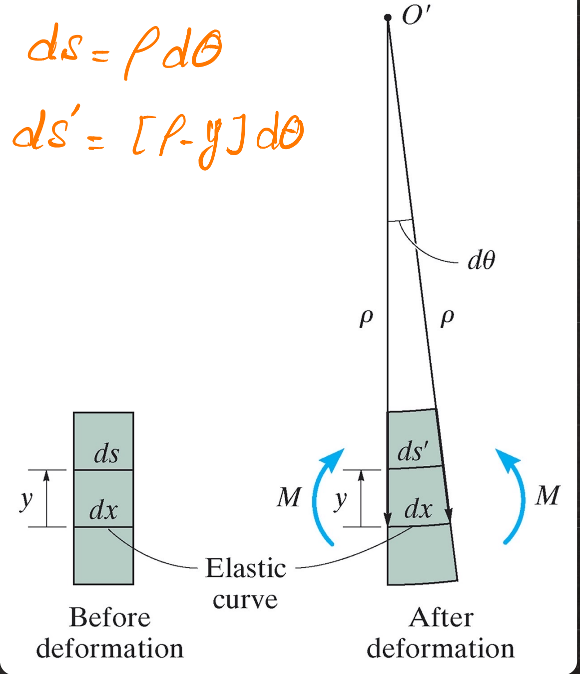
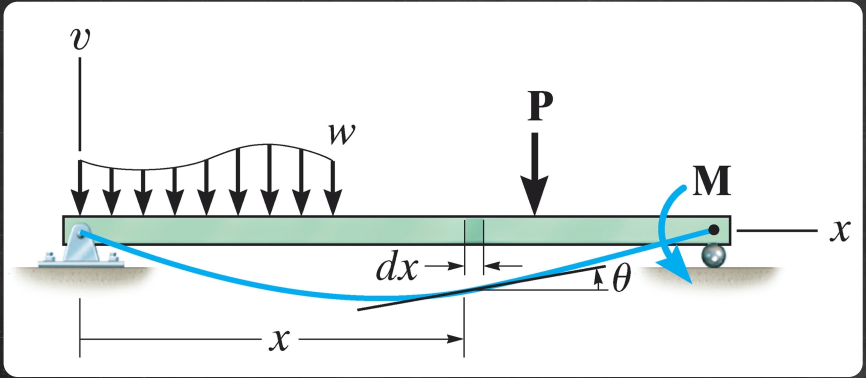
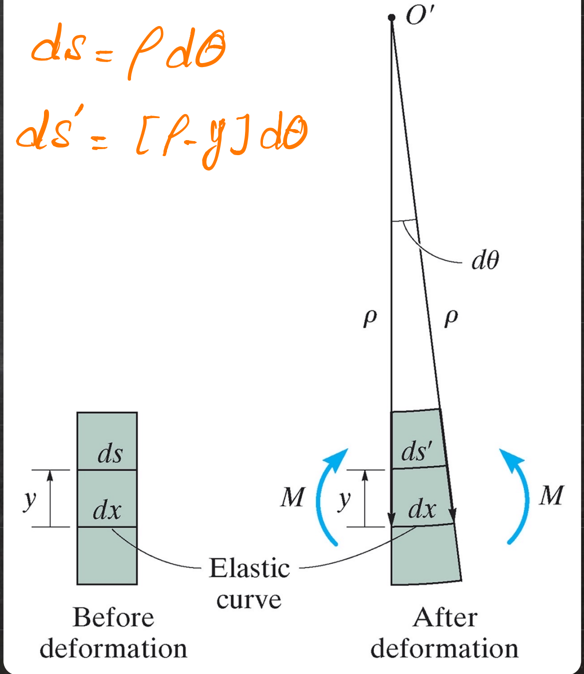


$$\frac{1}{\rho} = -\frac{\varepsilon}{y}$$

MOMENT-CURVATURE RELATIONSHIP

$$\varepsilon = \frac{ds' - ds}{ds} \Rightarrow \varepsilon = -\frac{y}{\rho}$$





$$ds = \rho\, d\theta$$

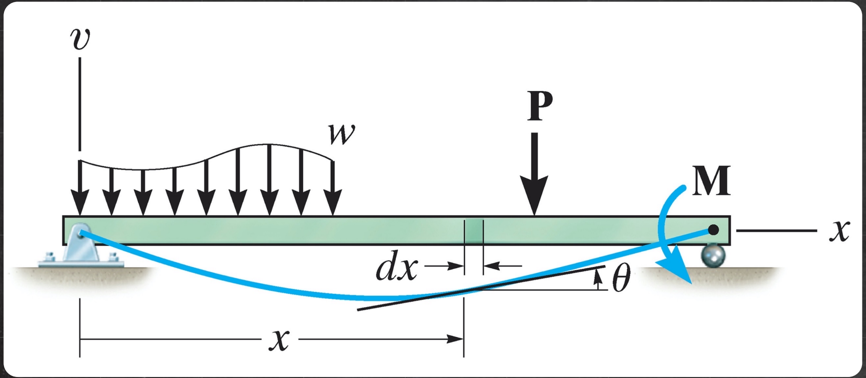
$$ds' = [\rho - y]\, d\theta$$

$$\sigma = E\varepsilon$$

$$\frac{1}{\rho} = -\frac{\varepsilon}{y} = -\frac{\sigma}{Ey}$$

MOMENT-CURVATURE RELATIONSHIP

$$\varepsilon = \frac{ds' - ds}{ds} \Rightarrow \varepsilon = -\frac{y}{\rho}$$



$\sigma = E\varepsilon$ $\quad$ $\sigma = -My/I$ FLEXURE FORMULA

$$\frac{1}{\rho} = -\frac{\varepsilon}{y} = -\frac{\sigma}{Ey} = \frac{M}{EI}$$

$ds = \rho\, d\theta$

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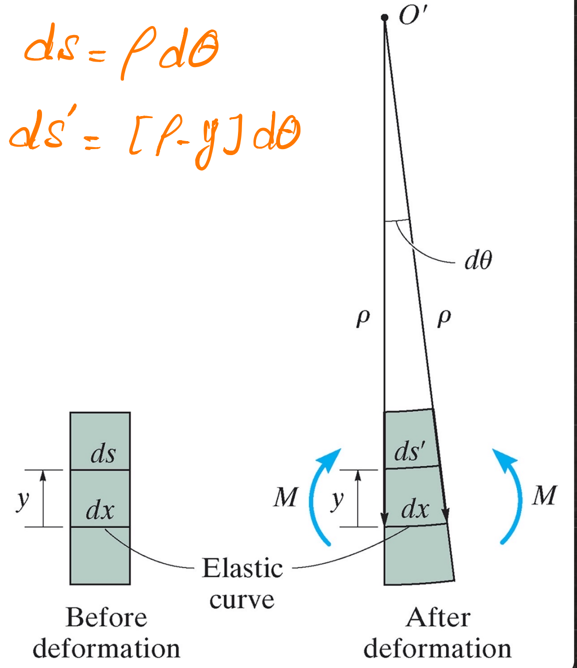

MOMENT-CURVATURE RELATIONSHIP

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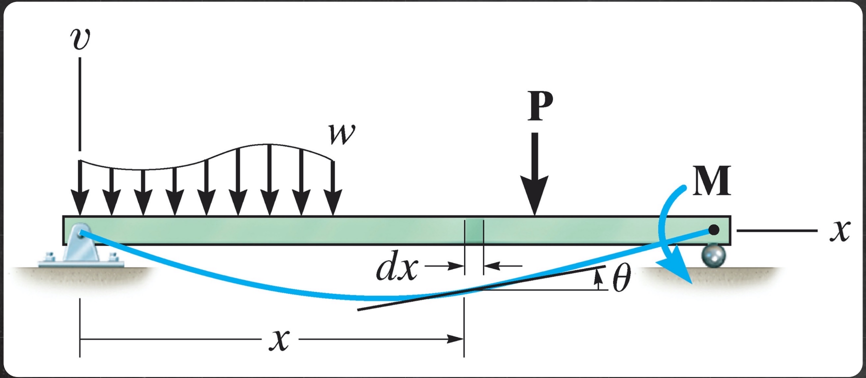
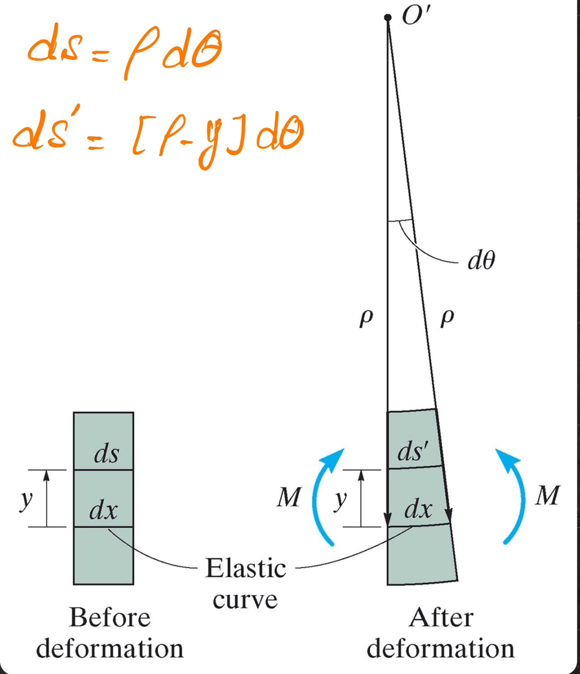




$$ds = \rho\, d\theta$$

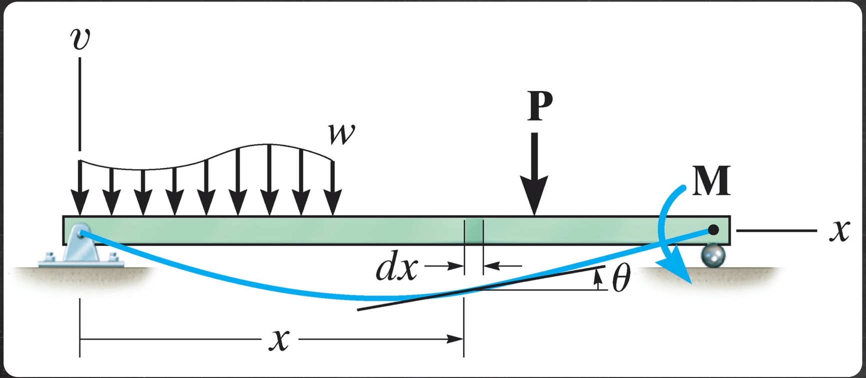
$$ds' = [\rho - y]\, d\theta$$

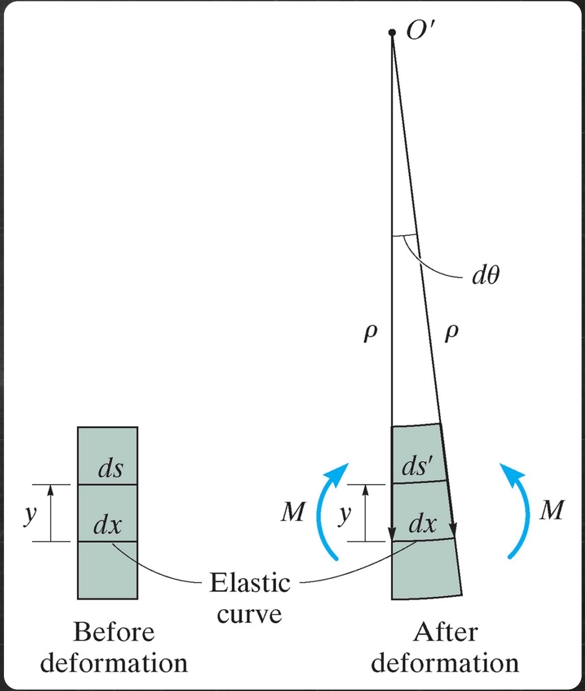
$\sigma = E\varepsilon$ $\quad \sigma = -My/I$ ← FLEXURE FORMULA

$$\frac{1}{\rho} = -\frac{\varepsilon}{y} = -\frac{\sigma}{Ey} = \frac{M}{EI}$$

$$\boxed{\frac{1}{\rho} = \frac{M}{EI}}$$

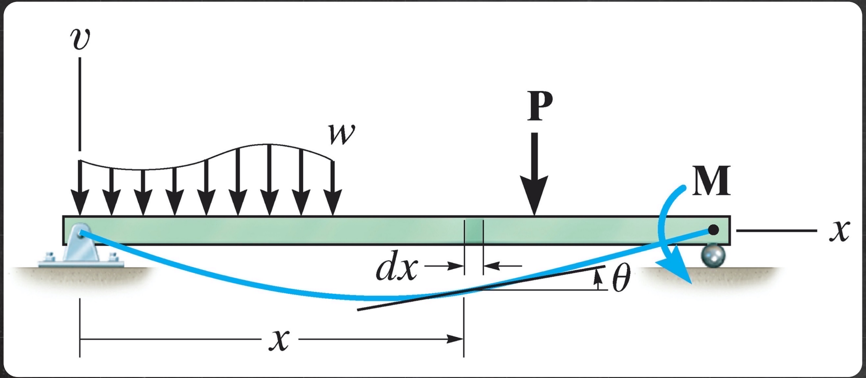
Moment-Curvature Relationship

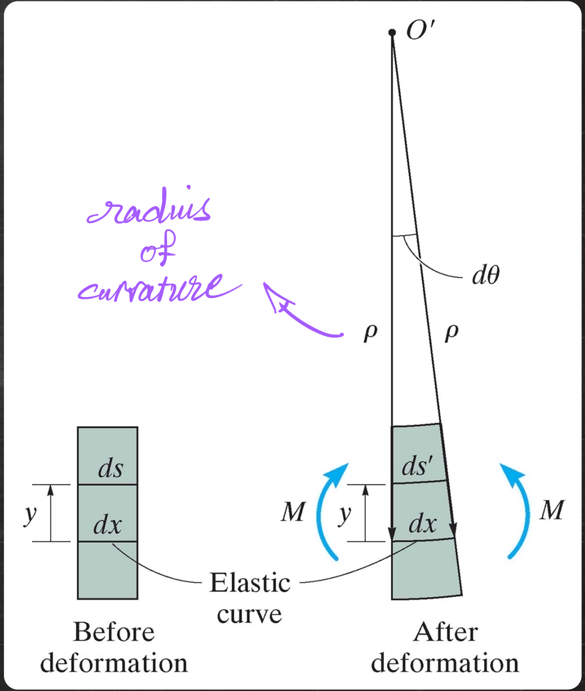




$$\boxed{\frac{1}{\rho} = \frac{M}{EI}}$$

MOMENT-CURVATURE RELATIONSHIP

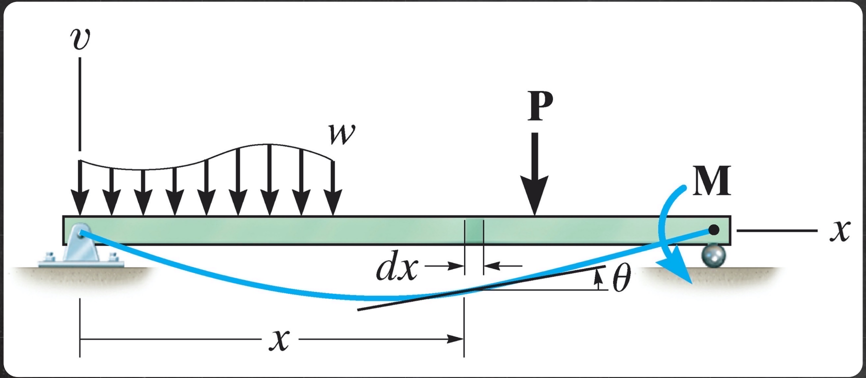


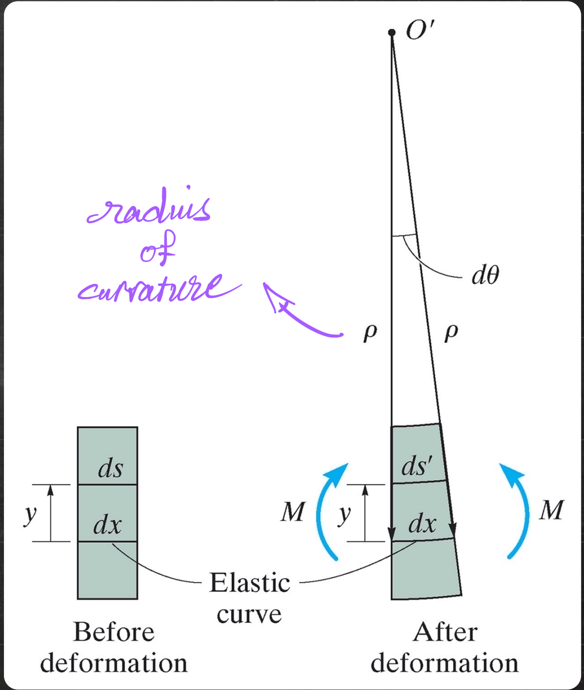


ρ: radius of curvature

$$\boxed{\frac{1}{\rho} = \frac{M}{EI}}$$

MOMENT-CURVATURE RELATIONSHIP





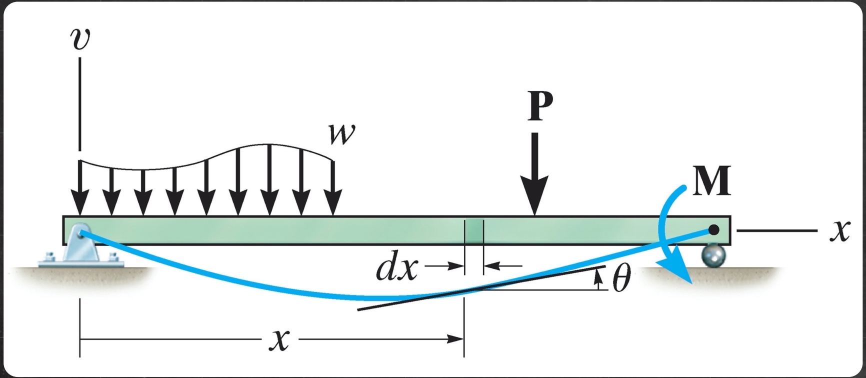
$$\frac{1}{\rho} = \frac{d^2v/dx^2}{\left[1+\left(dv/dx\right)^2\right]^{3/2}}$$

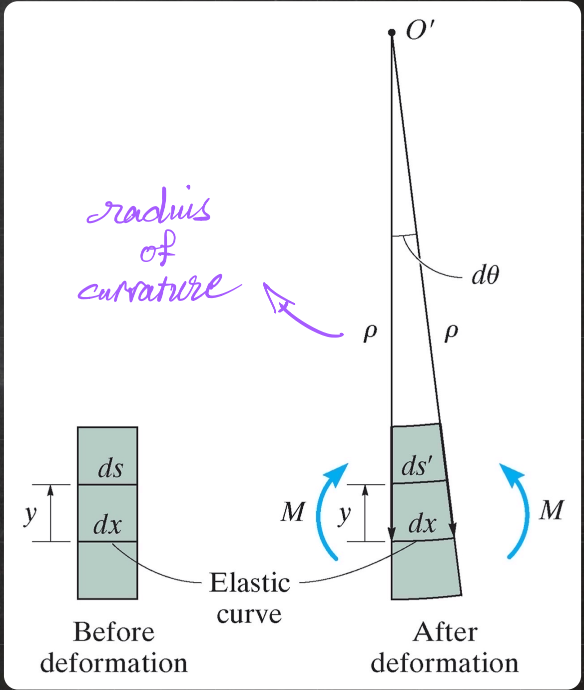
ρ: radius of curvature

$$\boxed{\frac{1}{\rho} = \frac{M}{EI}}$$

MOMENT-CURVATURE RELATIONSHIP

$$\Rightarrow \frac{v''}{[1+v'^2]^{3/2}} = \frac{M}{EI}$$





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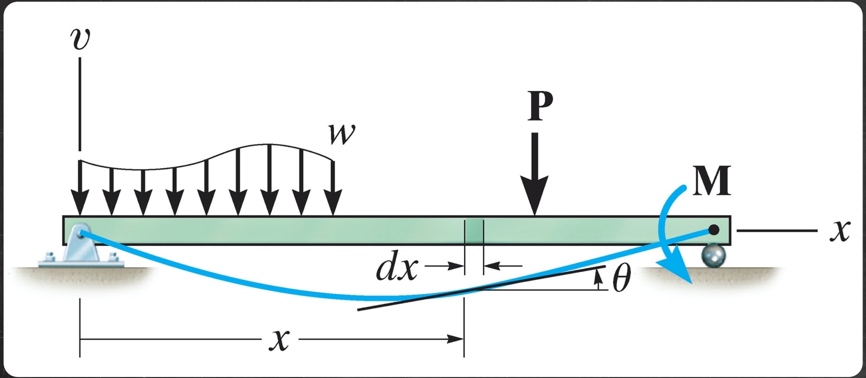
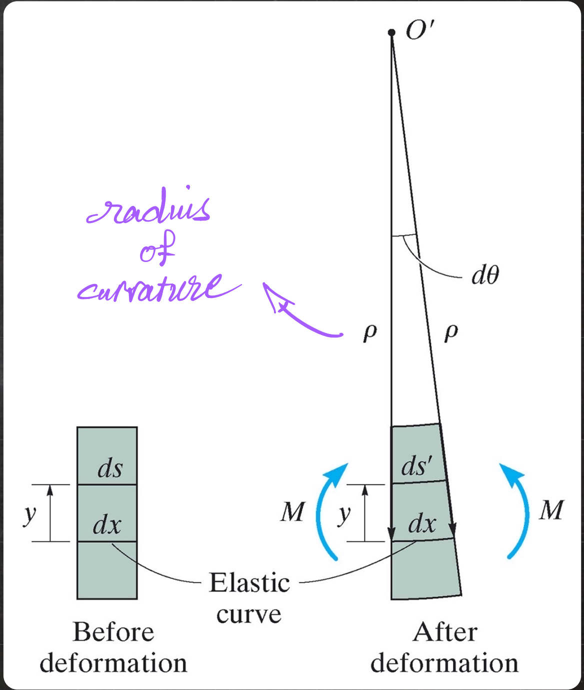




radius of curvature

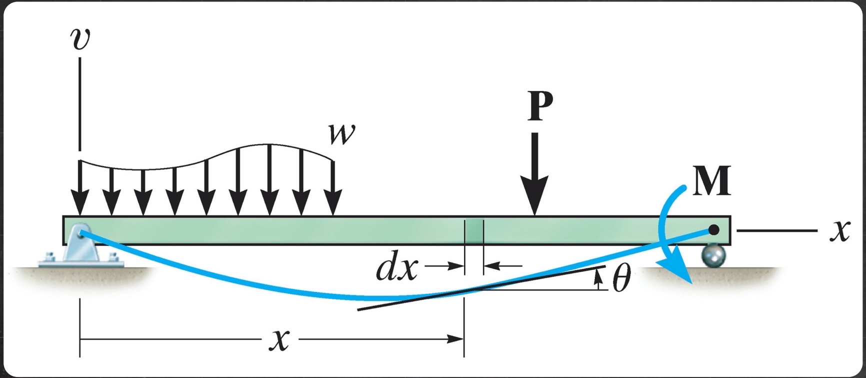
$v' \ll 1$

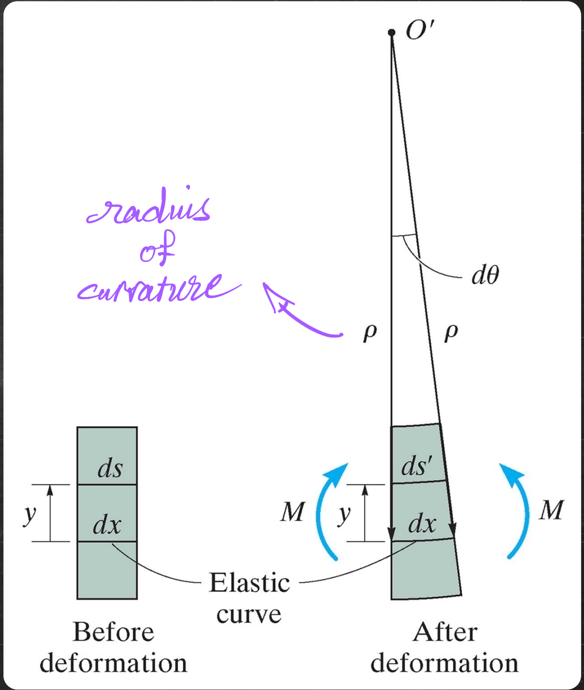
$\Downarrow$

v'^2 : negligible

MOMENT-CURVATURE RELATIONSHIP

$$\Rightarrow \frac{v''}{[1+v'^2]^{3/2}} = \frac{M}{EI}$$





radius of curvature

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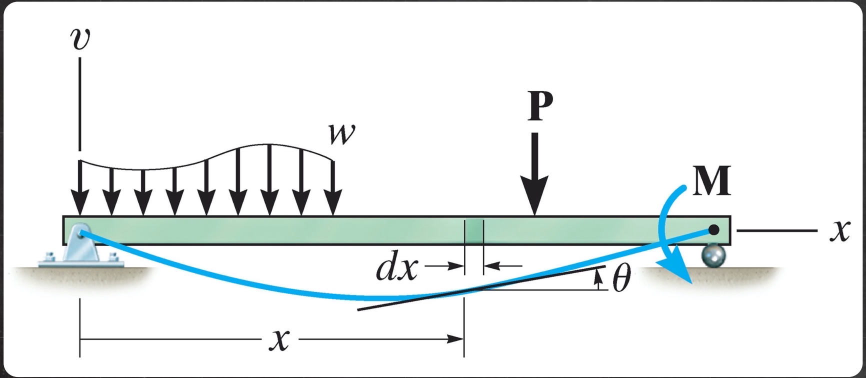
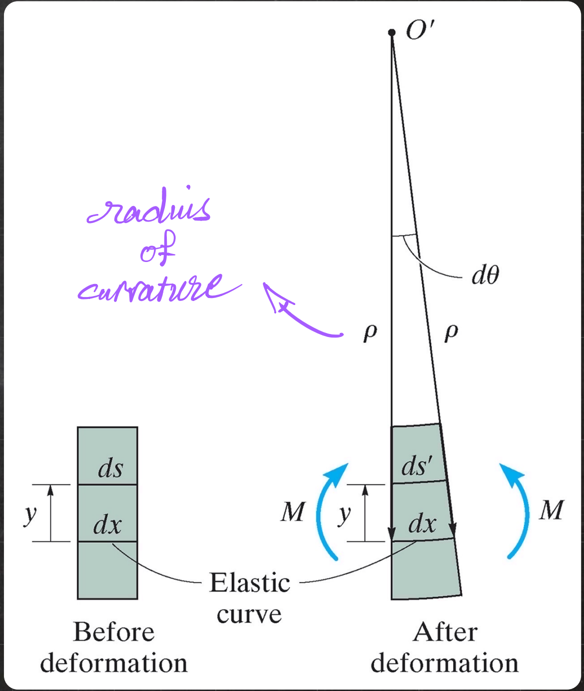
$$\Rightarrow \frac{d^2 v}{dx^2} = \frac{M}{EI}$$

MOMENT-CURVATURE RELATIONSHIP

$$\Rightarrow \frac{v''}{[1+v'^2]^{3/2}} = \frac{M}{EI}$$





$v' \ll 1$

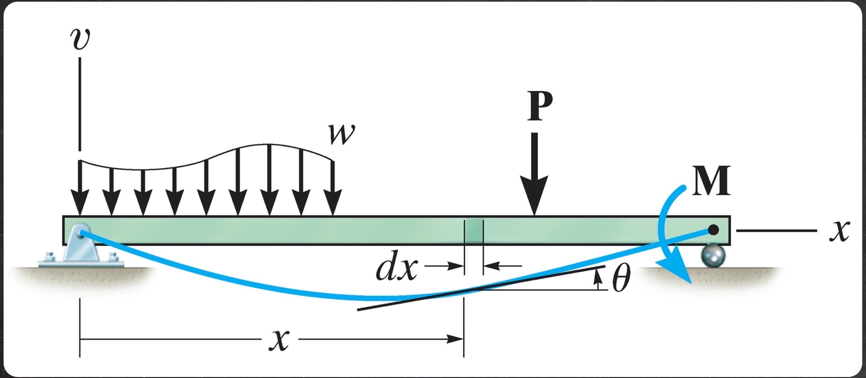
$\Downarrow$

v'^2 : negligible

$$\Rightarrow \frac{d^2 v}{dx^2} = \frac{M}{\underbrace{EI}}$$

FLEXURAL STIFFNESS

MOMENT-CURVATURE RELATIONSHIP



$$EI \frac{d^2v}{dx^2} = M(x)$$

$v' \ll 1$

$\Downarrow$

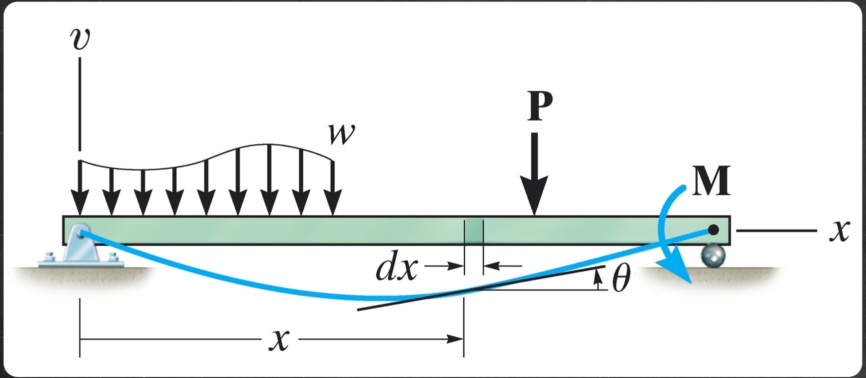
v'^2 : negligible

$$\Rightarrow \frac{d^2v}{dx^2} = \frac{M}{\underbrace{EI}}$$

FLEXURAL STIFFNESS

MOMENT-CURVATURE RELATIONSHIP

RECALL: $\frac{dM}{dx} = V$, $\frac{dV}{dx} = w$



$$EI \frac{d^2v}{dx^2} = M(x)$$

$v' \ll 1$

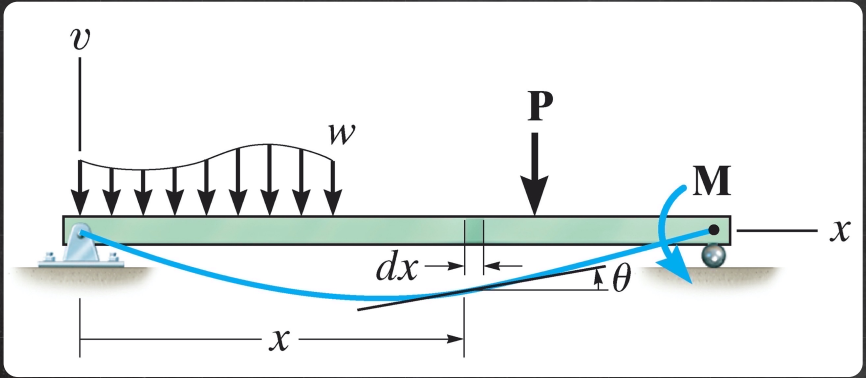
$\Downarrow$

v'^2 : negligible

$$\Rightarrow \frac{d^2v}{dx^2} = \frac{M}{\underbrace{EI}_{\text{FLEXURAL STIFFNESS}}}$$

MOMENT-CURVATURE RELATIONSHIP

RECALL: $\frac{dM}{dx} = V$, $\frac{dV}{dx} = w$



$$EI \frac{d^2 v}{dx^2} = M(x)$$

$$\frac{d}{dx}\left(EI \frac{d^2 v}{dx^2}\right) = V(x)$$

$v' \ll 1$

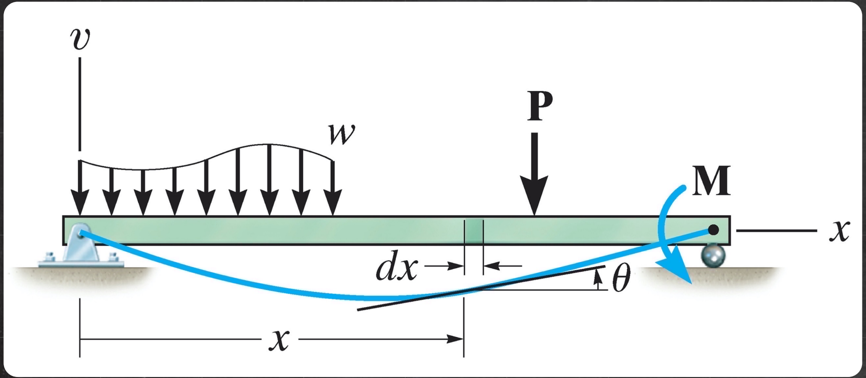
$\Downarrow$

v'^2 : negligible

$$\Rightarrow \frac{d^2 v}{dx^2} = \frac{M}{\underbrace{EI}_{\text{FLEXURAL STIFFNESS}}}$$

MOMENT-CURVATURE RELATIONSHIP

RECALL: $\frac{dM}{dx} = V$, $\frac{dV}{dx} = w$



$$EI \frac{d^2v}{dx^2} = M(x)$$

$$\frac{d}{dx}\left(EI \frac{d^2v}{dx^2}\right) = V(x)$$

$$\frac{d^2}{dx^2}\left(EI \frac{d^2v}{dx^2}\right) = w(x)$$

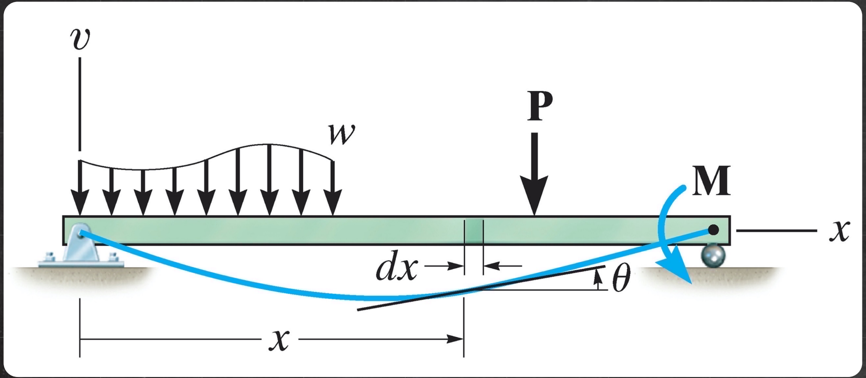
$v' \ll 1$

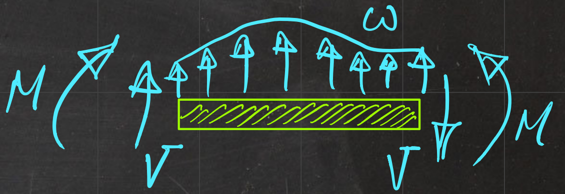
$\Downarrow$

v'^2 : negligible

$$\Rightarrow \frac{d^2v}{dx^2} = \frac{M}{\underbrace{EI}_{\text{FLEXURAL STIFFNESS}}}$$

MOMENT-CURVATURE RELATIONSHIP





$$EI\frac{d^2v}{dx^2} = M(x)$$

$$\frac{d}{dx}\left(EI\frac{d^2v}{dx^2}\right) = V(x)$$

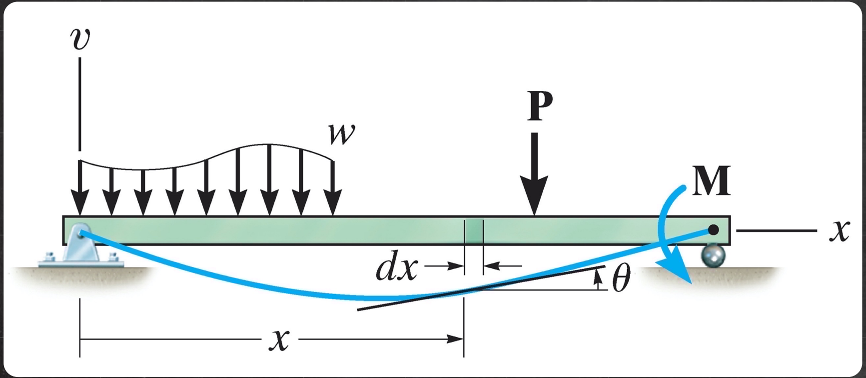
$$\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right) = \omega(x)$$

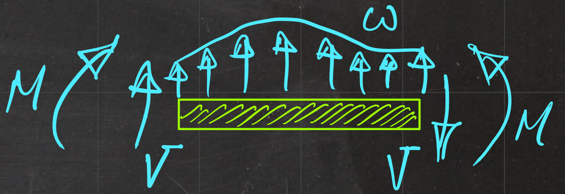
NOTE:

$v \rightarrow$ deflection (small v)

$V \rightarrow$ shear force (capital V)

Moment-Curvature Relationship





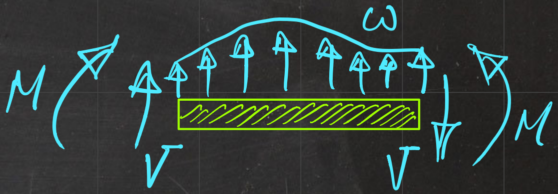
$$EI\,\frac{d^2v}{dx^2} = M(x)$$

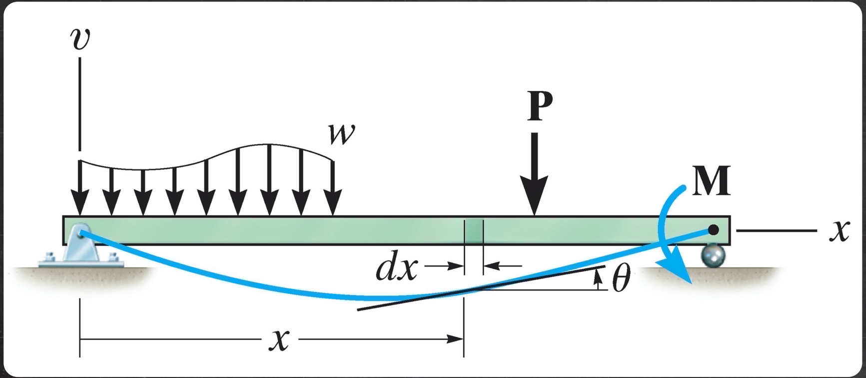
$$\frac{d}{dx}\left(EI\,\frac{d^2v}{dx^2}\right) = V(x)$$

$$\frac{d^2}{dx^2}\left(EI\,\frac{d^2v}{dx^2}\right) = \omega(x)$$

$$\omega(x) \xrightarrow{\int} V(x) \xrightarrow{\int} M(x)$$

Moment-Curvature Relationship





$$EI\frac{d^2v}{dx^2} = M(x)$$

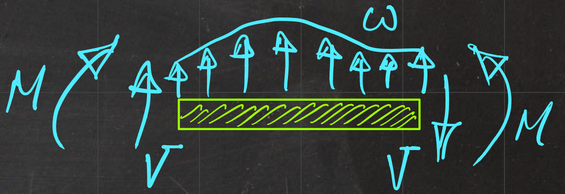
$$\frac{d}{dx}\left(EI\frac{d^2v}{dx^2}\right) = V(x)$$

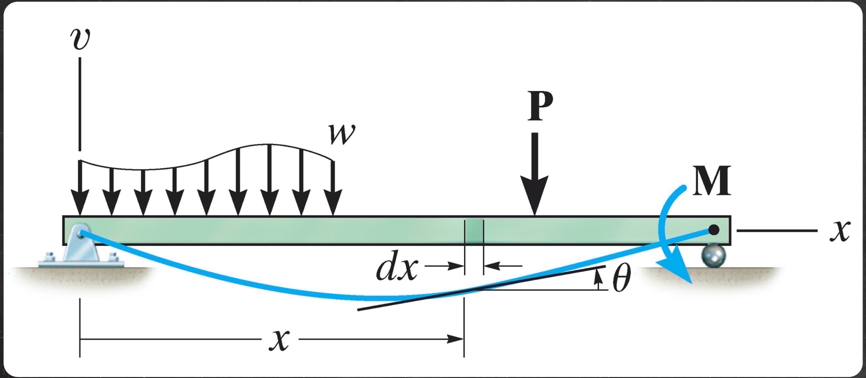
$$\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right) = \omega(x)$$

$$\omega(x) \xrightarrow{\int} V(x) \xrightarrow{\int} M(x)$$

$$\frac{M(x)}{EI} \xrightarrow{\int} \theta(x) \xrightarrow{\int} v(x)$$

Moment-Curvature Relationship





$$EI\,\frac{d^2 v}{dx^2} = M(x)$$

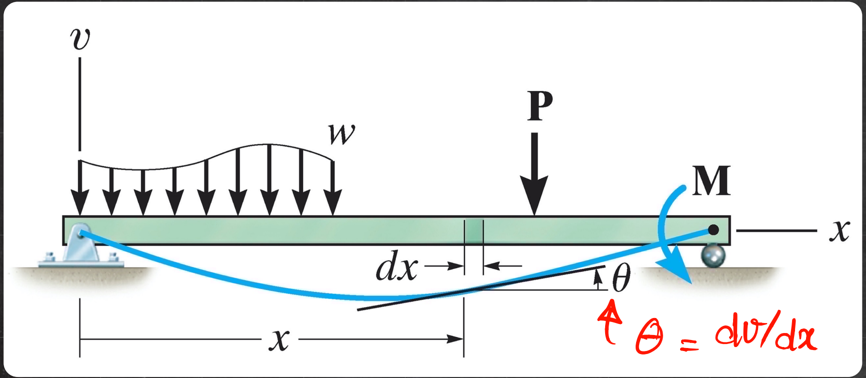
$$\frac{d}{dx}\left(EI\,\frac{d^2 v}{dx^2}\right) = V(x)$$

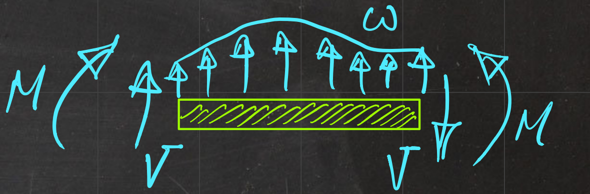
$$\frac{d^2}{dx^2}\left(EI\,\frac{d^2 v}{dx^2}\right) = \omega(x)$$

$$\omega(x) \underset{d/dx}{\overset{\int}{\rightleftarrows}} V(x) \underset{d/dx}{\overset{\int}{\rightleftarrows}} M(x)$$

$$\frac{M(x)}{EI} \underset{d/dx}{\overset{\int}{\rightleftarrows}} \theta(x) \underset{d/dx}{\overset{\int}{\rightleftarrows}} v(x)$$

Moment-Curvature Relationship





$$EI\,\frac{d^2v}{dx^2} = M(x)$$

$$\frac{d}{dx}\left(EI\,\frac{d^2v}{dx^2}\right) = V(x)$$

$$\frac{d^2}{dx^2}\left(EI\,\frac{d^2v}{dx^2}\right) = w(x)$$

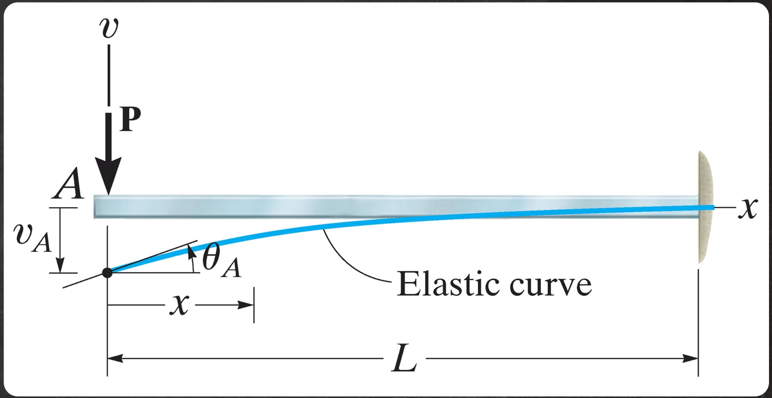
$$w(x) \underset{d/dx}{\overset{\int}{\rightleftarrows}} V(x) \underset{d/dx}{\overset{\int}{\rightleftarrows}} M(x)$$

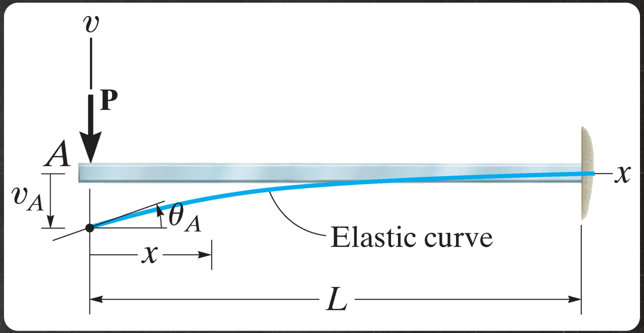
dv/dx

$$\frac{M(x)}{EI} \underset{d/dx}{\overset{\int}{\rightleftarrows}} \theta(x) \underset{d/dx}{\overset{\int}{\rightleftarrows}} v(x)$$

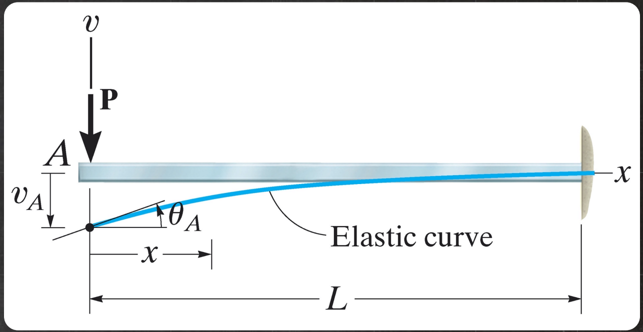
Exercise 1 . [similar to ... P. 582 ... 12.1]

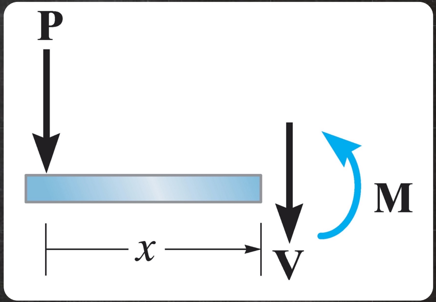
THE CANTILEVERED BEAM SHOWN IN THE FIGURE IS SUBJECTED TO A VERTICAL LOAD P AT ITS END. DETERMINE THE EQUATION OF THE ELASTIC CURVE. EI IS CONSTANT.



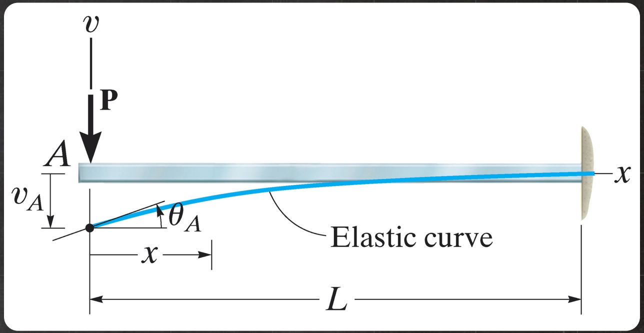

v
P
A
x
v_A
θ_A
Elastic curve
x
L

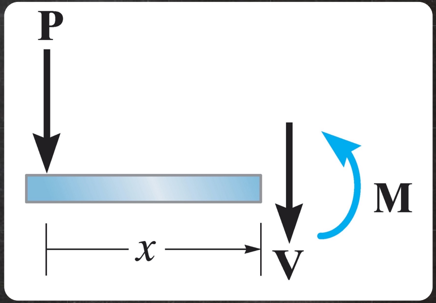
$$M(x) = -Px$$





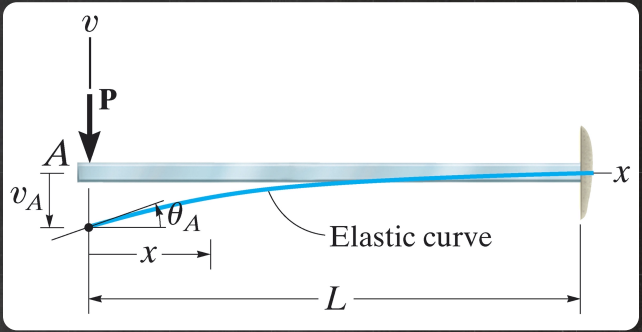
$EI\, v'' = M$ ✓ $M(x) = -Px$

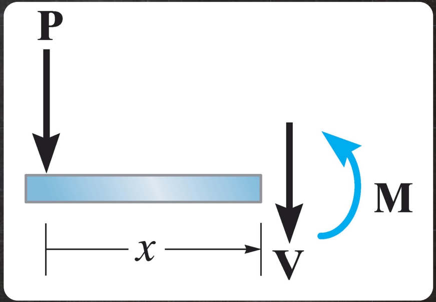




$$EI\,v'' = M \qquad M(x) = -Px$$

$$EI\,v'' = -Px$$

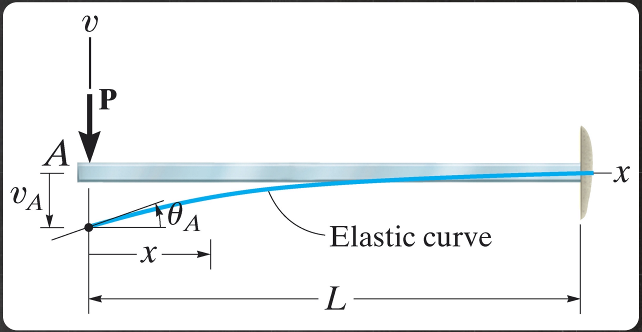


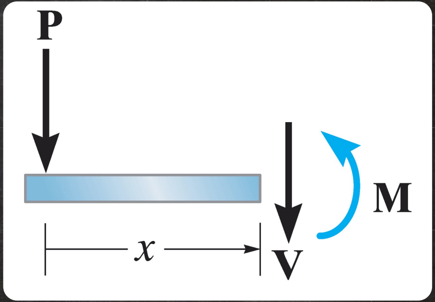


$EI\, v'' = M \checkmark \quad M(x) = -Px$

$EI\, v'' = -Px$

$EI\, v' = -\frac{1}{2}Px^2 + C_1$



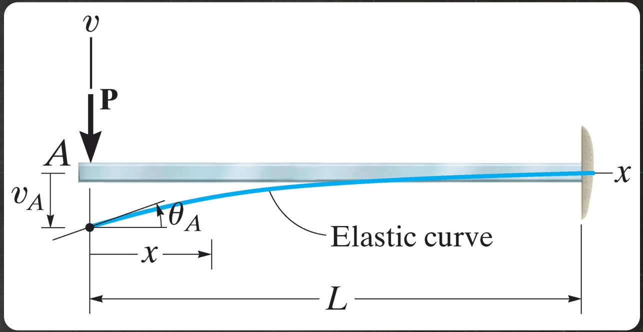


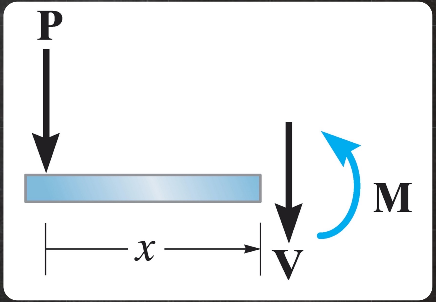
$$EI\,v'' = M \qquad M(x) = -Px$$

$$EI\,v'' = -Px$$

$$EI\,v' = -\frac{1}{2}Px^2 + C_1$$

$$EI\,v = -\frac{1}{6}Px^3 + C_1 x + C_2$$



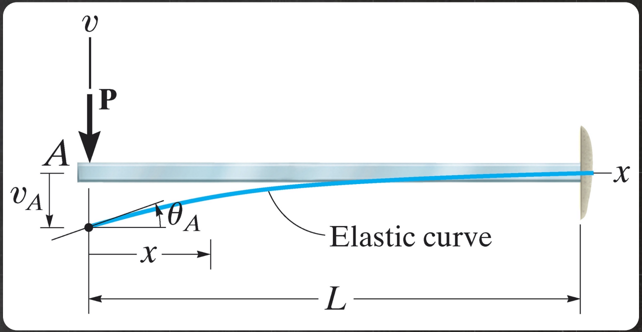


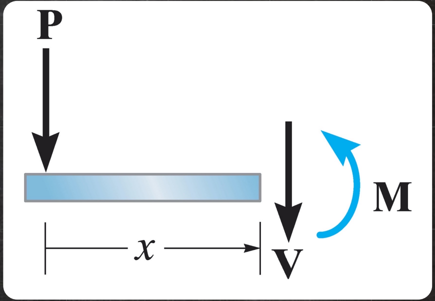
$$EI\, v'' = M \quad \checkmark \quad M(x) = -Px$$

$$EI\, v'' = -Px$$

$$EI\, v' = -\frac{1}{2} P x^2 + C_1$$

$$EI\, v = -\frac{1}{6} P x^3 + C_1 x + C_2 \quad \checkmark \quad C_1, C_2 = ?$$





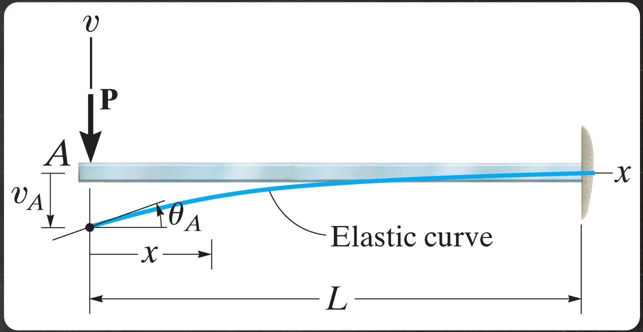
$EI\, v'' = M \quad M(x) = -Px$

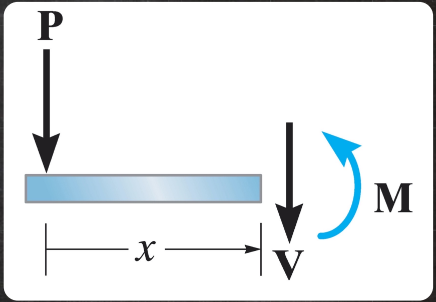
$EI\, v'' = -Px$

$EI\, v' = -\frac{1}{2}Px^2 + C_1$

$EI\, v = -\frac{1}{6}Px^3 + C_1 x + C_2 \quad C_1, C_2 = ?$

BCs: $v(x=L) = 0$, $v'(x=L) = 0$



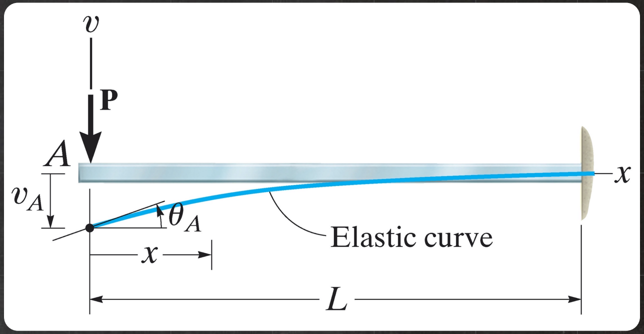


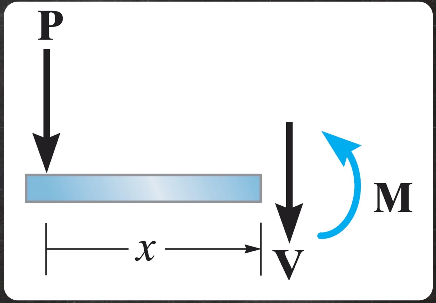
$$EI\, v'' = M \quad \checkmark \quad M(x) = -Px$$

$$EI\, v'' = -Px$$

$$EI\, v' = -\frac{1}{2}Px^2 + C_1$$

$$EI\, v = -\frac{1}{6}Px^3 + C_1 x + C_2 \quad \checkmark \quad C_1, C_2 = ?$$





BCs: $v(x=L) = 0 \quad , \quad v'(x=L) = 0$

$$\Rightarrow C_1 = PL^2/2 \quad , \quad C_2 = -PL^3/3$$

$EI\,v'' = M$ ✓ $M(x) = -Px$

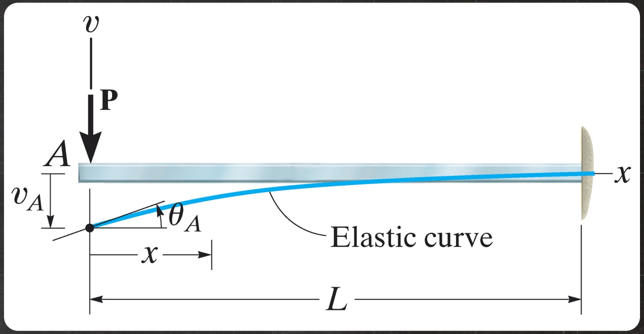
$EI\,v'' = -Px$

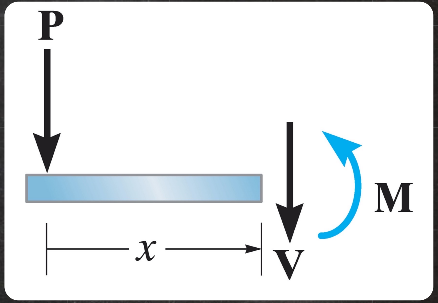
$EI\,v' = -\frac{1}{2}Px^2 + C_1$

$EI\,v = -\frac{1}{6}Px^3 + C_1x + C_2$

$\Rightarrow v = \frac{1}{6}\frac{P}{EI}\left(-x^3 + 3L^2x - 2L^3\right)$

↳ ELASTIC CURVE EQUATION





$EI\,v'' = M \quad M(x) = -Px$

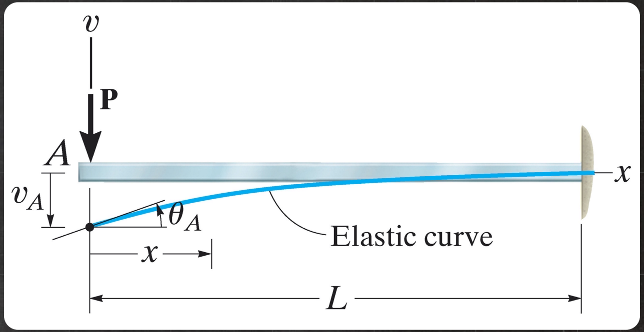
$EI\,v'' = -Px$

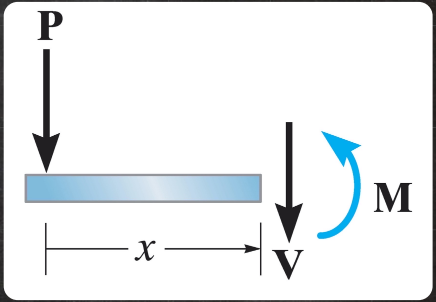
$EI\,v' = -\frac{1}{2}Px^2 + C_1$

$EI\,v = -\frac{1}{6}Px^3 + C_1 x + C_2$

$\Rightarrow v = \frac{1}{6}\frac{P}{EI}\left(-x^3 + 3L^2 x - 2L^3\right)$

↳ ELASTIC CURVE EQUATION $\Rightarrow v' = \theta = \frac{P}{2EI}\left(L^2 - x^2\right)$



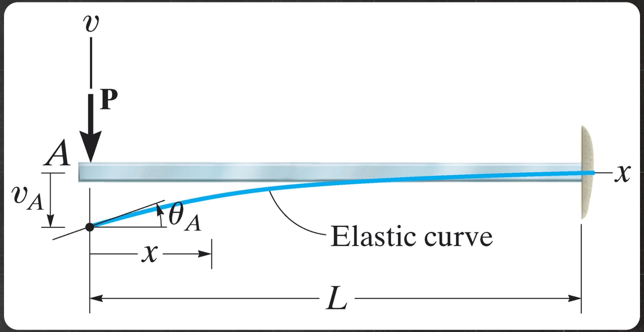


$$EI\,v'' = M \qquad M(x) = -Px$$

$$EI\,v'' = -Px$$

$$EI\,v' = -\frac{1}{2}Px^2 + C_1$$

$$EI\,v = -\frac{1}{6}Px^3 + C_1 x + C_2$$



max. deflection and slope

$$@\,x=0 \Rightarrow \begin{cases} v_A = -\dfrac{PL^3}{3EI} \\[2ex] \theta_A = \dfrac{PL^2}{2EI} \end{cases}$$

$$\Rightarrow v = \frac{1}{6}\frac{P}{EI}\left(-x^3 + 3L^2x - 2L^3\right)$$

ELASTIC CURVE EQUATION

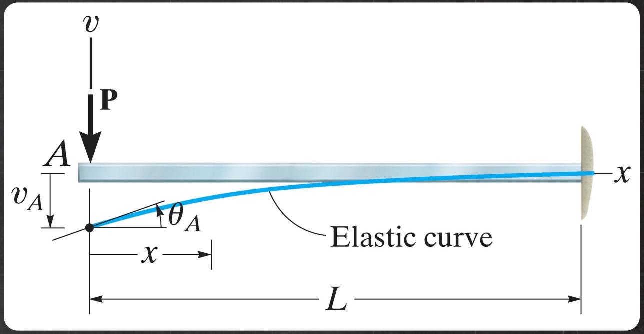
$$\Rightarrow v' = \theta = \frac{P}{2EI}\left(L^2 - x^2\right)$$

$$EI\,v'' = M \quad M(x) = -Px$$

$$EI\,v'' = -Px$$

$$EI\,v' = -\frac{1}{2}Px^2 + C_1$$

$$EI\,v = -\frac{1}{6}Px^3 + C_1 x + C_2$$



max. deflection and slope

$$@\, x=0 \Rightarrow \begin{cases} v_A = -\dfrac{PL^3}{3EI} \quad (-) \checkmark \\[2ex] \theta_A = \dfrac{PL^2}{2EI} \quad (+) \checkmark \end{cases}$$

$$\Rightarrow v = \frac{1}{6}\frac{P}{EI}\left(-x^3 + 3L^2x - 2L^3\right)$$

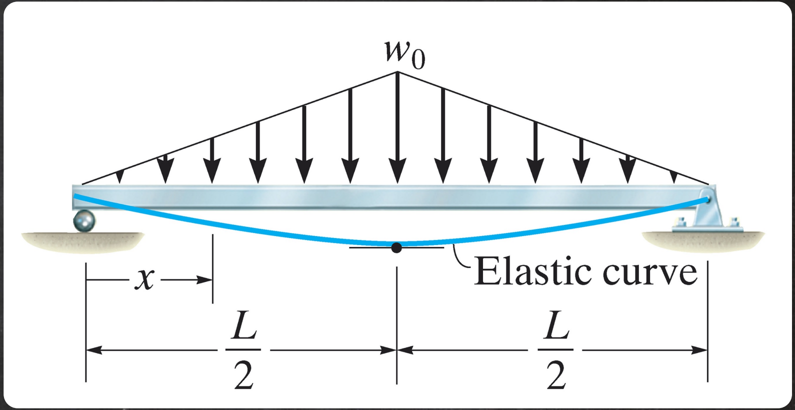
↳ ELASTIC CURVE EQUATION

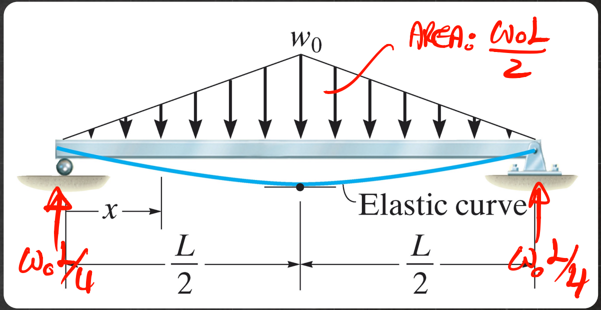
$$\Rightarrow v' = \theta = \frac{P}{2EI}\left(L^2 - x^2\right)$$

Exercise 2 . [similar to ... P. 584 ... 12.2]

DETERMINE THE EQUATION OF THE ELASTIC CURVE FOR THE BEAM SHOWN IN THE FIGURE.

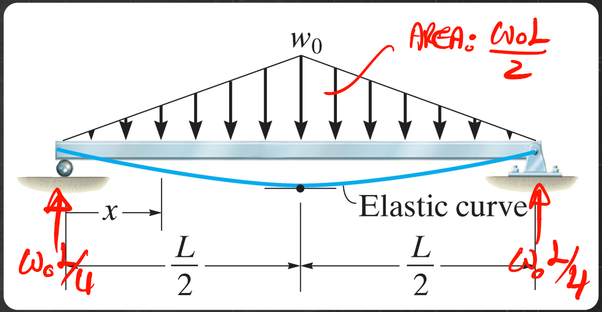
EI IS CONSTANT.





w_0
AREA: $\frac{w_0 L}{2}$
Elastic curve
x
$\frac{L}{2}$
$\frac{L}{2}$
$w_0 L/4$
$w_0 L/4$

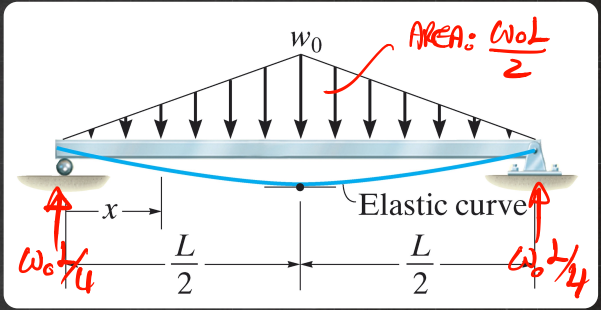
SYMMETRY: $0 \leq x \leq L/2$

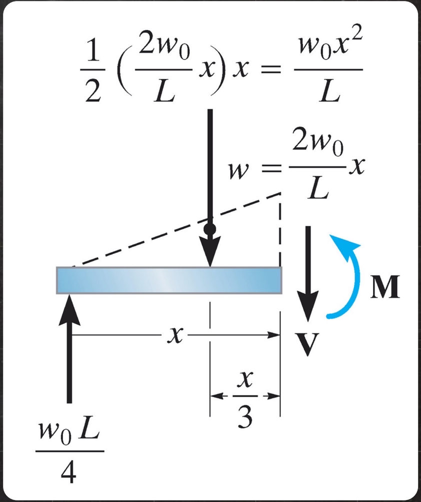


SYMMETRY: $0 \leqslant x \leqslant L/2$

$$\sum M = 0$$

$$\circlearrowleft_0 \quad M + \frac{w_0 x^2}{L}\frac{x}{3} - \frac{w_0 L}{4} x = 0$$





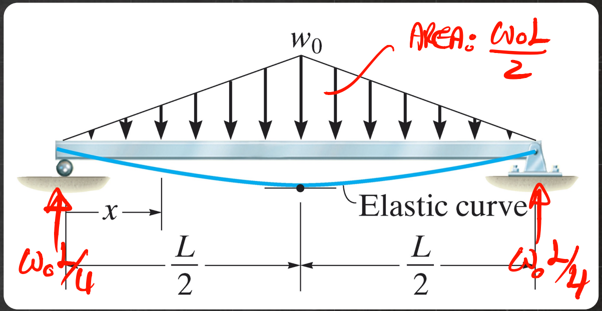
SYMMETRY: $0 \leq x \leq L/2$

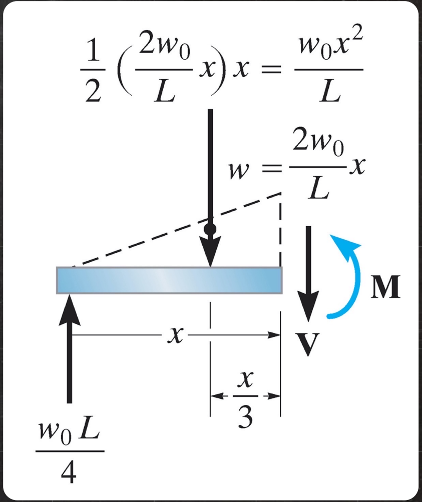
$$\sum M = 0$$

$$\circlearrowleft_o \; M + \frac{\omega_o x^2}{L}\frac{x}{3} - \frac{\omega_o L}{4} x = 0$$

$$\Downarrow$$

$$M = -\frac{\omega_o x^3}{3L} + \frac{\omega_o L}{4} x$$





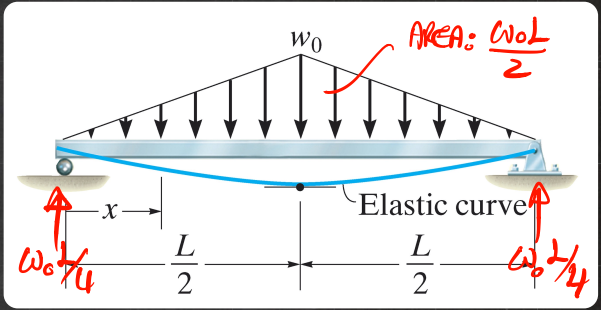
SYMMETRY: $0 \leq x \leq L/2$

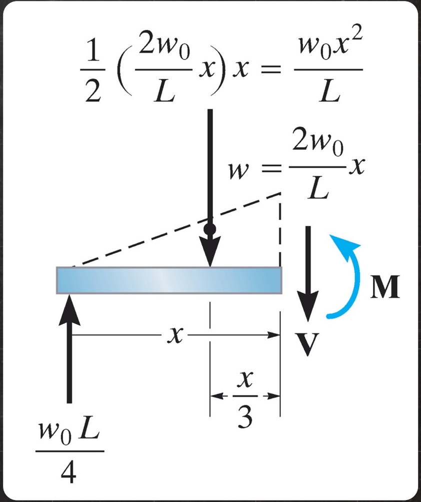
$$EI\, v'' = M \qquad \Sigma M = 0$$

$$\circlearrowleft_0 \quad M + \frac{w_0 x^2}{L}\frac{x}{3} - \frac{w_0 L}{4} x = 0$$

$$\Downarrow$$

$$M = -\frac{w_0 x^3}{3L} + \frac{w_0 L}{4} x$$

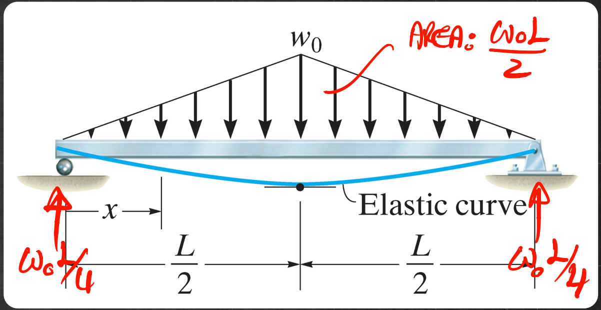


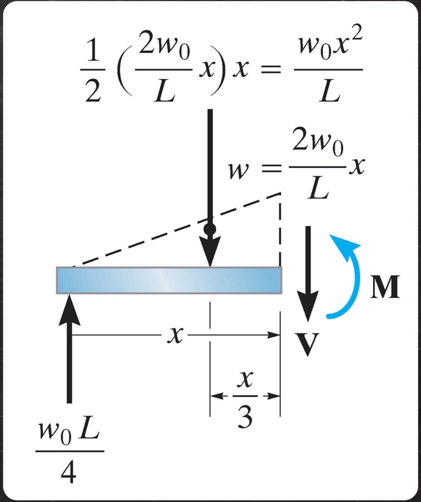


$$EI\, v'' = M$$

$$\Downarrow$$

$$EI\, v'' = -\frac{\omega_0 x^3}{3L} + \frac{\omega_0 L}{4} x$$



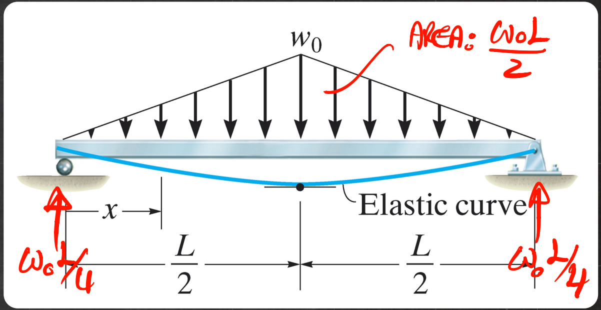


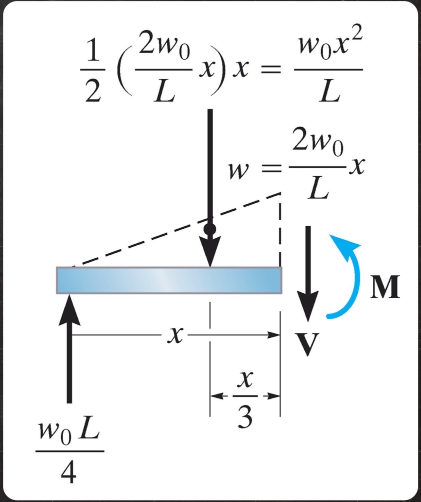
$$EI\,v'' = M$$

$$\Downarrow$$

$$EI\,v'' = -\frac{\omega_0 x^3}{3L} + \frac{\omega_0 L}{4}x$$

$$EI\,v' = -\frac{\omega_0}{12L}x^4 + \frac{\omega_0 L}{8}x^2 + C_1$$





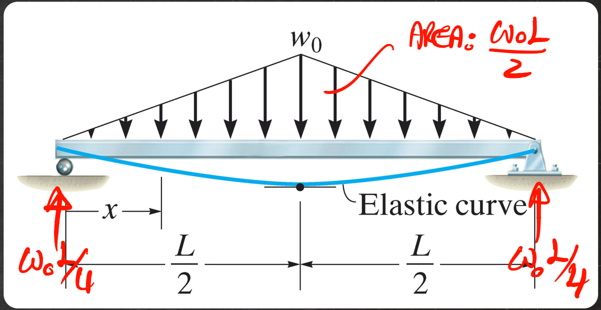
$$EI\,v'' = M$$

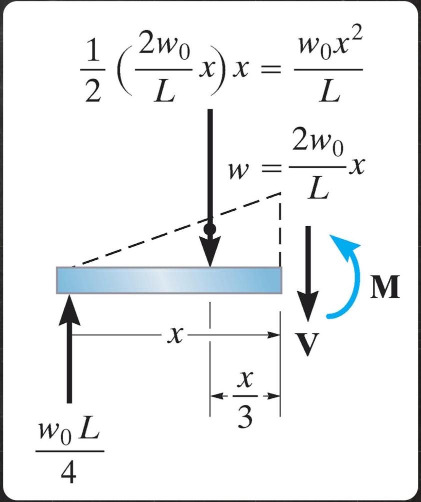
$$\Downarrow$$

$$EI\,v'' = -\frac{w_0 x^3}{3L} + \frac{w_0 L}{4} x$$

$$EI\,v' = -\frac{w_0}{12L} x^4 + \frac{w_0 L}{8} x^2 + C_1$$

$$EI\,v = -\frac{w_0}{60L} x^5 + \frac{w_0 L}{24} x^3 + C_1 x + C_2$$





$$EI\, v'' = M$$

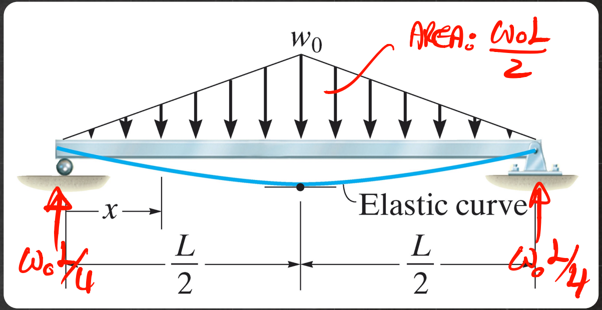
$$\Downarrow$$

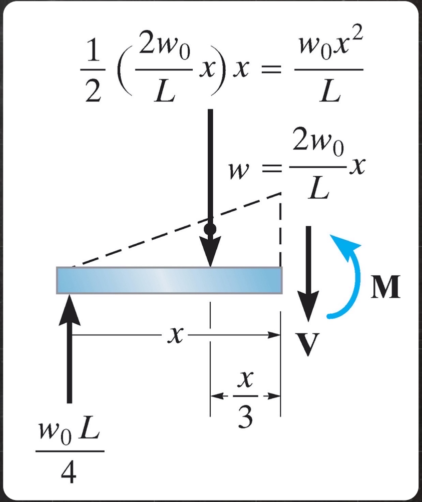
$$EI\, v'' = -\frac{w_0 x^3}{3L} + \frac{w_0 L}{4} x$$

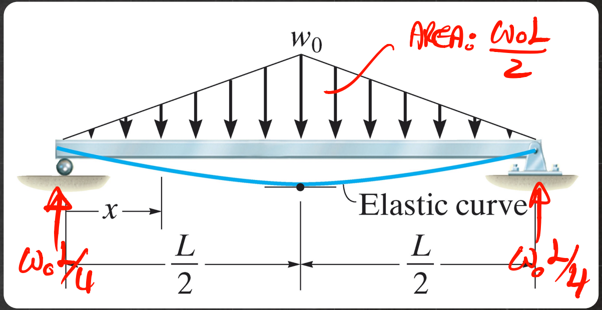
$$EI\, v' = -\frac{w_0}{12L} x^4 + \frac{w_0 L}{8} x^2 + C_1$$

from boundary conditions

$$EI\, v = -\frac{w_0}{60L} x^5 + \frac{w_0 L}{24} x^3 + C_1 x + C_2$$







$$EI\, v'' = M$$

$$\Downarrow$$

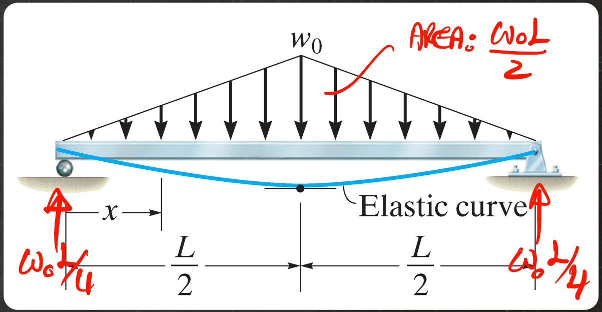
$$EI\, v'' = -\frac{w_0 x^3}{3L} + \frac{w_0 L}{4} x$$

$$EI\, v' = -\frac{w_0}{12L} x^4 + \frac{w_0 L}{8} x^2 + C_1$$

$$EI\, v = -\frac{w_0}{60L} x^5 + \frac{w_0 L}{24} x^3 + C_1 x + C_2$$

from boundary conditions

$$v(0) = 0, \quad v'(L/2) = 0$$



$$EI\,v'' = M$$

$$\Downarrow$$

$$EI\,v'' = -\frac{w_0 x^3}{3L} + \frac{w_0 L}{4} x$$

$$EI\,v' = -\frac{w_0}{12L} x^4 + \frac{w_0 L}{8} x^2 + C_1$$

$$EI\,v = -\frac{w_0}{60L} x^5 + \frac{w_0 L}{24} x^3 + C_1 x + C_2$$

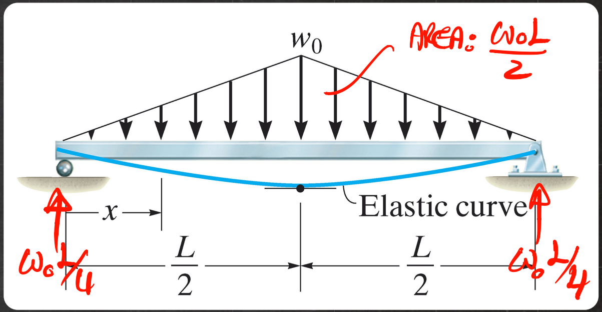
from boundary conditions

$$v(0) = 0, \quad v'(L/2) = 0$$

$$\Downarrow$$

$$C_1 = -\frac{5 w_0 L^3}{192}$$

$$C_2 = 0$$



$$v(0) = 0, \quad v'(L/2) = 0$$

$$\Downarrow$$

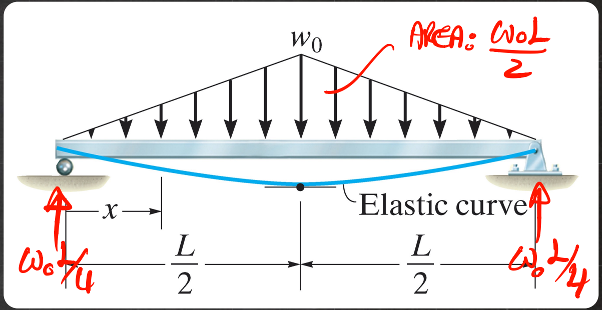
$$EI\,v = -\frac{w_0}{60L}x^5 + \frac{w_0 L}{24}x^3 - \frac{5 w_0 L^3}{192}x$$

$$C_1 = -\frac{5 w_0 L^3}{192}$$

$$\Uparrow$$

$$EI\,v = -\frac{w_0}{60L}x^5 + \frac{w_0 L}{24}x^3 + C_1 x + C_2$$

$$C_2 = 0$$



$$v_{max} = v(x = L/2)$$

$$\Uparrow$$

$$EI\,v = -\frac{\omega_0}{60L}x^5 + \frac{\omega_0 L}{24}x^3 - \frac{5\omega_0 L^3}{192}x$$

$$\Uparrow$$

$$EI\,v = -\frac{\omega_0}{60L}x^5 + \frac{\omega_0 L}{24}x^3 + C_1 x + C_2$$

$$v(0) = 0, \quad v'(L/2) = 0$$

$$\Downarrow$$

$$C_1 = -\frac{5\omega_0 L^3}{192}$$

$$C_2 = 0$$

$$v_{max} = -\frac{\omega_0 L^4}{120EI}$$

$$\Uparrow$$

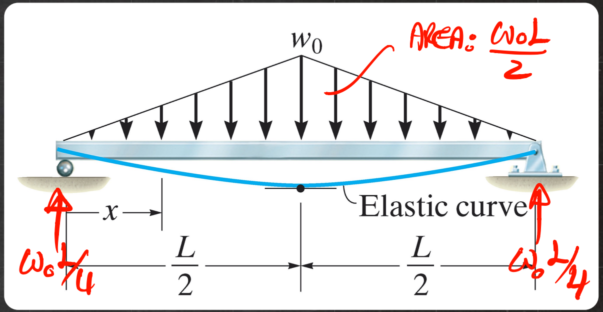
$$v_{max} = v(x = L/2)$$

$$\Uparrow$$

$$EIv = -\frac{\omega_0}{60L}x^5 + \frac{\omega_0 L}{24}x^3 - \frac{5\omega_0 L^3}{192}x$$

$$\Uparrow$$

$$EIv = -\frac{\omega_0}{60L}x^5 + \frac{\omega_0 L}{24}x^3 + C_1 x + C_2$$



$$v(0) = 0, \quad v'(L/2) = 0$$

$$\Downarrow$$

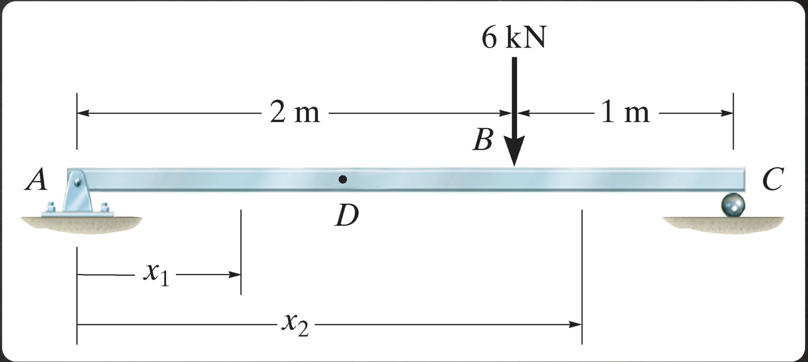
$$C_1 = -\frac{5\omega_0 L^3}{192}$$

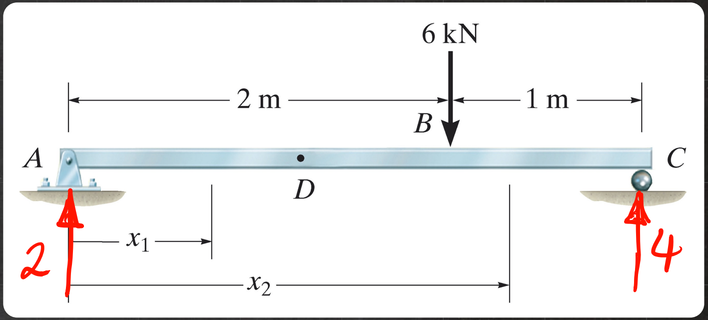
$$C_2 = 0$$

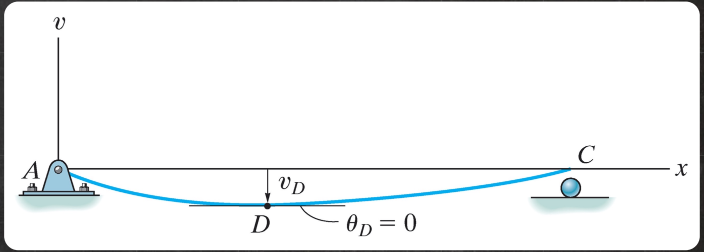
Exercise 3 . [similar to … P. 586 … 12.3]

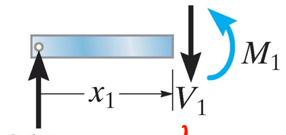
DETERMINE THE EQUATION OF THE ELASTIC CURVE FOR THE BEAM SHOWN IN THE FIGURE AND THE MAXIMUM DEFLECTION.

EI IS CONSTANT.




6 kN
2 m
1 m
B
A
C
D
x_1
x_2
2
4


v
A
C
x
v_D
D
$\theta_D = 0$



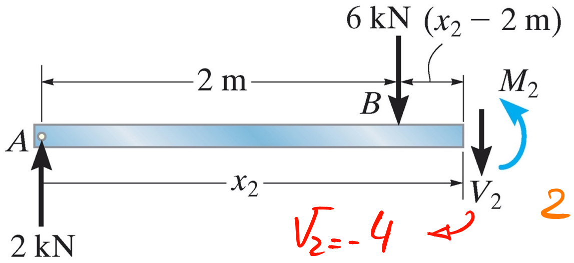
$V_1 = 2$

$0 \leqslant x_1 \leqslant 2$

$M_1 = 2x_1$

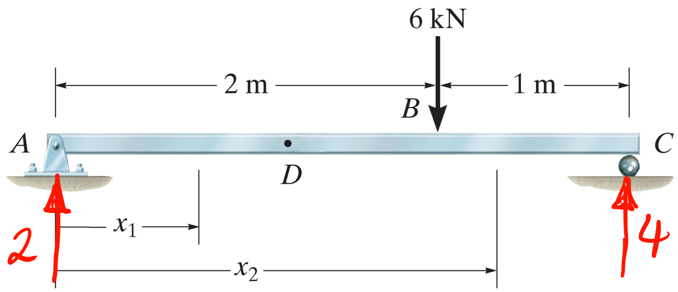
$M_2 = 2x_2 - 6(x_2 - 2)$

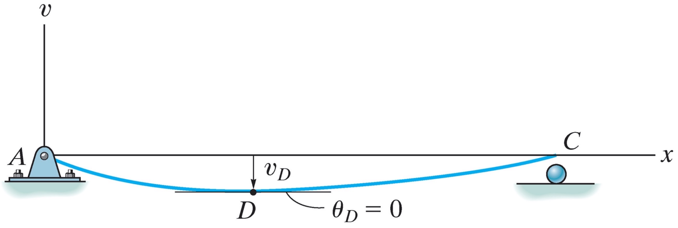
$\Rightarrow M_2 = 4(3 - x_2)$



$V_2 = -4$

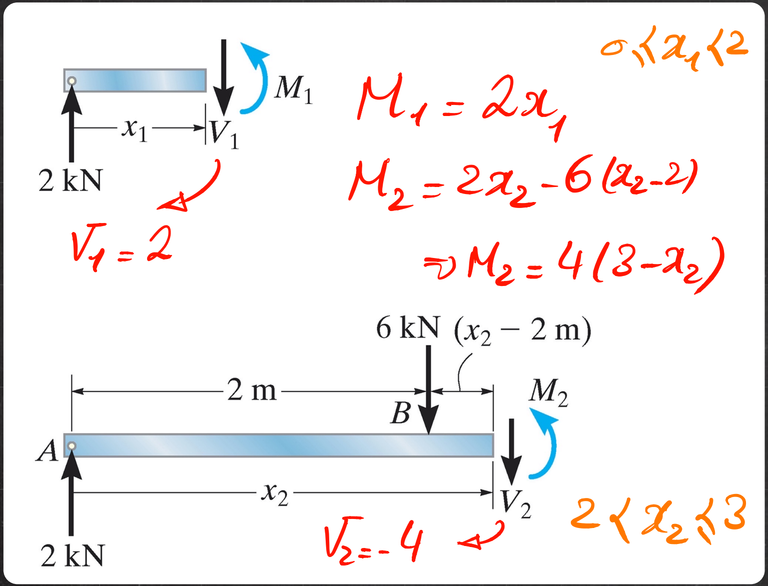
$2 < x_2 \leqslant 3$

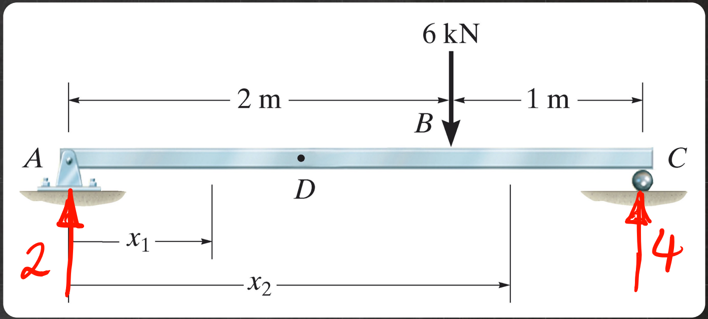


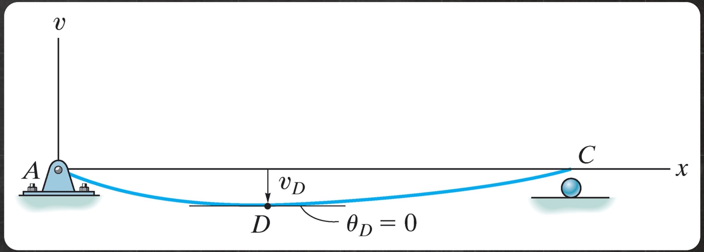


$0 \leqslant x_1 \leqslant 2$

$2 < x_2 \leqslant 3$





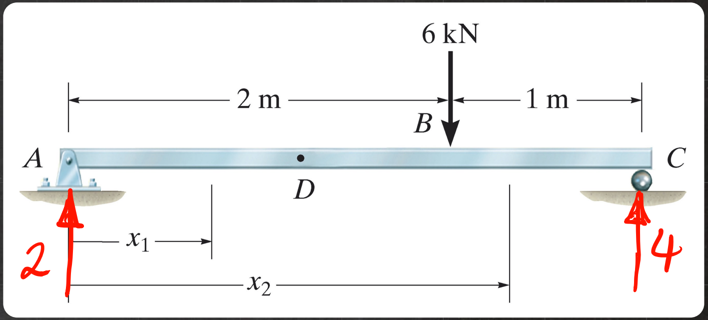


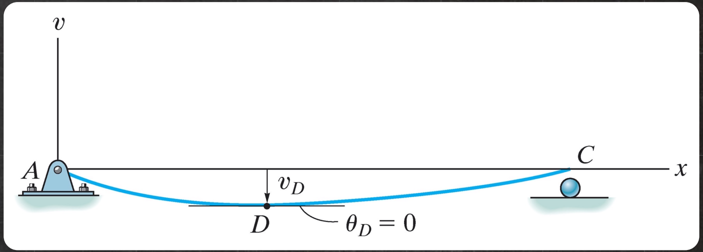
$$0 \leqslant x_1 \leqslant 2 \Rightarrow M_1 = 2x_1 \Rightarrow EI\, v_1'' = 2x_1$$

$$2 \leqslant x_2 \leqslant 3 \Rightarrow M_2 = 4(3 - x_2) \Rightarrow EI\, v_2'' = -4x_2 + 12$$

$0 \leq x_1 \leq 2$ $\quad$ $2 \leq x_2 \leq 3$

$M_1 = 2x_1$ $\quad$ $M_2 = 4(3 - x_2)$

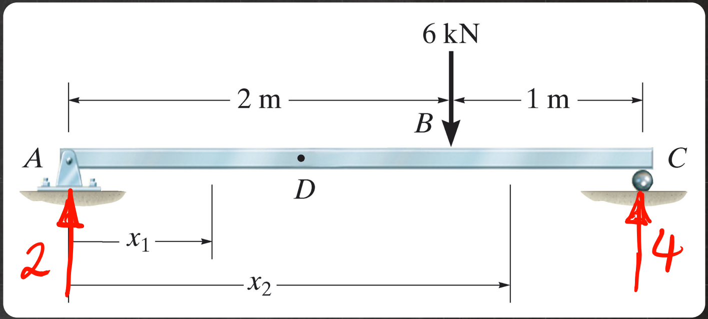


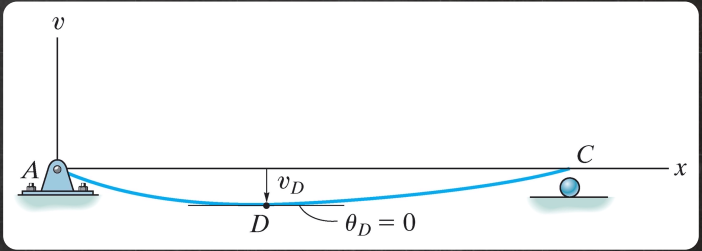


$0 \leqslant x_1 \leqslant 2$ $\qquad$ $2 \leqslant x_2 \leqslant 3$

$M_1 = 2x_1$ $\qquad$ $M_2 = 4(3 - x_2)$

$EI\, v_1'' = 2x_1$ $\qquad$ $EI\, v_2'' = -4x_2 + 12$



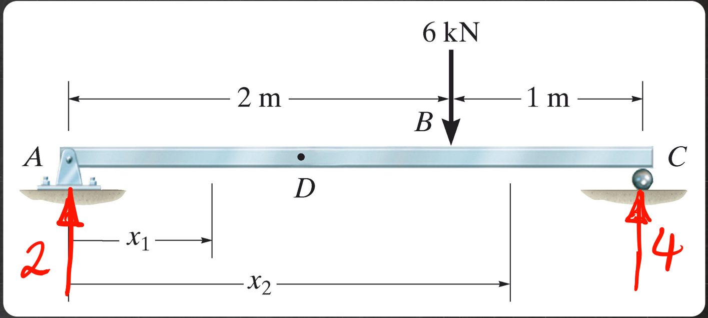


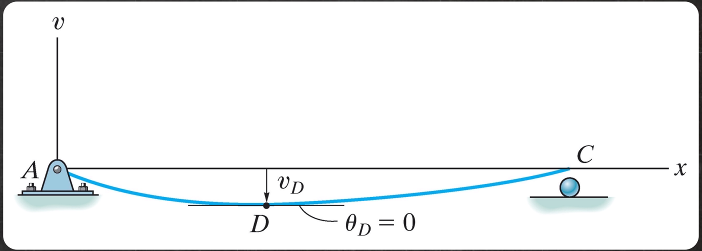
$0 \leqslant x_1 \leqslant 2$ $\qquad$ $2 \leqslant x_2 \leqslant 3$

$M_1 = 2x_1$ $\qquad$ $M_2 = 4(3 - x_2)$

$EI\, v_1'' = 2x_1$ $\qquad$ $EI\, v_2'' = -4x_2 + 12$

$EI\, v_1' = x_1^2 + C_1$ $\qquad$ $EI\, v_2' = -2x_2^2 + 12x_2 + C_3$





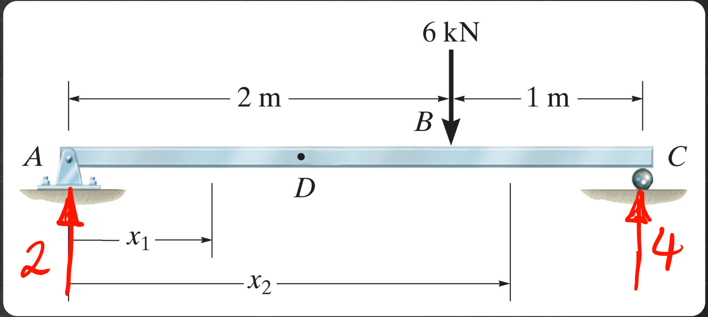
$0 \leq x_1 < 2$ $\qquad$ $2 < x_2 \leq 3$

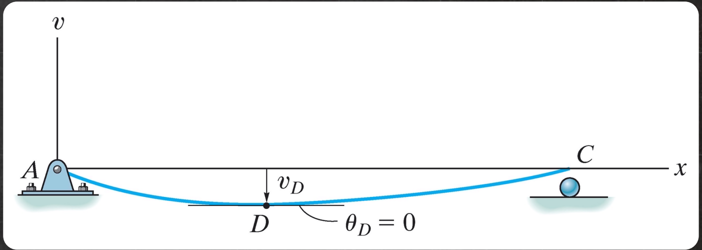
$M_1 = 2x_1$ $\qquad$ $M_2 = 4(3 - x_2)$

$EI\, v_1'' = 2x_1$ $\qquad$ $EI\, v_2'' = -4x_2 + 12$

$EI\, v_1' = x_1^2 + C_1$ $\qquad$ $EI\, v_2' = -2x_2^2 + 12x_2 + C_3$

$EI\, v_1 = \dfrac{x_1^3}{3} + C_1 x + C_2$ $\qquad$ $EI\, v_2 = -\dfrac{2}{3}x_2^3 + 6x_2^2 + C_3 x + C_4$





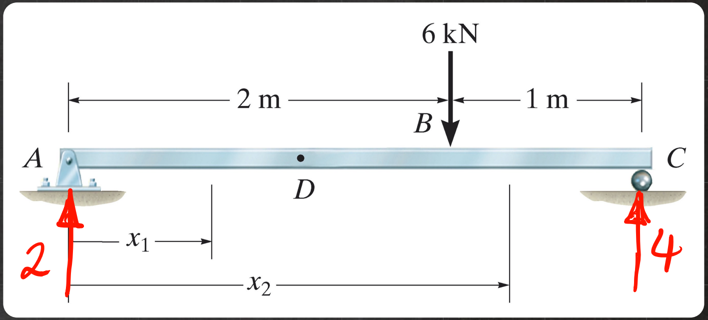
$0 \leqslant x_1 \leqslant 2$ $\qquad$ $2 \leqslant x_2 \leqslant 3$

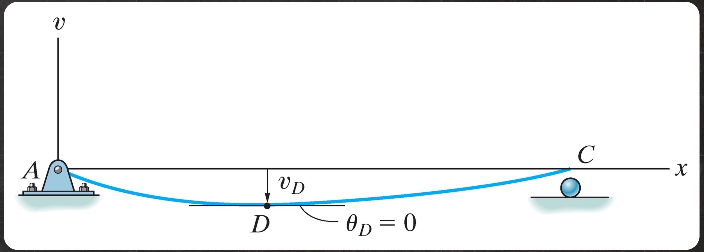
$M_1 = 2x_1$ $\qquad$ $M_2 = 4(3 - x_2)$

$EI\,v_1'' = 2x_1$ $\qquad$ $EI\,v_2'' = -4x_2 + 12$

$EI\,v_1' = x_1^2 + C_1$ $\qquad$ $EI\,v_2' = -2x_2^2 + 12x_2 + C_3$

$EI\,v_1 = \dfrac{x_1^3}{3} + C_1 x + C_2$ $\qquad$ $EI\,v_2 = -\dfrac{2}{3}x_2^3 + 6x_2^2 + C_3 x + C_4$





BCs: $v_1(x_1 = 0) = 0$

$v_2(x_2 = 3) = 0$

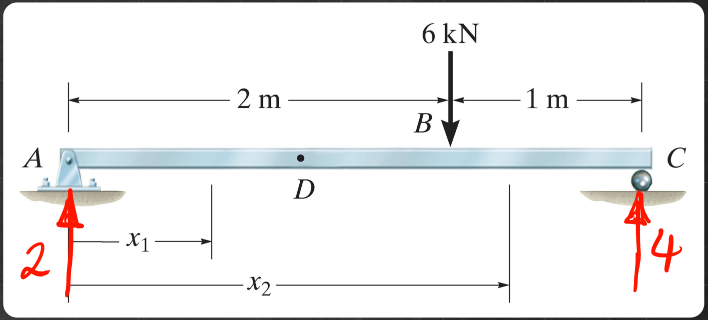
$0 \leqslant x_1 \leqslant 2$ $\quad$ $2 \leqslant x_2 \leqslant 3$

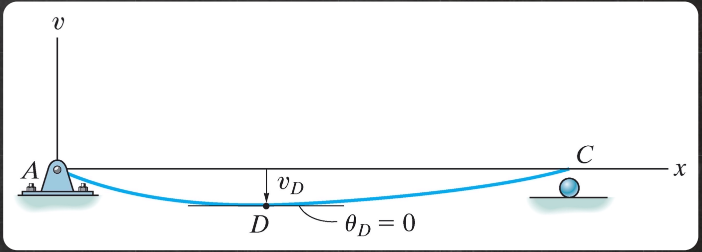
$M_1 = 2x_1$ $\quad$ $M_2 = 4(3 - x_2)$

$EI\, v_1'' = 2x_1$ $\quad$ $EI\, v_2'' = -4x_2 + 12$

$EI\, v_1' = x_1^2 + C_1$ $\quad$ $EI\, v_2' = -2x_2^2 + 12x_2 + C_3$

$EI\, v_1 = \dfrac{x_1^3}{3} + C_1 x_1 + C_2$ $\quad$ $EI\, v_2 = -\dfrac{2}{3}x_2^3 + 6x_2^2 + C_3 x_2 + C_4$





BCs: $v_1(x_1 = 0) = 0$

$v_2(x_2 = 3) = 0$

$v_1(x_1 = 2) = v_2(x_2 = 2)$

$v_1'(x_1 = 2) = v_2'(x_2 = 2)$

COMPATIBILITY CONSTRAINTS

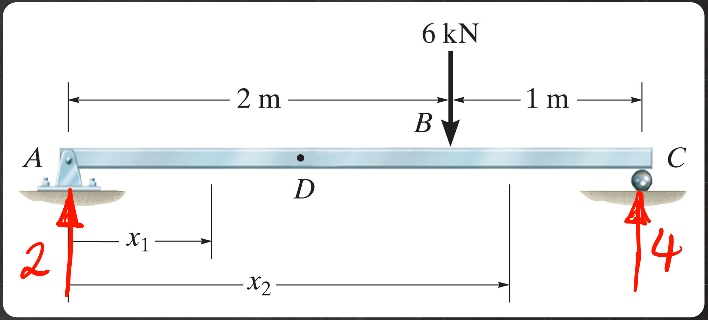
$0 \leqslant x_1 \leqslant 2$ $\qquad$ $2 \leqslant x_2 \leqslant 3$

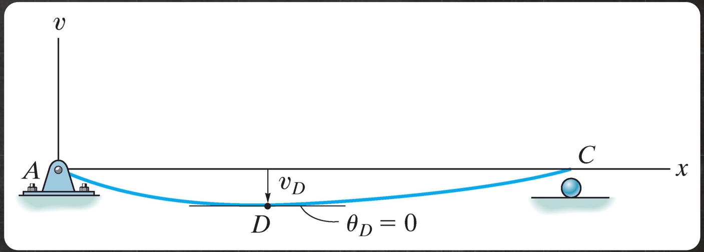
$M_1 = 2x_1$ $\qquad$ $M_2 = 4(3 - x_2)$

$EI\, v_1'' = 2x_1$ $\qquad$ $EI\, v_2'' = -4x_2 + 12$

$EI\, v_1' = x_1^2 + C_1$ $\qquad$ $EI\, v_2' = -2x_2^2 + 12x_2 + C_3$

$EI\, v_1 = \dfrac{x_1^3}{3} + C_1 x + C_2$ $\qquad$ $EI\, v_2 = -\dfrac{2}{3}x_2^3 + 6x_2^2 + C_3 x + C_4$





BCs: $v_1(x_1 = 0) = 0$ $\qquad$ $v_1(x_1 = 2) = v_2(x_2 = 2)$

$v_2(x_2 = 3) = 0$ $\qquad$ $v_1'(x_1 = 2) = v_2'(x_2 = 2)$

$\Rightarrow$ $C_1 = -8/3$, $C_3 = -44/3$

$C_2 = 0$, $C_4 = 8$

$0 \leq x_1 \leq 2$ $\qquad$ $2 \leq x_2 \leq 3$

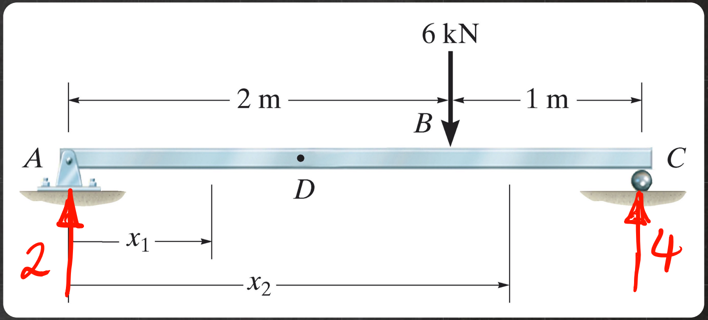
$M_1 = 2x_1$ $\qquad$ $M_2 = 4(3 - x_2)$

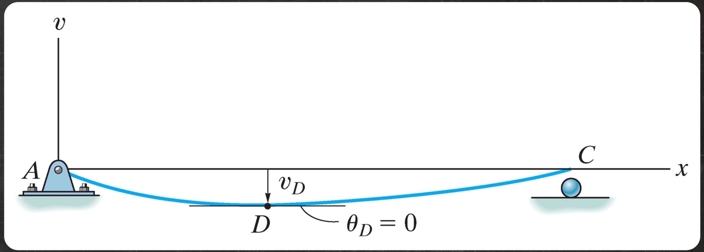
$EI\, v_1'' = 2x_1$ $\qquad$ $EI\, v_2'' = -4x_2 + 12$

$EI\, v_1' = x_1^2 + C_1$ $\qquad$ $EI\, v_2' = -2x_2^2 + 12x_2 + C_3$

$EI\, v_1 = \frac{x_1^3}{3} + C_1 x + C_2$ $\qquad$ $EI\, v_2 = -\frac{2}{3}x_2^3 + 6x_2^2 + C_3 x + C_4$

$v_1'(x = x_D) \overset{!}{=} 0$





$C_1 = -8/3$, $C_3 = -44/3$

$C_2 = 0$, $C_4 = 8$

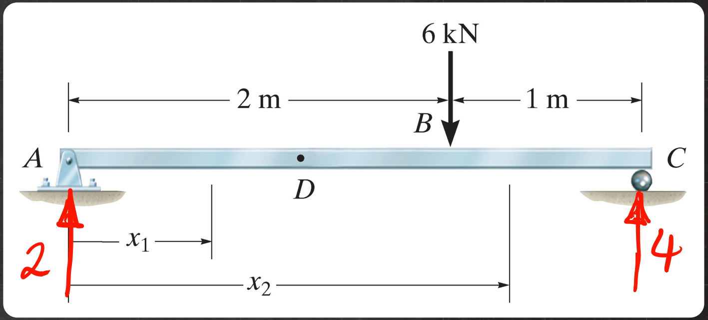
$0 \leqslant x_1 \leqslant 2$ $\qquad$ $2 \leqslant x_2 \leqslant 3$

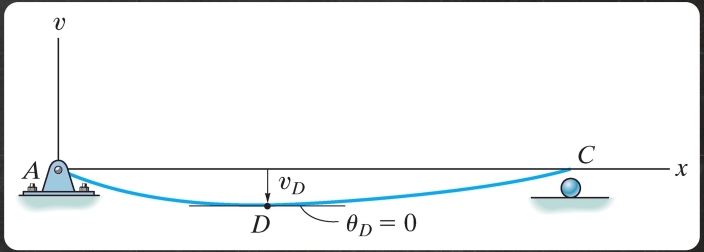
$M_1 = 2x_1$ $\qquad$ $M_2 = 4(3 - x_2)$

$EI\, v_1'' = 2x_1$ $\qquad$ $EI\, v_2'' = -4x_2 + 12$

$EI\, v_1' = x_1^2 + C_1$ $\qquad$ $EI\, v_2' = -2x_2^2 + 12x_2 + C_3$

$EI\, v_1 = \dfrac{x_1^3}{3} + C_1 x_1 + C_2$ $\qquad$ $EI\, v_2 = -\dfrac{2}{3}x_2^3 + 6x_2^2 + C_3 x_2 + C_4$





$\to v_1'(x = x_D) \overset{!}{=} 0 \Rightarrow x_D = 1.633 < 2$ ✓

$C_1 = -8/3$, $C_3 = -44/3$

$C_2 = 0$, $C_4 = 8$

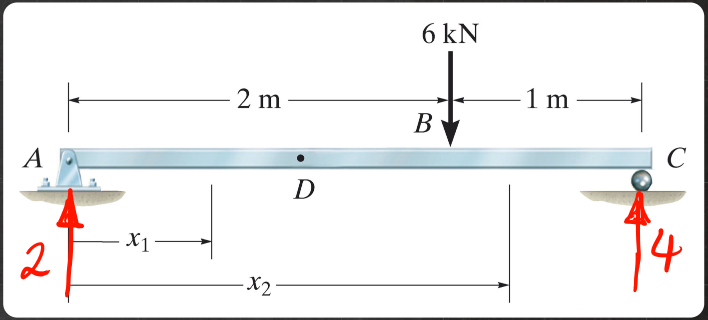
$0 \leqslant x_1 \leqslant 2$ $\qquad$ $2 \leqslant x_2 \leqslant 3$

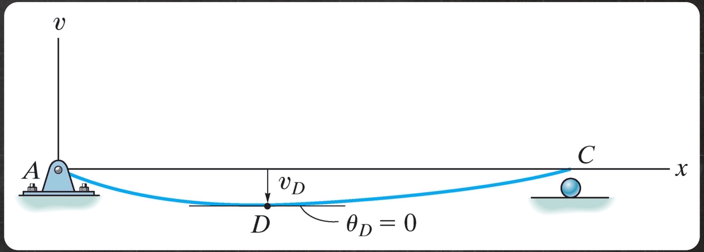
$M_1 = 2x_1$ $\qquad$ $M_2 = 4(3 - x_2)$

$EI\, v_1'' = 2x_1$ $\qquad$ $EI\, v_2'' = -4x_2 + 12$

$EI\, v_1' = x_1^2 + C_1$ $\qquad$ $EI\, v_2' = -2x_2^2 + 12x_2 + C_3$

$EI\, v_1 = \frac{x_1^3}{3} + C_1 x + C_2$ $\qquad$ $EI\, v_2 = -\frac{2}{3}x_2^3 + 6x_2^2 + C_3 x + C_4$





$v_1'(x = x_D) \overset{!}{=} 0 \Rightarrow x_D = 1.633 < 2$ ✓

$\Rightarrow v_{max} = v(x = x_D) = -\frac{2.90}{EI}$

$C_1 = -8/3$, $C_3 = -44/3$

$C_2 = 0$, $C_4 = 8$

DISCONTINUITY FUNCTIONS → TO AVOID DEFINING MULTIPLE COORDINATES SYSTEMS

↳ IN "EXERCISE 3", WE DEFINED x_1 & x_2 !

DISCONTINUITY FUNCTIONS → TO AVOID DEFINING MULTIPLE COORDINATES SYSTEMS

↳ IN "EXERCISE 3", WE DEFINED x_1 & x_2!

MACAULAY FUNCTIONS:

DISCONTINUITY FUNCTIONS → TO AVOID DEFINING MULTIPLE COORDINATES SYSTEMS

↳ IN "EXERCISE 3", WE DEFINED x_1 & x_2 !

MACAULAY FUNCTIONS:

$$\langle x-a \rangle^n = \begin{cases} 0 & x < a \\ (x-a)^n & x \geqslant a \end{cases} \qquad n \geqslant 0$$

↳ Macaulay Bracket

DISCONTINUITY FUNCTIONS → TO AVOID DEFINING MULTIPLE COORDINATES SYSTEMS

↳ IN "EXERCISE 3", WE DEFINED x_1 & x_2 !

MACAULAY FUNCTIONS:

$$\langle x-a\rangle^n = \begin{cases} 0 & x < a \\ (x-a)^n & x \geqslant a \end{cases} \qquad n \geqslant 0$$

↳ Macaulay Bracket

$$\int \langle x-a\rangle^n \, dx = \frac{\langle x-a\rangle^{n+1}}{n+1} + C$$

Discontinuity Functions

Singularity Functions:

$$\langle x-a \rangle^{n} = \begin{cases} 0 & x \neq a \\ 1 & x = a \end{cases} \qquad n = -1, -2, \ldots$$

$$\int \langle x-a \rangle^{n} \, dx = \langle x-a \rangle^{n+1}$$

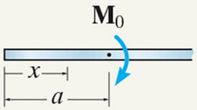
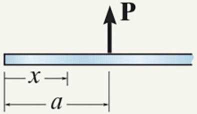
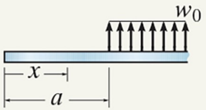
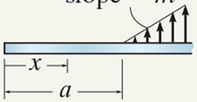
$$\langle x-a\rangle^n = \begin{cases} 0 & x < a \\ (x-a)^n & x \geqslant a \end{cases} \qquad n \geqslant 0$$

$$\int \langle x-a\rangle^n \, dx = \frac{\langle x-a\rangle^{n+1}}{n+1} + C$$

$$\langle x-a\rangle^n = \begin{cases} 0 & x \neq a \\ 1 & x = a \end{cases} \qquad n = -1, -2, \dots$$

$$\int \langle x-a\rangle^n \, dx = \langle x-a\rangle^{n+1}$$

TABLE 12–2

Loading	Loading Function $w = w(x)$	Shear $V = \int w(x)dx$	Moment $M = \int V dx$
	$w = M_0\langle x-a\rangle^{-2}$	$V = M_0\langle x-a\rangle^{-1}$	$M = M_0\langle x-a\rangle^{0}$
	$w = P\langle x-a\rangle^{-1}$	$V = P\langle x-a\rangle^{0}$	$M = P\langle x-a\rangle^{1}$
	$w = w_0\langle x-a\rangle^{0}$	$V = w_0\langle x-a\rangle^{1}$	$M = \frac{w_0}{2}\langle x-a\rangle^{2}$
	$w = m\langle x-a\rangle^{1}$	$V = \frac{m}{2}\langle x-a\rangle^{2}$	$M = \frac{m}{6}\langle x-a\rangle^{3}$

WE SOLVE "EXERCISE 3" USING THIS APPROACH

&

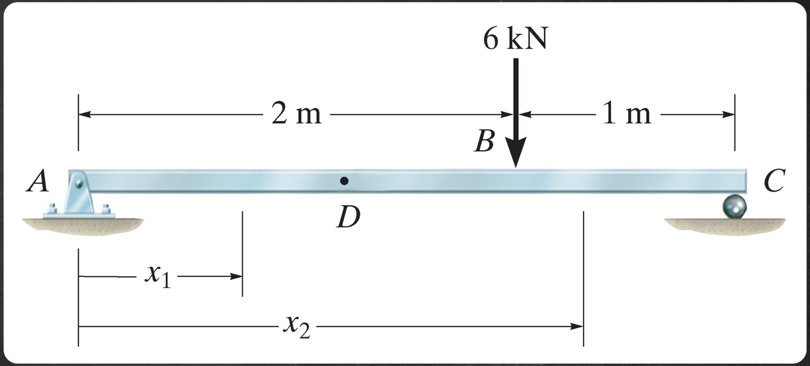
↓

ALSO, CALLED DOUBLE INTEGRATION METHOD

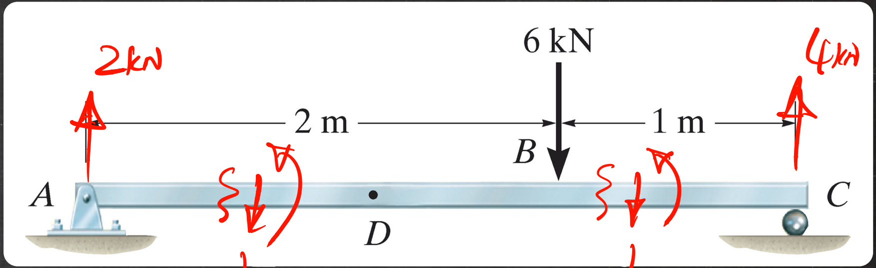
Exercise 4 . [similar to ... P. 586 ... 12.3]

DETERMINE THE EQUATION OF THE ELASTIC CURVE FOR THE BEAM SHOWN IN THE FIGURE AND THE MAXIMUM DEFLECTION.

EI IS CONSTANT.



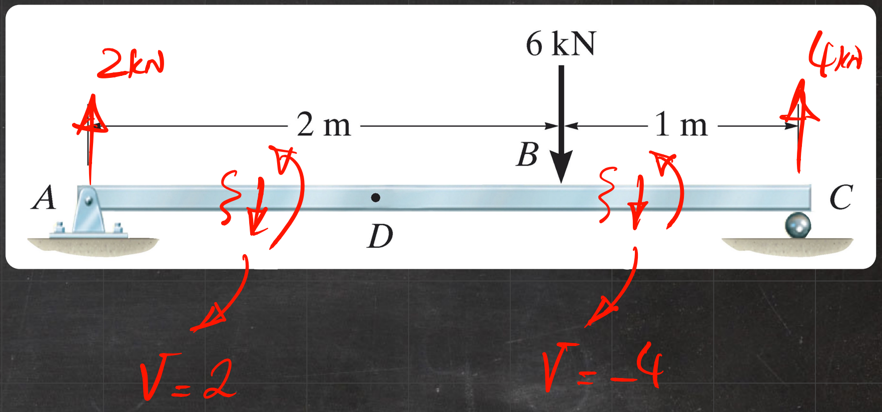
↳ SOLVE USING DISCONTINUITY FUNCTIONS.



V = 2

V = -4

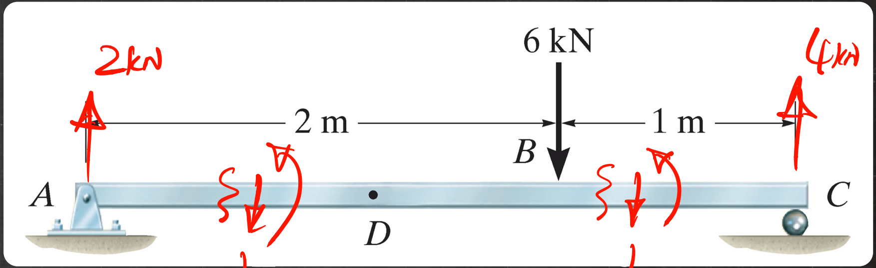
$$V = 2 - \langle x-2 \rangle^0 \, 6$$



$V = 2 - \langle x-2 \rangle^0 6$

$M = \int V \, dx$

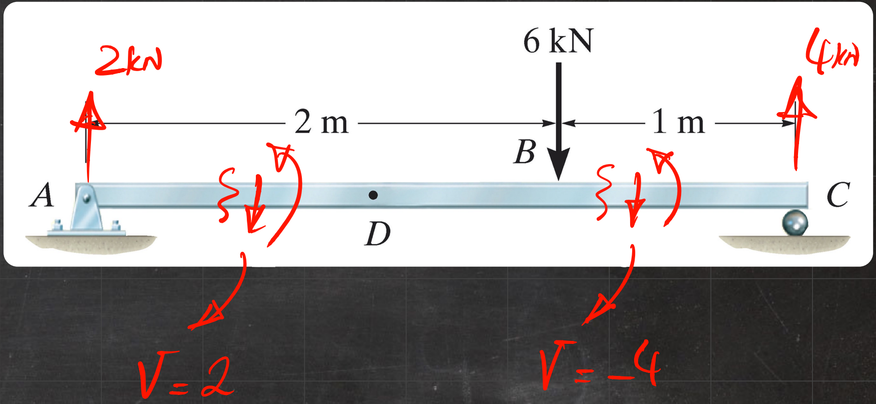
$= 2x - \langle x-2 \rangle^1 6 + C_1$



$V = 2$

$V = -4$

$$V = 2 - \langle x-2 \rangle^0 6$$



$$M = \int V \, dx$$

$$= 2x - \langle x-2 \rangle^1 6 + C_1$$

$$C_1 = 0 \Leftarrow M(0) = 0$$

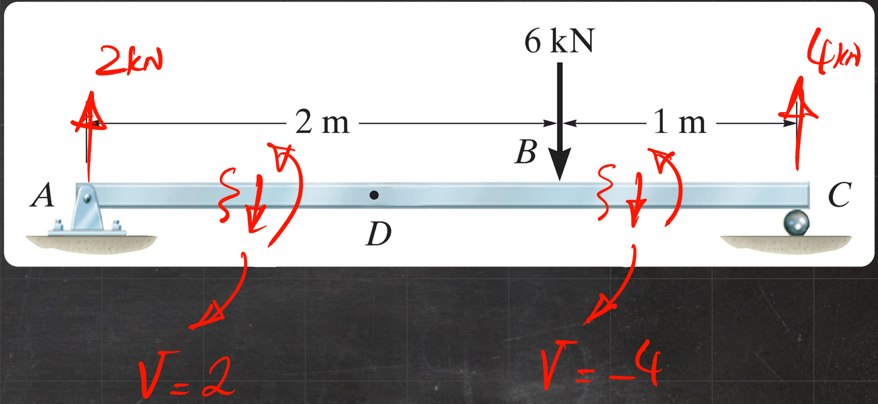
$$V = 2 - \langle x-2 \rangle^0 6$$

$$M = \int V \, dx$$

$$= 2x - \langle x-2 \rangle^1 6 + C_1$$

$$C_1 = 0 \Leftarrow M(0) = 0$$

$$EI v'' = M$$



$V = 2 - \langle x-2 \rangle^0 6$

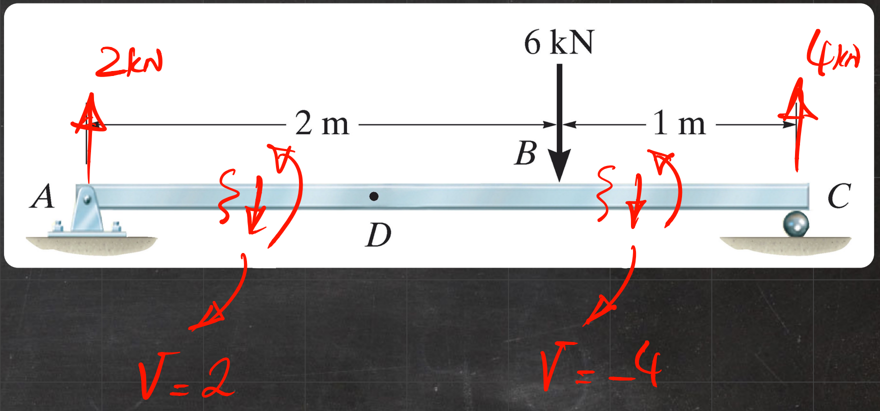
$M = \int V\,dx$

$= 2x - \langle x-2 \rangle^1 6 + C_1$

$C_1 = 0 \Leftarrow M(0) = 0$

$EI v'' = M$

$EI v'' = 2x - \langle x-2 \rangle^1 6$



$$V = 2 - \langle x-2\rangle^0 6$$

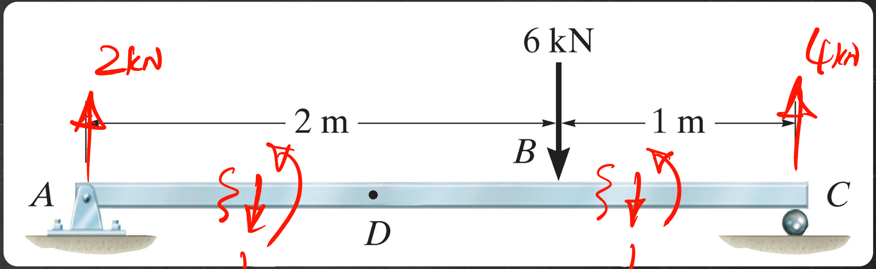
$$M = \int V\,dx$$

$$= 2x - \langle x-2\rangle^1 6 + C_1$$

$$C_1 = 0 \Leftarrow M(0) = 0$$

$$EI v'' = M$$

$$EI v'' = 2x - \langle x-2\rangle^1 6$$



$V = 2$ $\qquad V = -4$

$$\langle x-a\rangle^n = \begin{cases} 0 & x < a \\ (x-a)^n & x \geq a \end{cases} \qquad n \geq 0$$

$$\int \langle x-a\rangle^n dx = \frac{\langle x-a\rangle^{n+1}}{n+1} + C$$

$V = 2 - \langle x-2 \rangle^0 6$

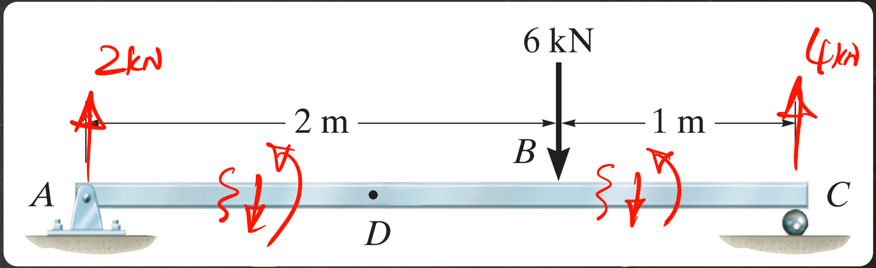
$M = \int V\,dx$

$= 2x - \langle x-2 \rangle^1 6 + C_1$

$C_1 = 0 \Leftarrow M(0) = 0$

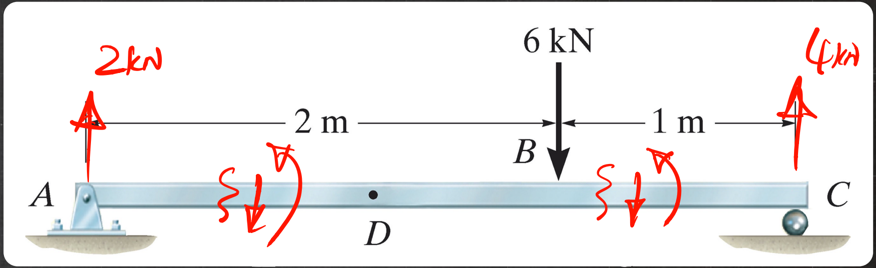
$EIv'' = M$

$EIv'' = 2x - \langle x-2 \rangle^1 6$



$EIv' = x^2 - 3\langle x-2 \rangle^2 + C_2$

$$V = 2 - \langle x-2 \rangle^0 \, 6$$



$$M = \int V \, dx$$

$$= 2x - \langle x-2 \rangle^1 \, 6 + C_1$$

$$C_1 = 0 \Leftarrow M(0) = 0$$

$$EI v'' = M$$

$$EI v'' = 2x - \langle x-2 \rangle^1 \, 6$$

$$EI v' = x^2 - 3 \langle x-2 \rangle^2 + C_2$$

$$EI v = \frac{x^3}{3} - \langle x-2 \rangle^3 + C_2 x + C_3$$

$V = 2 - \langle x-2 \rangle^0 6$

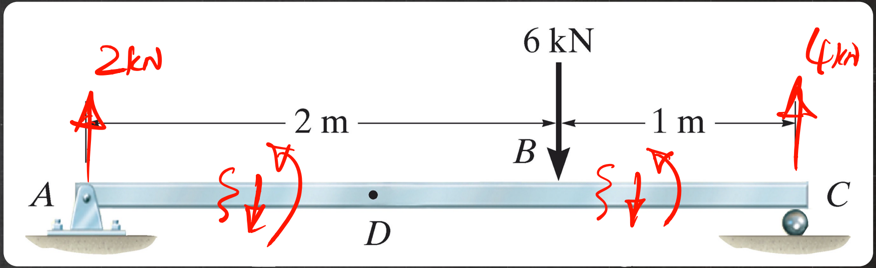
$M = \int V\,dx$

$= 2x - \langle x-2 \rangle^1 6 + C_1$

$C_1 = 0 \Leftarrow M(0) = 0$

$EIv'' = M$

$EIv'' = 2x - \langle x-2 \rangle^1 6$



$EIv' = x^2 - 3\langle x-2 \rangle^2 + C_2$

$EIv = \frac{x^3}{3} - \langle x-2 \rangle^3 + C_2 x + C_3$

BCs : $v(0) = 0$, $v(3) = 0$

$$V = 2 - \langle x-2\rangle^0 6$$

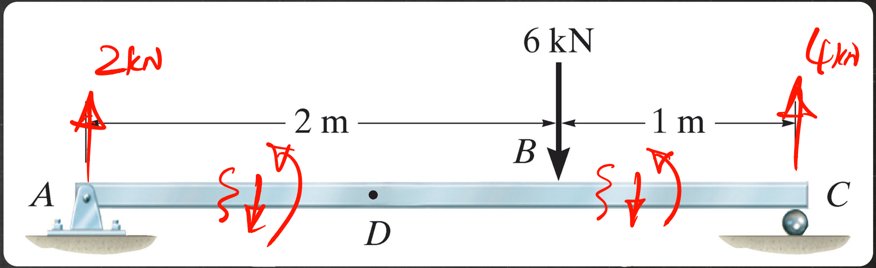
$$M = \int V\,dx$$

$$= 2x - \langle x-2\rangle^1 6 + C_1$$

$$C_1 = 0 \Leftarrow M(0) = 0$$

$$EI v'' = M$$

$$EI v'' = 2x - \langle x-2\rangle^1 6$$



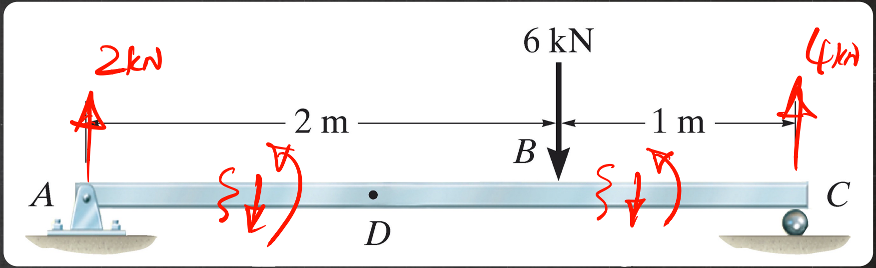
$$EI v' = x^2 - 3\langle x-2\rangle^2 + C_2$$

$$EI v = \frac{x^3}{3} - \langle x-2\rangle^3 + C_2 x + C_3$$

BCs : $v(0) = 0$, $v(3) = 0$

$$\Rightarrow C_3 = 0 \;,\; C_2 = -8/3$$

$V = 2 - \langle x-2 \rangle^0 6$



$M = \int V \, dx$

$= 2x - \langle x-2 \rangle^1 6 + C_1$

$C_1 = 0 \Leftarrow M(0) = 0$

$EI v'' = M$

$EI v'' = 2x - \langle x-2 \rangle^1 6$

$EI v' = x^2 - 3 \langle x-2 \rangle^2 + C_2$

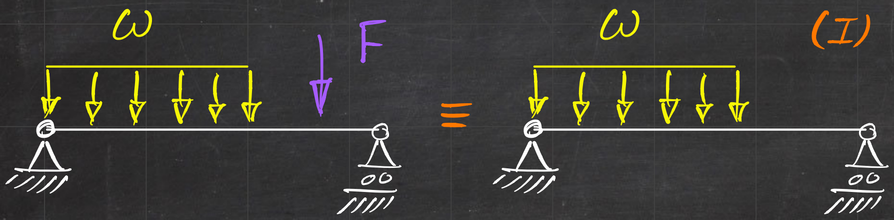
$EI v = \frac{x^3}{3} - \langle x-2 \rangle^3 + C_2 x + C_3$

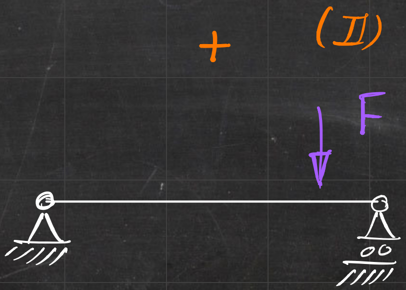
BCs : $v(0) = 0$, $v(3) = 0$

$\Rightarrow C_3 = 0$, $C_2 = -8/3$

$\Rightarrow v(x) = \frac{1}{EI} \left[\frac{x^3}{3} - \langle x-2 \rangle^3 - \frac{8}{3} x \right]$

METHOD OF SUPERPOSITION





$$v(x) = \overset{I}{v}(x) + \overset{II}{v}(x)$$

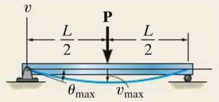
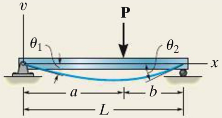
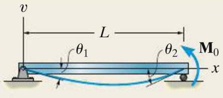
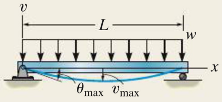
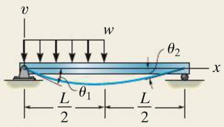
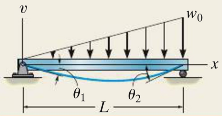
Cantilevered Beam Slopes and Deflections

Beam	Slope	Deflection	Elastic Curve
	$\theta_{max} = \frac{-PL^2}{2EI}$	$v_{max} = \frac{-PL^3}{3EI}$	$v = \frac{-Px^2}{6EI}(3L - x)$
	$\theta_{max} = \frac{-PL^2}{8EI}$	$v_{max} = \frac{-5PL^3}{48EI}$	$v = \frac{-Px^2}{12EI}(3L - 2x) \quad 0 \le x \le L/2$ $v = \frac{-PL^2}{48EI}(6x - L) \quad L/2 \le x \le L$
	$\theta_{max} = \frac{-wL^3}{6EI}$	$v_{max} = \frac{-wL^4}{8EI}$	$v = \frac{-wx^2}{24EI}(x^2 - 4Lx + 6L^2)$
	$\theta_{max} = \frac{M_0L}{EI}$	$v_{max} = \frac{M_0L^2}{2EI}$	$v = \frac{M_0x^2}{2EI}$
	$\theta_{max} = \frac{-wL^3}{48EI}$	$v_{max} = \frac{-7wL^4}{384EI}$	$v = \frac{-wx^2}{24EI}\left(x^2 - 2Lx + \frac{3}{2}L^2\right)$ $0 \le x \le L/2$ $v = \frac{-wL^3}{384EI}(8x - L)$ $L/2 \le x \le L$
	$\theta_{max} = \frac{-w_0L^3}{24EI}$	$v_{max} = \frac{-w_0L^4}{30EI}$	$v = \frac{-w_0x^2}{120EIL}(10L^3 - 10L^2x + 5Lx^2 - x^3)$

Simply Supported Beam Slopes and Deflections

Beam	Slope	Deflection	Elastic Curve
	$\theta_{max} = \frac{-PL^2}{16EI}$	$v_{max} = \frac{-PL^3}{48EI}$	$v = \frac{-Px}{48EI}(3L^2 - 4x^2)$ $0 \le x \le L/2$
	$\theta_1 = \frac{-Pab(L + b)}{6EIL}$ $\theta_2 = \frac{Pab(L + a)}{6EIL}$	$v\Big\vert_{x=a} = \frac{-Pba}{6EIL}(L^2 - b^2 - a^2)$	$v = \frac{-Pbx}{6EIL}(L^2 - b^2 - x^2)$ $0 \le x \le a$
	$\theta_1 = \frac{-M_0L}{6EI}$ $\theta_2 = \frac{M_0L}{3EI}$	$v_{max} = \frac{-M_0L^2}{9\sqrt{3}EI}$ at $x = 0.5774L$	$v = \frac{-M_0x}{6EIL}(L^2 - x^2)$
	$\theta_{max} = \frac{-wL^3}{24EI}$	$v_{max} = \frac{-5wL^4}{384EI}$	$v = \frac{-wx}{24EI}(x^3 - 2Lx^2 + L^3)$
	$\theta_1 = \frac{-3wL^3}{128EI}$ $\theta_2 = \frac{7wL^3}{384EI}$	$v\Big\vert_{x=L/2} = \frac{-5wL^4}{768EI}$ $v_{max} = -0.006563\frac{wL^4}{EI}$ at $x = 0.4598L$	$v = \frac{-wx}{384EI}(16x^3 - 24Lx^2 + 9L^3)$ $0 \le x \le L/2$ $v = \frac{-wL}{384EI}(8x^3 - 24Lx^2 + 17L^2x - L^3)$ $L/2 \le x < L$
	$\theta_1 = \frac{-7w_0L^3}{360EI}$ $\theta_2 = \frac{w_0L^3}{45EI}$	$v_{max} = -0.00652\frac{w_0L^4}{EI}$ at $x = 0.5193L$	$v = \frac{-w_0x}{360EIL}(3x^4 - 10L^2x^2 + 7L^4)$

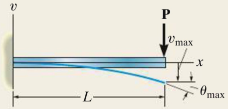
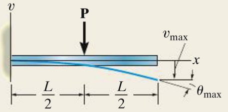
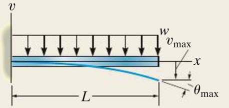
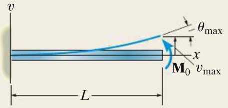
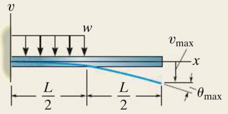
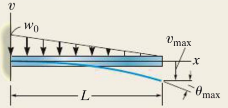
Simply Supported Beam Slopes and Deflections

Beam	Slope	Deflection	Elastic Curve
	$\theta_{max} = \dfrac{-PL^2}{16EI}$	$v_{max} = \dfrac{-PL^3}{48EI}$	$v = \dfrac{-Px}{48EI}(3L^2 - 4x^2)$ $0 \le x \le L/2$
	$\theta_1 = \dfrac{-Pab(L+b)}{6EIL}$ $\theta_2 = \dfrac{Pab(L+a)}{6EIL}$	$v\Big\rvert_{x=a} = \dfrac{-Pba}{6EIL}(L^2 - b^2 - a^2)$	$v = \dfrac{-Pbx}{6EIL}(L^2 - b^2 - x^2)$ $0 \le x \le a$
	$\theta_1 = \dfrac{-M_0 L}{6EI}$ $\theta_2 = \dfrac{M_0 L}{3EI}$	$v_{max} = \dfrac{-M_0 L^2}{9\sqrt{3}EI}$ at $x = 0.5774L$	$v = \dfrac{-M_0 x}{6EIL}(L^2 - x^2)$
	$\theta_{max} = \dfrac{-wL^3}{24EI}$	$v_{max} = \dfrac{-5wL^4}{384EI}$	$v = \dfrac{-wx}{24EI}(x^3 - 2Lx^2 + L^3)$
	$\theta_1 = \dfrac{-3wL^3}{128EI}$ $\theta_2 = \dfrac{7wL^3}{384EI}$	$v\Big\rvert_{x=L/2} = \dfrac{-5wL^4}{768EI}$ $v_{max} = -0.006563\dfrac{wL^4}{EI}$ at $x = 0.4598L$	$v = \dfrac{-wx}{384EI}(16x^3 - 24Lx^2 + 9L^3)$ $0 \le x \le L/2$ $v = \dfrac{-wL}{384EI}(8x^3 - 24Lx^2 + 17L^2 x - L^3)$ $L/2 \le x < L$
	$\theta_1 = \dfrac{-7w_0 L^3}{360EI}$ $\theta_2 = \dfrac{w_0 L^3}{45EI}$	$v_{max} = -0.00652\dfrac{w_0 L^4}{EI}$ at $x = 0.5193L$	$v = \dfrac{-w_0 x}{360EIL}(3x^4 - 10L^2x^2 + 7L^4)$

Appendix C

SIMPLY SUPPORTED BEAMS

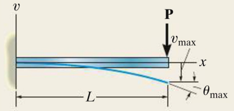
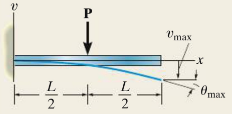
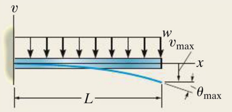
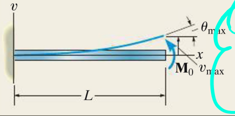
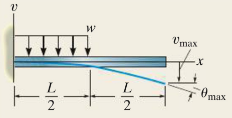
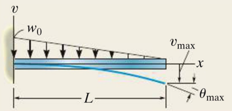
Cantilevered Beam Slopes and Deflections

Beam	Slope	Deflection	Elastic Curve
	$\theta_{max} = \dfrac{-PL^2}{2EI}$	$v_{max} = \dfrac{-PL^3}{3EI}$	$v = \dfrac{-Px^2}{6EI}(3L - x)$
	$\theta_{max} = \dfrac{-PL^2}{8EI}$	$v_{max} = \dfrac{-5PL^3}{48EI}$	$v = \dfrac{-Px^2}{12EI}(3L - 2x) \quad 0 \le x \le L/2$ $v = \dfrac{-PL^2}{48EI}(6x - L) \quad L/2 \le x \le L$
	$\theta_{max} = \dfrac{-wL^3}{6EI}$	$v_{max} = \dfrac{-wL^4}{8EI}$	$v = \dfrac{-wx^2}{24EI}(x^2 - 4Lx + 6L^2)$
	$\theta_{max} = \dfrac{M_0 L}{EI}$	$v_{max} = \dfrac{M_0 L^2}{2EI}$	$v = \dfrac{M_0 x^2}{2EI}$
	$\theta_{max} = \dfrac{-wL^3}{48EI}$	$v_{max} = \dfrac{-7wL^4}{384EI}$	$v = \dfrac{-wx^2}{24EI}\left(x^2 - 2Lx + \frac{3}{2}L^2\right) \quad 0 \le x \le L/2$ $v = \dfrac{-wL^3}{384EI}(8x - L) \quad L/2 \le x \le L$
	$\theta_{max} = \dfrac{-w_0 L^3}{24EI}$	$v_{max} = \dfrac{-w_0 L^4}{30EI}$	$v = \dfrac{-w_0 x^2}{120EIL}(10L^3 - 10L^2 x + 5Lx^2 - x^3)$

Appendix C

CANTILEVERED BEAMS

Cantilevered Beam Slopes and Deflections

Beam	Slope	Deflection	Elastic Curve
	$\theta_{max} = \frac{-PL^2}{2EI}$	$v_{max} = \frac{-PL^3}{3EI}$	$v = \frac{-Px^2}{6EI}(3L - x)$
	$\theta_{max} = \frac{-PL^2}{8EI}$	$v_{max} = \frac{-5PL^3}{48EI}$	$v = \frac{-Px^2}{12EI}(3L - 2x) \quad 0 \le x \le L/2$ $v = \frac{-PL^2}{48EI}(6x - L) \quad L/2 \le x \le L$
	$\theta_{max} = \frac{-wL^3}{6EI}$	$v_{max} = \frac{-wL^4}{8EI}$	$v = \frac{-wx^2}{24EI}(x^2 - 4Lx + 6L^2)$
	$\theta_{max} = \frac{M_0 L}{EI}$	$v_{max} = \frac{M_0 L^2}{2EI}$	$v = \frac{M_0 x^2}{2EI}$
	$\theta_{max} = \frac{-wL^3}{48EI}$	$v_{max} = \frac{-7wL^4}{384EI}$	$v = \frac{-wx^2}{24EI}\left(x^2 - 2Lx + \frac{3}{2}L^2\right) \quad 0 \le x \le L/2$ $v = \frac{-wL^3}{384EI}(8x - L) \quad L/2 \le x \le L$
	$\theta_{max} = \frac{-w_0 L^3}{24EI}$	$v_{max} = \frac{-w_0 L^4}{30EI}$	$v = \frac{-w_0 x^2}{120EIL}(10L^3 - 10L^2 x + 5Lx^2 - x^3)$

$$\delta = \frac{PL}{EA}$$

$$\phi = \frac{TL}{GJ}$$

$$\theta = \frac{ML}{EI}$$

Statically Indeterminate Beams and Shafts

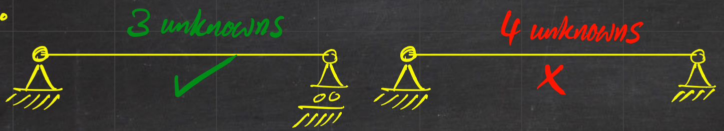
STATICALLY INDETERMINATE BEAMS AND SHAFTS

↳ More unknowns than static equilibrium equations ↝ $\sum \mathbb{F} = 0$, $\sum M = 0$

Statically Indeterminate Beams and Shafts

↳ More unknowns than static equilibrium equations ⤳ $\Sigma \mathbb{F} = \emptyset$, $\Sigma M = \emptyset$

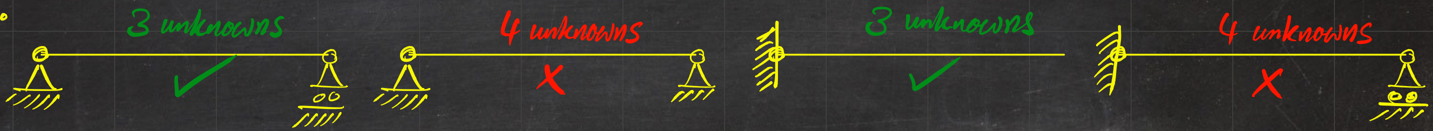
e.g.



STATICALLY INDETERMINATE BEAMS AND SHAFTS

↳ More unknowns than static equilibrium equations ↝ $\sum \mathbb{F} = \emptyset$, $\sum M = \emptyset$

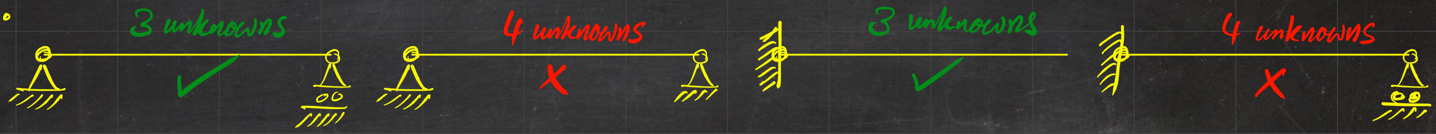
e.g.



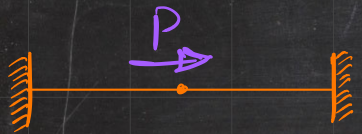
STATICALLY INDETERMINATE BEAMS AND SHAFTS

↳ More unknowns than static equilibrium equations ↝ $\Sigma \mathbb{F} = \emptyset$, $\Sigma M = \emptyset$

e.g.



RECALL:

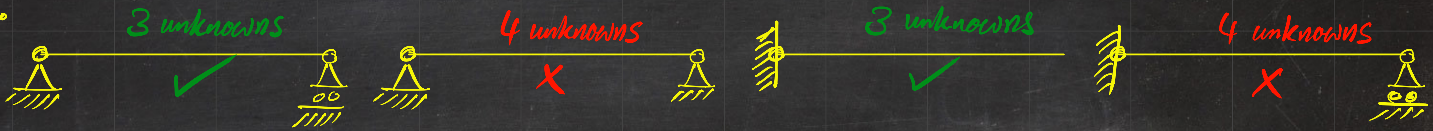


AXIAL LOADING ↝ Compute reactions at the supports

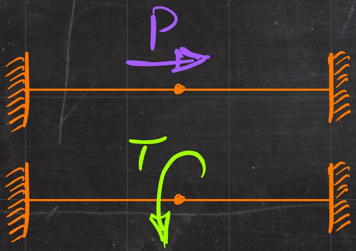
Statically Indeterminate Beams and Shafts

↳ More unknowns than static equilibrium equations ↝ $\Sigma \mathbb{F} = \emptyset$, $\Sigma M = \emptyset$

e.g.



RECALL:



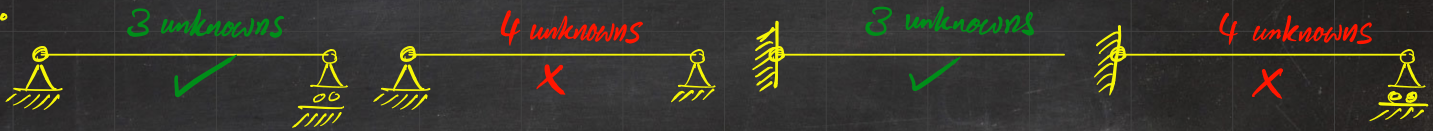
AXIAL LOADING ↝ Compute reactions at the supports

TORSION ↝ Compute reactions at the supports

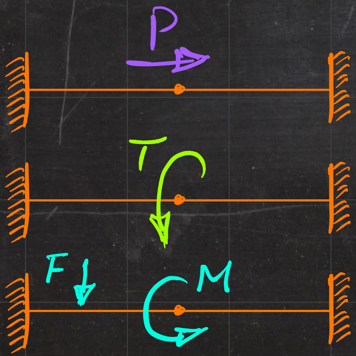
Statically Indeterminate Beams and Shafts

↳ More unknowns than static equilibrium equations ↳ $\Sigma \mathbb{F} = 0$, $\Sigma M = 0$

e.g.



Recall:



Axial loading ↳ Compute reactions at the supports

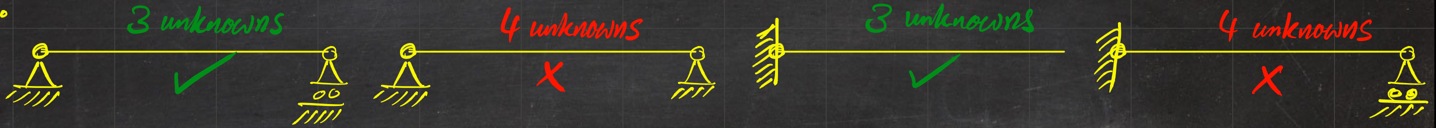
Torsion ↳ Compute reactions at the supports

Bending ↳ Compute reactions at the supports

STATICALLY INDETERMINATE BEAMS AND SHAFTS

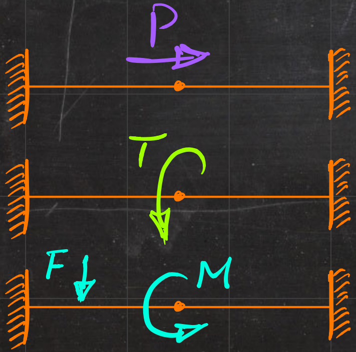
↳ More unknowns than static equilibrium equations ↳ $\sum F=0, \sum M=0$

e.g.



RECALL:

2 STEPS ⟶ (1) EQUILIBRIUM, (2) COMPATIBILITY



AXIAL LOADING ⟶ Compute reactions at the supports

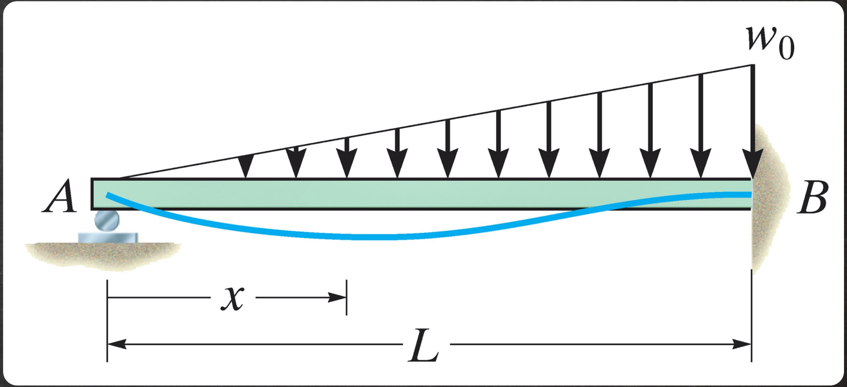
TORSION ⟶ Compute reactions at the supports

BENDING ⟶ Compute reactions at the supports

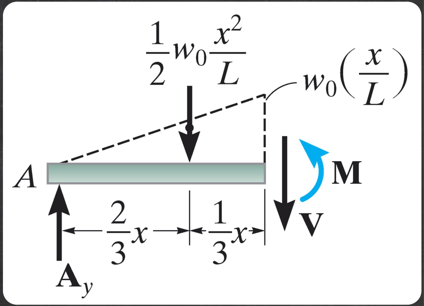
Exercise 5 . [similar to ... P. 633 ... 12.17]

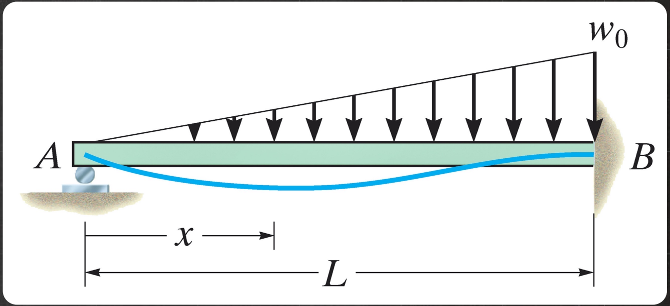
FOR THE BEAM SHOWN IN THE FIGURE, DETERMINE THE RACTION AT A AND B.

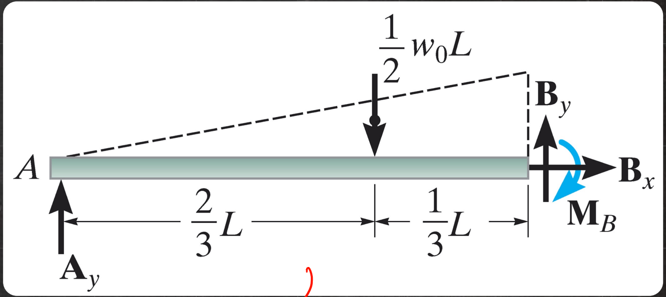
EI IS CONSTANT.



$$M = A_y x - \frac{1}{6}\frac{\omega_0 x^3}{L}$$



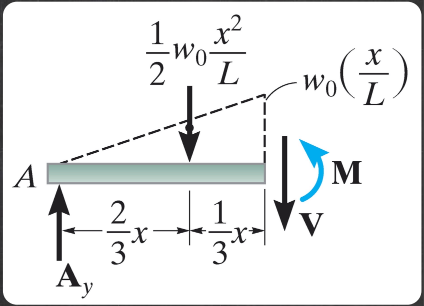


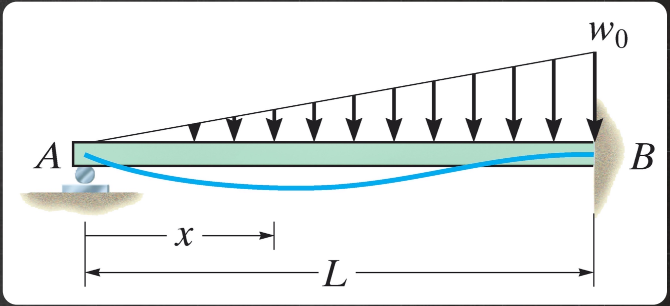


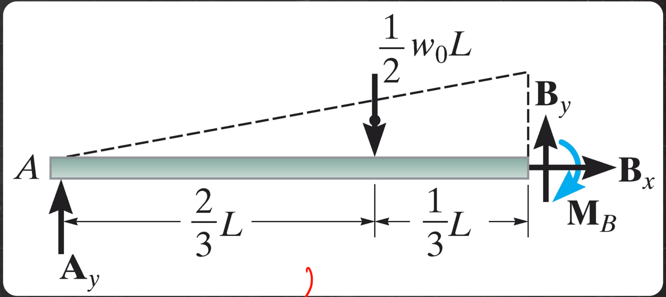
↳ 4 unknowns

$$M = A_y x - \frac{1}{6}\frac{w_0 x^3}{L}$$

$$EI\, v'' = M$$

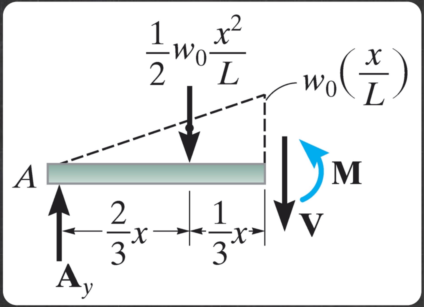


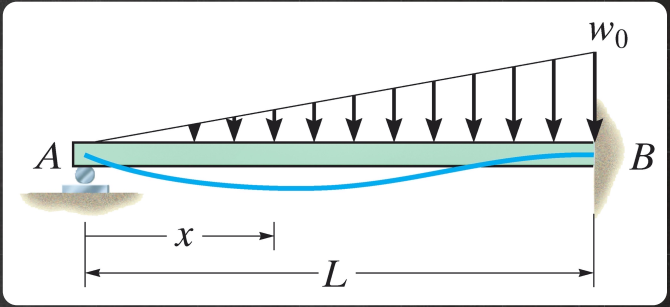




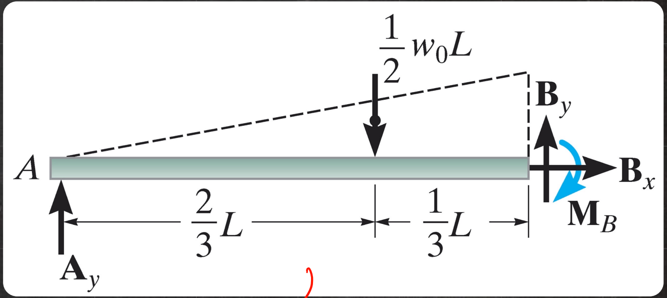
$\rightarrow$ 4 unknowns

$$M = A_y x - \frac{1}{6}\frac{w_0 x^3}{L}$$





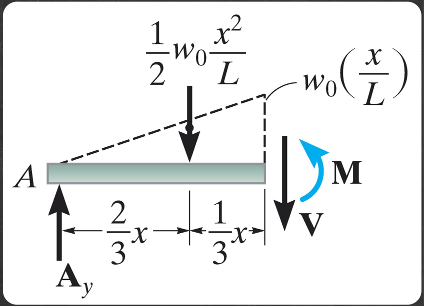
$$EI\, v'' = M$$

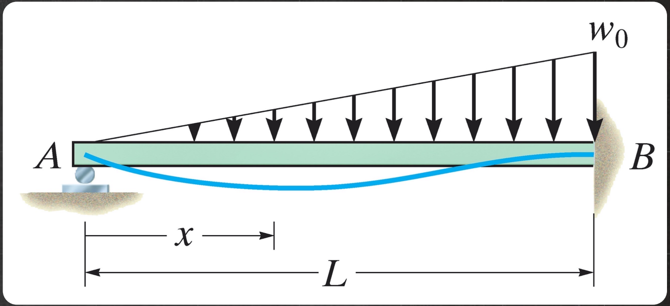


$$EI\, v'' = A_y x - \frac{1}{6} w_0 \frac{x^3}{L}$$

$\rhd$ 4 unknowns

$$M = A_y x - \frac{1}{6}\frac{\omega_0 x^3}{L}$$

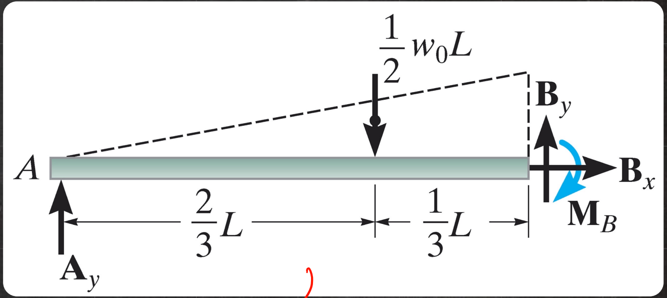




$$EI\, v'' = M$$

$$EI\, v'' = A_y x - \frac{1}{6}\omega_0 \frac{x^3}{L}$$

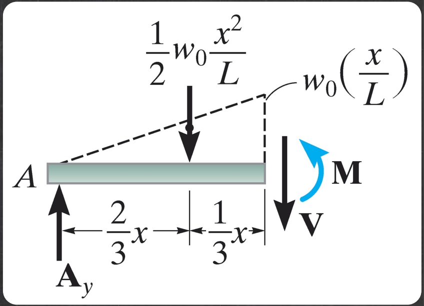
$$EI\, v' = \frac{1}{2} A_y x^2 - \frac{1}{24}\omega_0 \frac{x^4}{L} + C_1$$

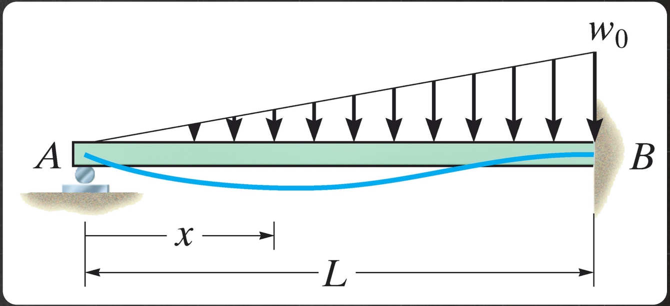


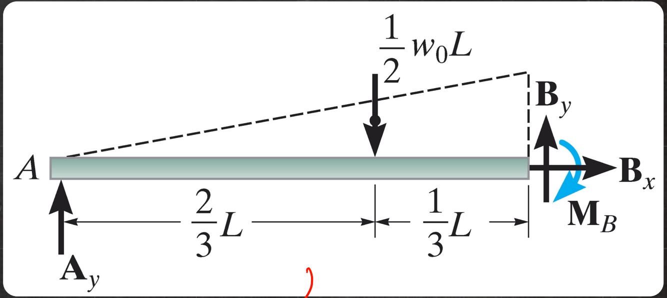
$\rhd$ 4 unknowns

$$M = A_y x - \frac{1}{6}\frac{w_0 x^3}{L}$$

$$EI v'' = M$$







$$EI v'' = A_y x - \frac{1}{6} w_0 \frac{x^3}{L}$$

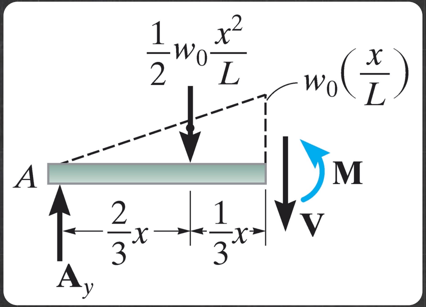
$$EI v' = \frac{1}{2} A_y x^2 - \frac{1}{24} w_0 \frac{x^4}{L} + C_1$$

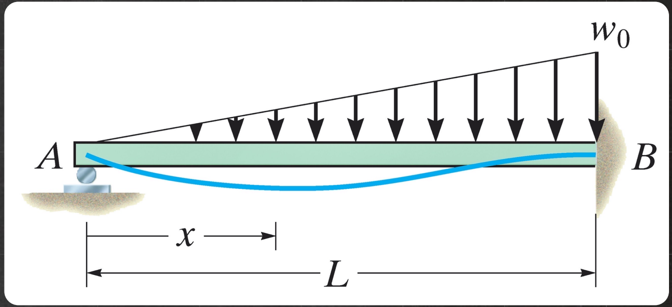
$\rightarrow$ 4 unknowns

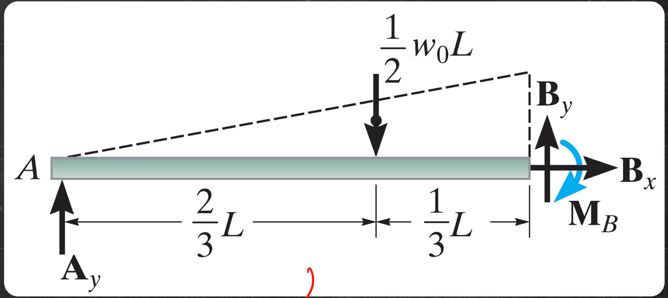
$$EI v = \frac{1}{6} A_y x^3 - \frac{1}{120} w_0 \frac{x^5}{L} + C_1 x + C_2$$

$$M = A_y x - \frac{1}{6}\frac{\omega_0 x^3}{L}$$

$$EI\,v'' = M$$







$$EI\,v'' = A_y x - \frac{1}{6}\omega_0\frac{x^3}{L}$$

$\rightarrow$ 4 unknowns

$$EI\,v' = \frac{1}{2}A_y x^2 - \frac{1}{24}\omega_0\frac{x^4}{L} + C_1$$

$$EI\,v = \frac{1}{6}A_y x^3 - \frac{1}{120}\omega_0\frac{x^5}{L} + C_1 x + C_2$$ $\leftarrow$ 3 unknowns $\leftarrow$ 3 BCs

$$M = A_y x - \frac{1}{6}\frac{\omega_o x^3}{L}$$

$$EI v'' = M$$

$$EI v'' = A_y x - \frac{1}{6}\omega_o \frac{x^3}{L}$$

$$EI v' = \frac{1}{2} A_y x^2 - \frac{1}{24}\omega_o \frac{x^4}{L} + C_1$$

$$EI v = \frac{1}{6} A_y x^3 - \frac{1}{120}\omega_o \frac{x^5}{L} + C_1 x + C_2$$

3 unknowns ← 3 BCs

BCs :

$v(x=0) = 0$

$v(x=L) = 0$

$v'(x=L) = 0$

$$M = A_y x - \frac{1}{6}\frac{w_0 x^3}{L}$$

$$EI\, v'' = M$$

$$EI\, v'' = A_y x - \frac{1}{6} w_0 \frac{x^3}{L}$$

$$EI\, v' = \frac{1}{2} A_y x^2 - \frac{1}{24} w_0 \frac{x^4}{L} + C_1$$

$$EI\, v = \frac{1}{6} A_y x^3 - \frac{1}{120} w_0 \frac{x^5}{L} + C_1 x + C_2$$

3 unknowns ← 3 BCs

BCs:

$$\left.\begin{aligned} v(x=0) &= 0 \\ v(x=L) &= 0 \\ v'(x=L) &= 0 \end{aligned}\right\}$$

$$\left.\begin{aligned} A_y &= \frac{1}{10} w_0 L \\ C_1 &= -\frac{1}{120} w_0 L^3 \\ C_2 &= 0 \end{aligned}\right\}$$

$$M = A_y x - \frac{1}{6}\frac{w_0 x^3}{L}$$

$$EI\,v'' = M$$

$$EI\,v'' = A_y x - \frac{1}{6} w_0 \frac{x^3}{L}$$

$$EI\,v' = \frac{1}{2} A_y x^2 - \frac{1}{24} w_0 \frac{x^4}{L} + C_1$$

$$EI\,v = \frac{1}{6} A_y x^3 - \frac{1}{120} w_0 \frac{x^5}{L} + C_1 x + C_2$$

3 unknowns $\leftarrow$ 3 BCs

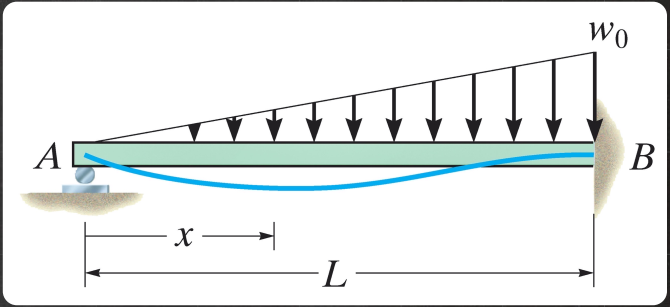
BCs : $\left.\begin{array}{l} v(x=0) = 0 \\ v(x=L) = 0 \\ v'(x=L) = 0 \end{array}\right\}$

$$\left\{\begin{array}{l} A_y = \frac{1}{10} w_0 L \\ C_1 = -\frac{1}{120} w_0 L^3 \\ C_2 = 0 \end{array}\right\}$$

$$B_y = \frac{2}{5} w_0 L \qquad M_B = w_0 L^2 / 15$$

ALTERNATIVE APPROACH

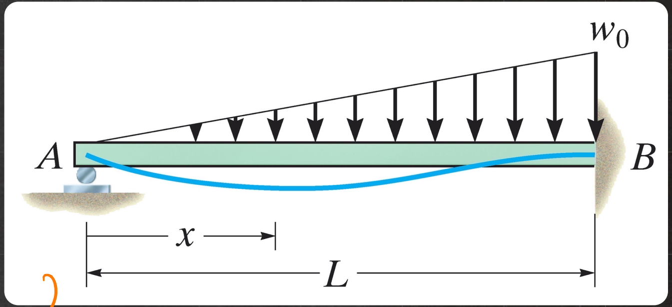
USING SUPERPOSITION

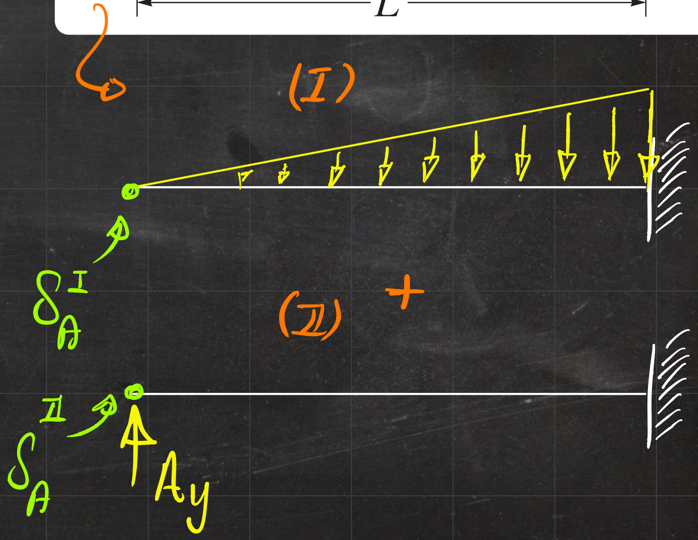


ALTERNATIVE APPROACH

USING SUPERPOSITION

$$\delta_A^{tot} \overset{!}{=} 0 \Rightarrow \delta_A^{I} + \delta_A^{II} = 0$$





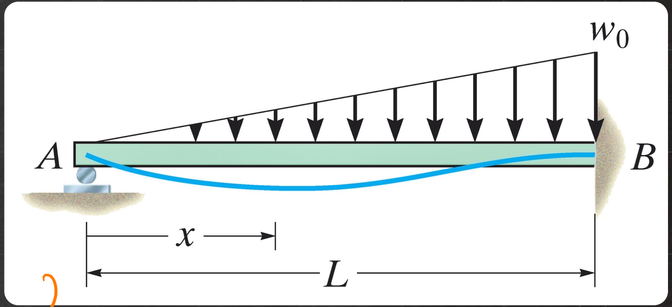
ALTERNATIVE APPROACH

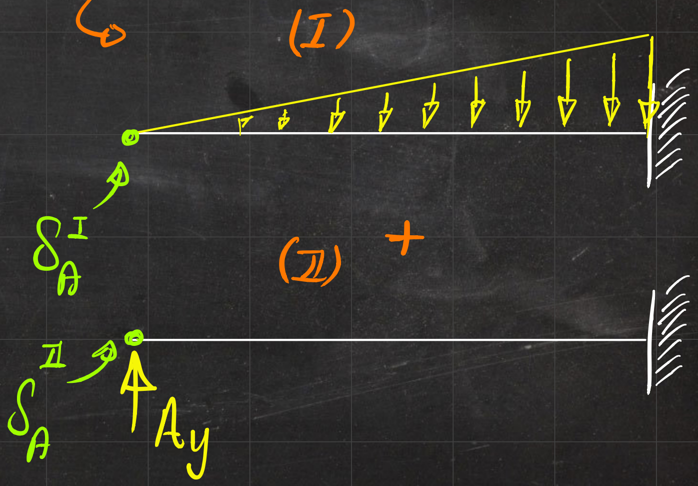
USING SUPERPOSITION

$$\delta_A^{tot} \overset{!}{=} 0 \Rightarrow \delta_A^{I} + \delta_A^{II} = 0$$

FROM TABLE :

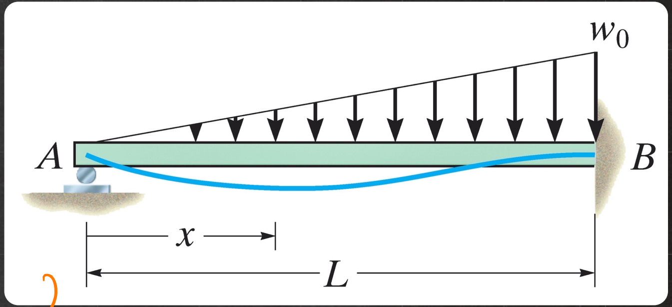
$$\delta_A^{I} = -\frac{w_0 L^4}{30\,EI}\,, \quad \delta_A^{II} = \frac{A_y L^3}{3EI}$$



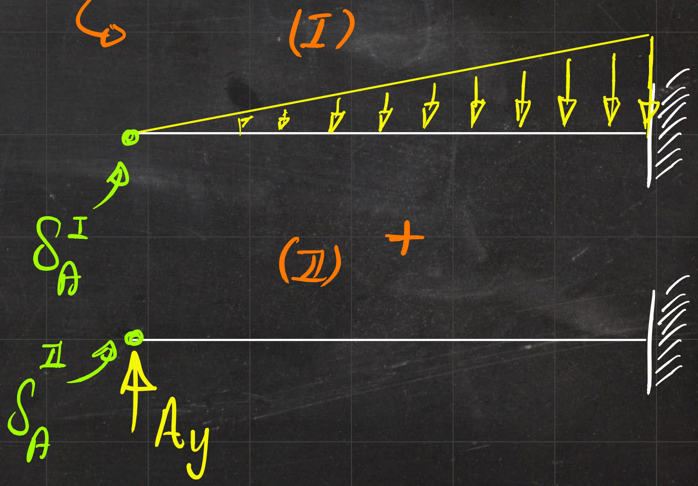


ALTERNATIVE APPROACH

USING SUPERPOSITION



$$\delta_A^{tot} \overset{!}{=} 0 \Rightarrow \delta_A^{I} + \delta_A^{II} = 0$$



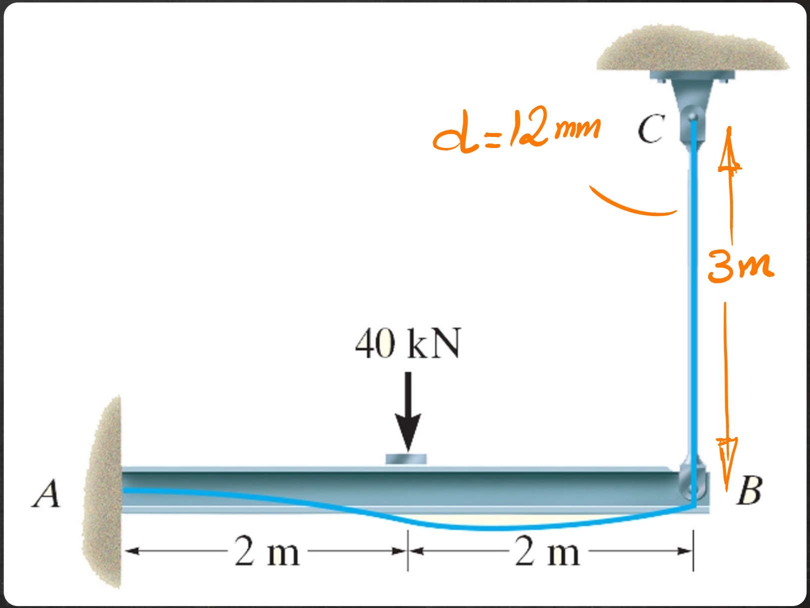
FROM TABLE:

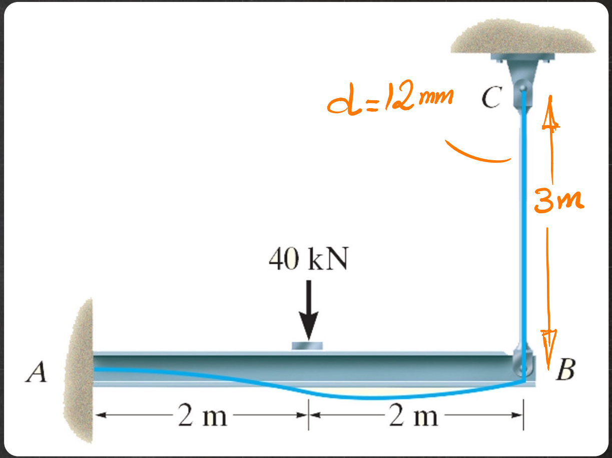
$$\delta_A^{I} = -\frac{w_0 L^4}{30\,EI}, \quad \delta_A^{II} = \frac{A_y L^3}{3EI}$$

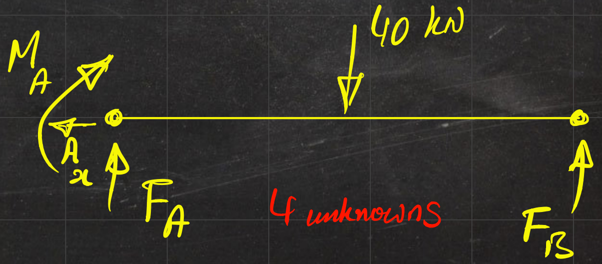
$$\Rightarrow A_y = \frac{w_0 L}{10}$$

Exercise 6 . [similar to … P. 648 … 12.22]

FOR THE BEAM-AND-ROD SYSTEM SHOWN IN THE FIGURE, DETERMINE THE FORCE IN THE ROD DUE TO THE LOADING. FOR BOTH MEMBERS $E = 210\ GPa$ AND FOR BEAM $I = 186 \times 10^{6}\ mm^{4}$.




d=12mm
C
3m
40 kN
A
B
2 m
2 m

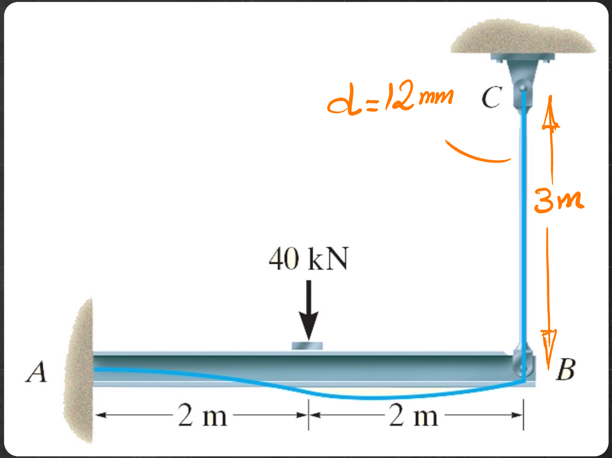

40 kN
M_A
A_x
F_A
F_B
4 unknowns

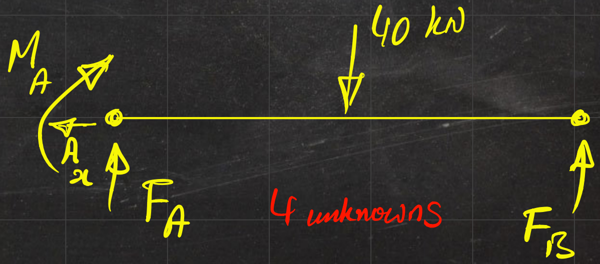
3 EQNS :

$$\sum F_x = 0$$

$$\sum F_y = 0$$

$$\sum M_A = 0$$





3 EQNS:

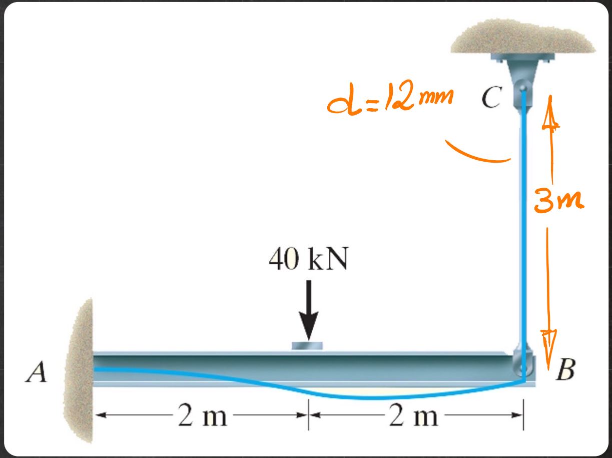
$$\sum F_x = 0$$

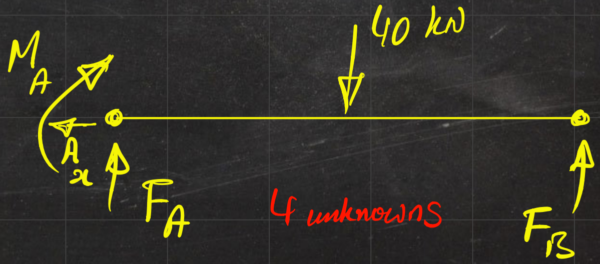
$$\sum F_y = 0$$

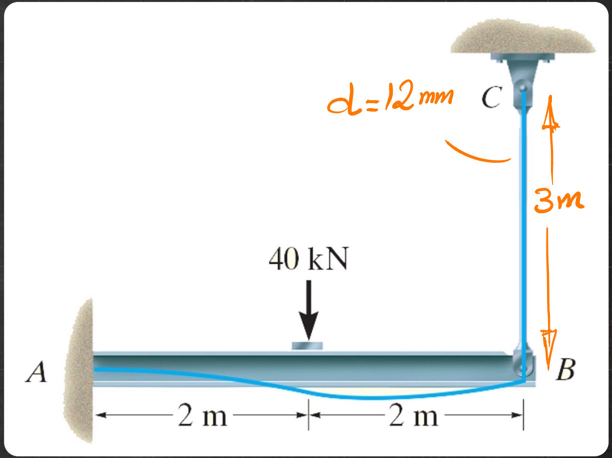
$$\sum M_A = 0$$

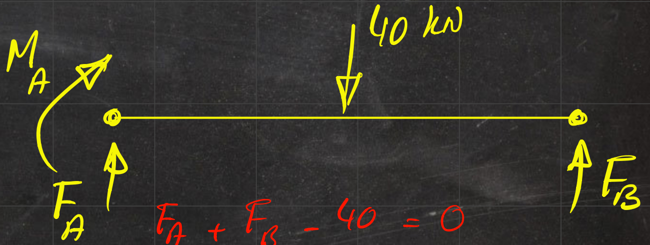
$$\Downarrow$$

$$\begin{cases} A_x = 0 \\ F_A + F_B - 40 = 0 \\ -M_A - 2 \times 40 + 4F_B = 0 \end{cases}$$





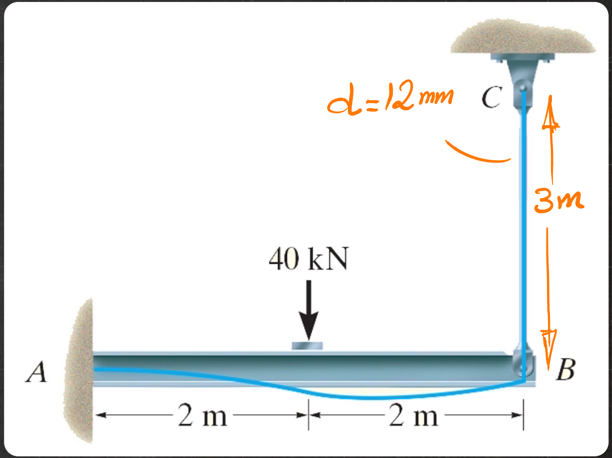


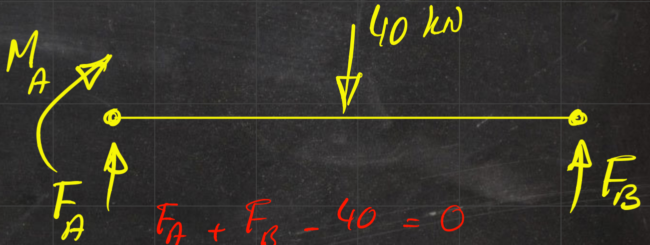


$$F_A + F_B - 40 = 0$$

$$-M_A - 2 \times 40 + 4 F_B = 0$$

$$V(x) = F_A - \langle x-2\rangle^0 \times 40$$





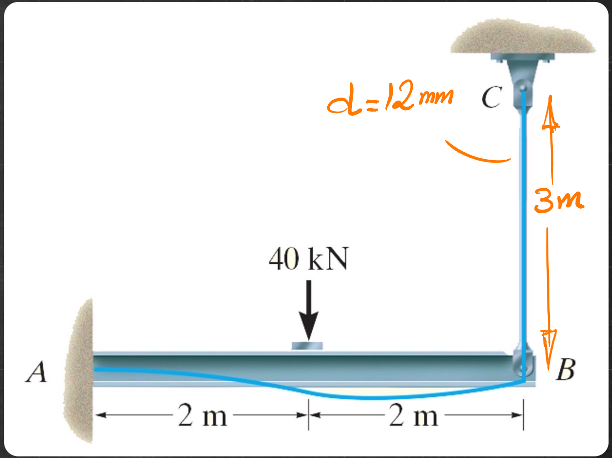
$$F_A + F_B - 40 = 0$$

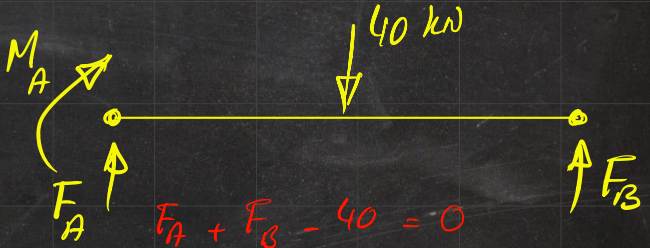
$$-M_A - 2 \times 40 + 4F_B = 0$$

$$V(x) = F_A - \langle x-2\rangle^0 \times 40$$

unknown
Constant
$M(0)$

$$M(x) = \int V\,dx = F_A\,x - 40\langle x-2\rangle^1 + M_A$$





$$F_A + F_B - 40 = 0$$

$$-M_A - 2 \times 40 + 4\,F_B = 0$$

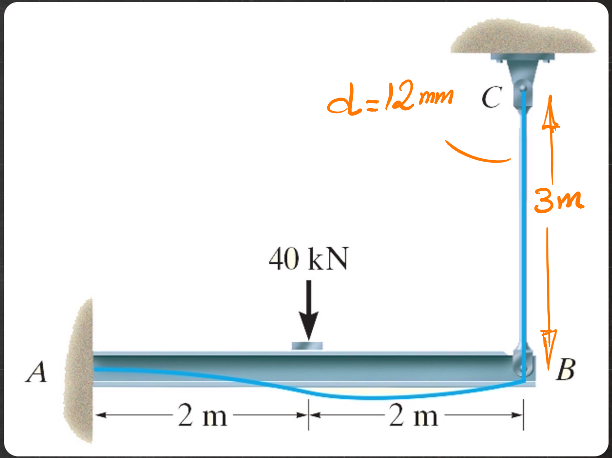
$$V(x) = F_A - \langle x-2\rangle^0 \times 40$$

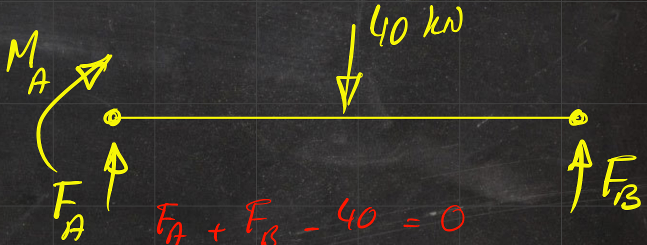
unknown Constant

$M(0)$

$$M(x) = \int V\,dx = F_A x - 40\langle x-2\rangle^1 + M_A$$

$$EI\, v'' = M$$





$$F_A + F_B - 40 = 0$$

$$-M_A - 2 \times 40 + 4 F_B = 0$$

$$V(x) = F_A - \langle x-2\rangle^0 \times 40$$

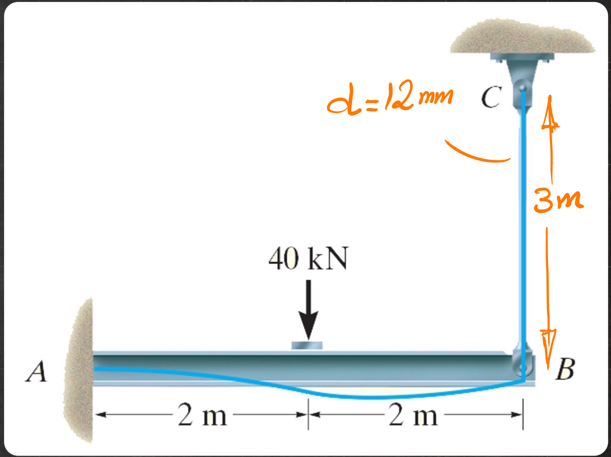
unknown Constant

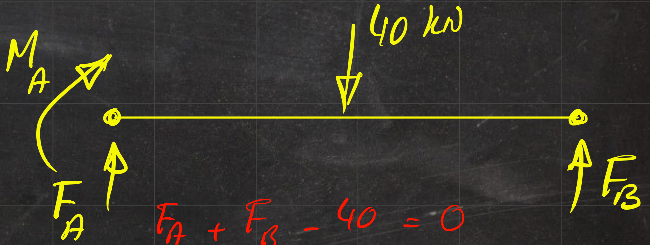
M(0)

$$M(x) = \int V\,dx = F_A x - 40\langle x-2\rangle^1 + M_A$$

$$EI\,v'' = M$$

$$EI\,v'' = F_A x - 40\langle x-2\rangle^1 + M_A$$





$$F_A + F_B - 40 = 0$$

$$-M_A - 2 \times 40 + 4F_B = 0$$

$$V(x) = F_A - \langle x-2\rangle^0 \times 40$$

unknown Constant

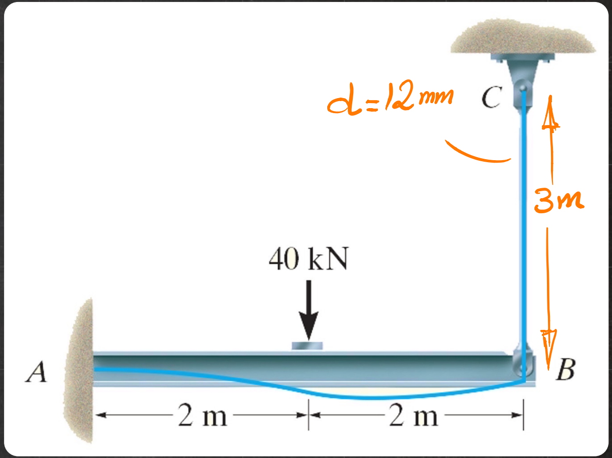
$M(0)$

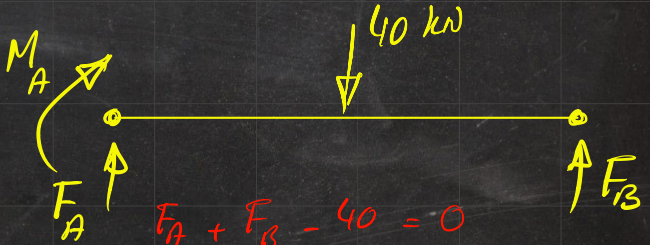
$$M(x) = \int V\,dx = F_A x - 40\langle x-2\rangle^1 + M_A$$

$$EI\,v'' = M$$

$$EI\,v'' = F_A x - 40\langle x-2\rangle^1 + M_A$$

$$EI\,v' = \frac{1}{2} F_A x^2 - 20\langle x-2\rangle^2 + M_A x + C_1$$





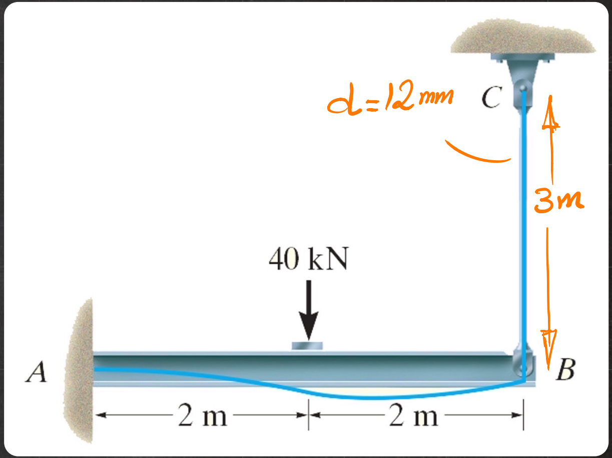
$$F_A + F_B - 40 = 0$$

$$-M_A - 2\times 40 + 4F_B = 0$$

$$V(x) = F_A - \langle x-2 \rangle^0 \times 40$$

$M(0)$ unknown Constant

$$M(x) = \int V\,dx = F_A x - 40\langle x-2\rangle^1 + M_A$$

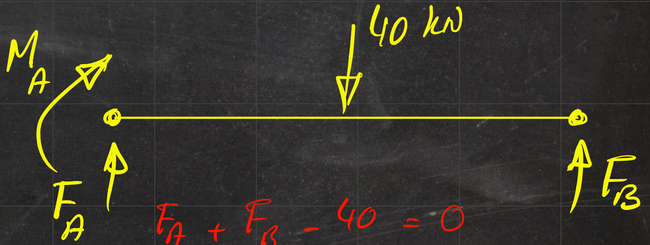


$$EI\,v'' = M$$

$$EI\,v'' = F_A x - 40\langle x-2\rangle^1 + M_A$$

$$EI\,v' = \frac{1}{2}F_A x^2 - 20\langle x-2\rangle^2 + M_A x + C_1$$

$$EI\,v = \frac{1}{6}F_A x^3 - \frac{20}{3}\langle x-2\rangle^3 + \frac{1}{2}M_A x^2 + C_1 x + C_2$$

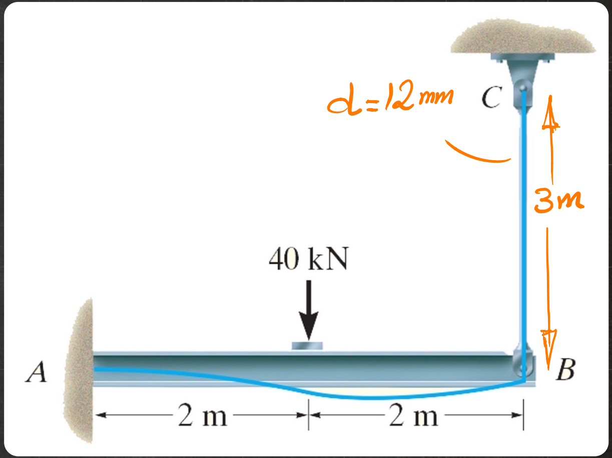


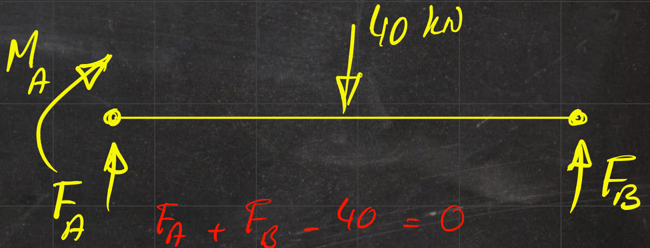
$$F_A + F_B - 40 = 0$$

$$-M_A - 2\times 40 + 4F_B = 0$$

BCs: $v(x=0)=0$

$v'(x=0)=0$





$F_A + F_B - 40 = 0$

$-M_A - 2\times 40 + 4F_B = 0$

$EIv'' = F_A x - 40\langle x-2\rangle^1 + M_A$

$EIv' = \frac{1}{2}F_A x^2 - 20\langle x-2\rangle^2 + M_A x + C_1$

$EIv = \frac{1}{6}F_A x^3 - \frac{20}{3}\langle x-2\rangle^3 + \frac{1}{2}M_A x^2 + C_1 x + C_2$

ROD

BEAM F_B F_B

BCs: $v(x=0)=0$

$v'(x=0)=0$

$v(x=4) = -\frac{F_B L}{EA}$

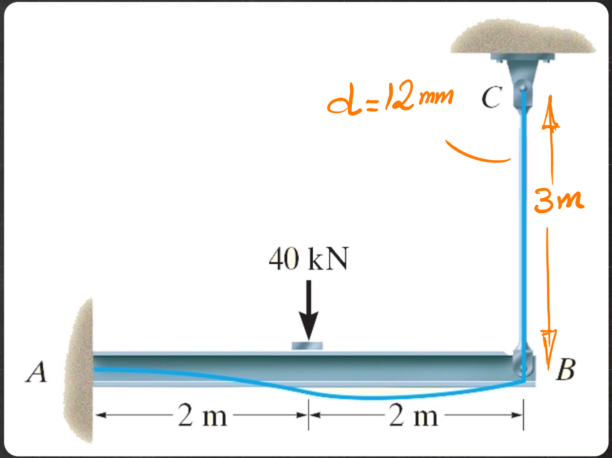
C

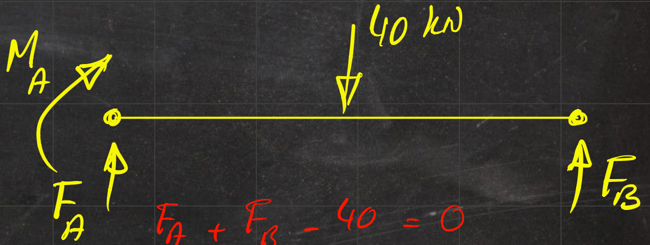
B

P

$\delta_B = -v(x=4)$

$\delta_B = \frac{PL}{EA}$





$$EI\,v'' = F_A x - 40\langle x-2\rangle^1 + M_A$$

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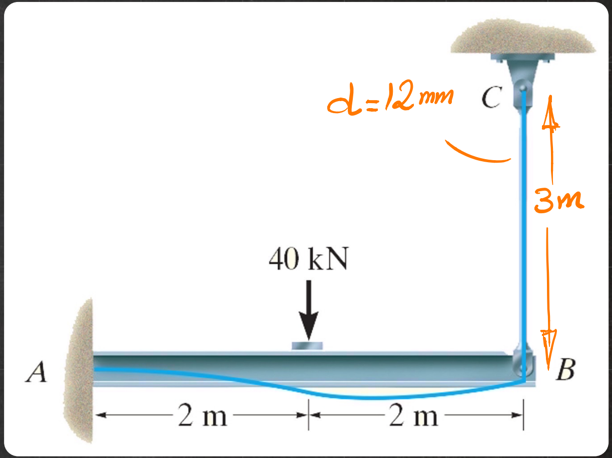
$$F_A + F_B - 40 = 0$$

$$-M_A - 2\times 40 + 4F_B = 0$$

BCs: $v(x=0) = 0$, $v'(x=0) = 0$ $\Rightarrow$ $\begin{cases} C_1 = 0 \\ C_2 = 0 \end{cases}$

$$v(x=4) = -\frac{F_B L}{EA}$$



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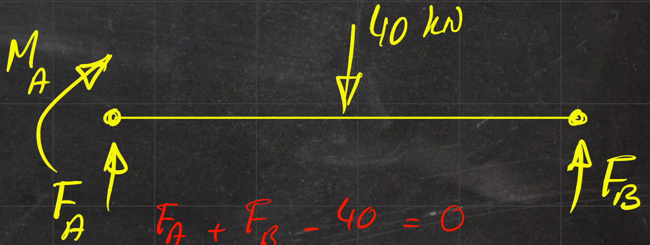


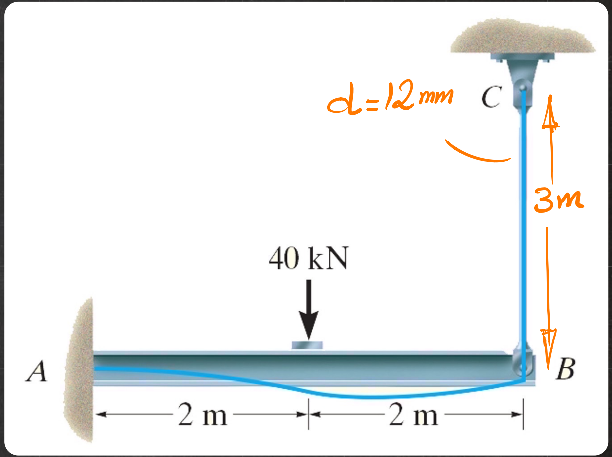
$$F_A + F_B - 40 = 0$$

$$-M_A - 2 \times 40 + 4 F_B = 0$$

$$\text{BCs:}\quad \left.\begin{aligned} v(x=0) &= 0 \\ v'(x=0) &= 0 \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} C_1 &= 0 \\ C_2 &= 0 \end{aligned}\right.$$

$$v(x=4) = -\frac{F_B L}{EA}$$

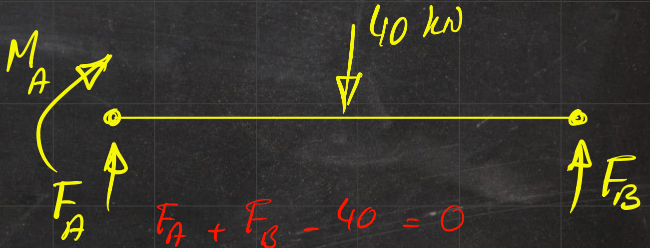
$$x=4 \Rightarrow EIv = -\frac{I F_B L}{A} = \frac{32}{3}F_A - \frac{160}{3} + 8M_A$$



$$EIv'' = F_A x - 40\langle x-2\rangle^1 + M_A$$

$$EIv' = \frac{1}{2}F_A x^2 - 20\langle x-2\rangle^2 + M_A x + C_1$$

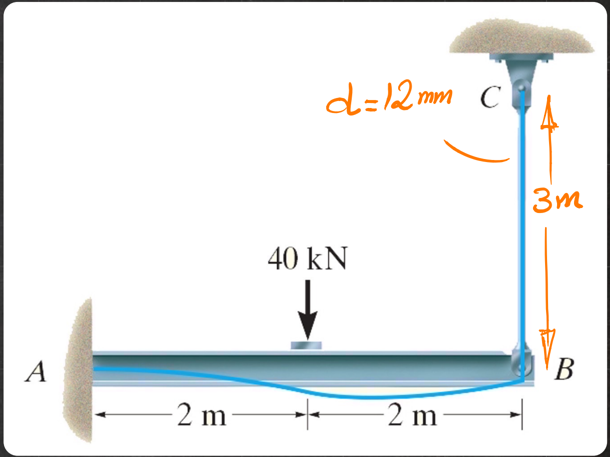
$$EIv = \frac{1}{6}F_A x^3 - \frac{20}{3}\langle x-2\rangle^3 + \frac{1}{2}M_A x^2 + C_1 x + C_2$$



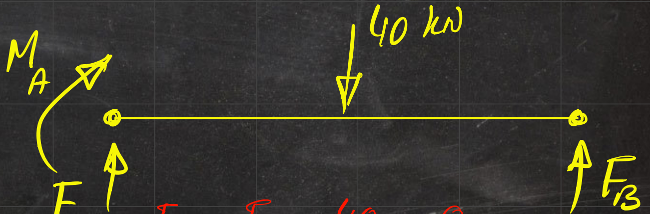
$$F_A + F_B - 40 = 0$$

$$-M_A - 2\times 40 + 4F_B = 0$$

3 Unknowns : F_A , F_B , M_A



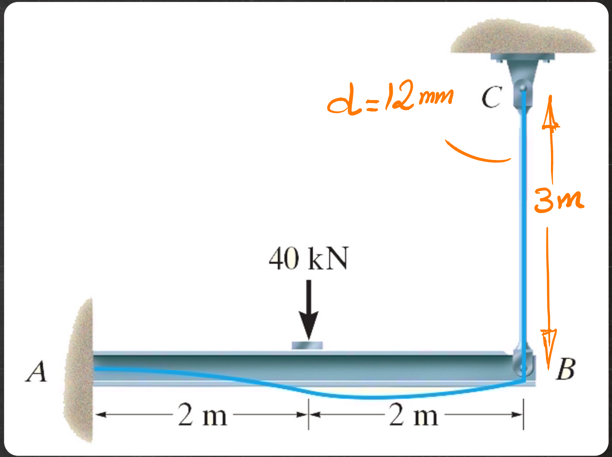
EQ. 3 $\rightarrow$ $EIv = -\dfrac{I F_B L}{A} = \dfrac{32}{3}F_A - \dfrac{160}{3} + 8M_A$



EQ. 1, 2 $\rightarrow$

$F_A + F_B - 40 = 0$

$-M_A - 2 \times 40 + 4F_B = 0$



$40 - F_B \qquad 4F_B - 80$

$$EIv = -\frac{I F_B L}{A} = \frac{32}{3}F_A - \frac{160}{3} + 8M_A$$

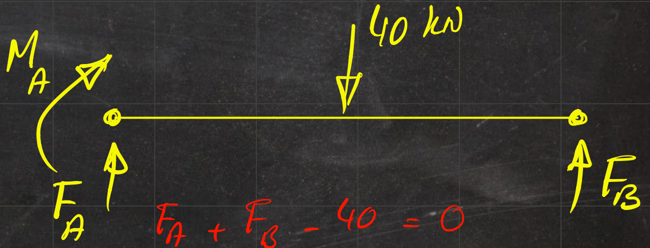
186×10^{-6}

$$-\frac{I F_B L}{A} = \frac{32}{3}(40 - F_B) - \frac{160}{3} + 8(4F_B - 80)$$

$L = 3$

$A = \frac{\pi}{4} d^2$

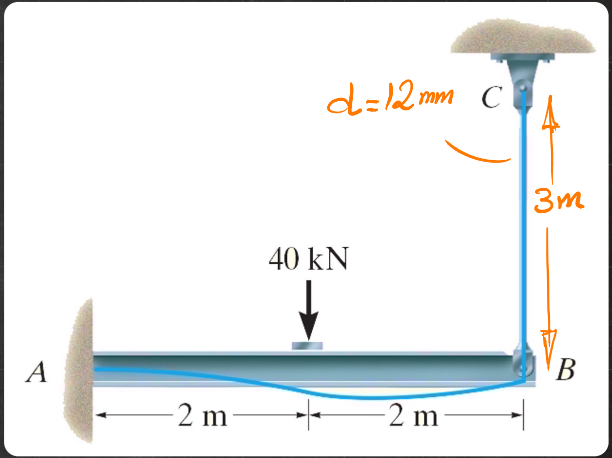
$d = 12\ mm$



$F_A + F_B - 40 = 0$

$-M_A - 2 \times 40 + 4F_B = 0$

$$\Rightarrow F_B = 10.15\ kN$$



$40 - F_B$ $\quad\quad$ $4F_B - 80$

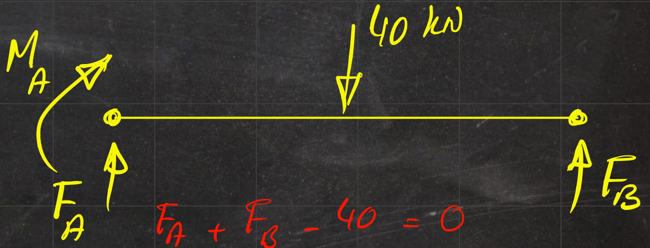
$$186 \times 10^{-6}$$

$$EIv = -\frac{I F_B L}{A} = \frac{32}{3} F_A - \frac{160}{3} + 8 M_A$$

$$-\frac{I F_B L^{3}}{A} = \frac{32}{3}(40 - F_B) - \frac{160}{3} + 8(4F_B - 80)$$

$$\frac{\pi}{4} d^2$$

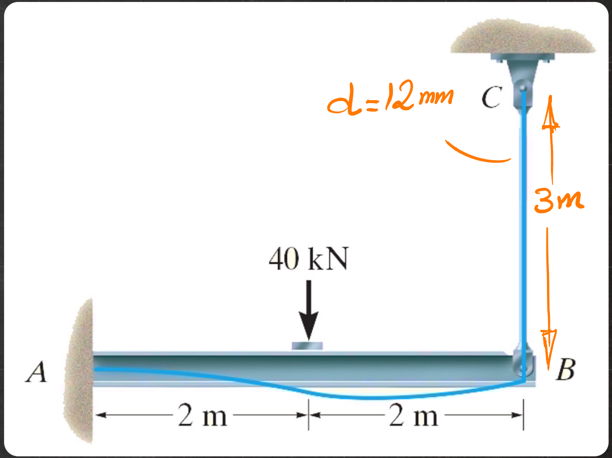
$d = 12\ mm$



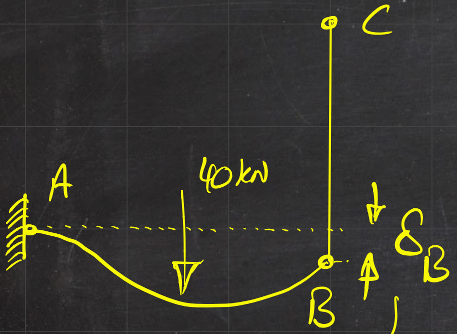
$$F_A + F_B - 40 = 0$$

$$-M_A - 2 \times 40 + 4F_B = 0$$

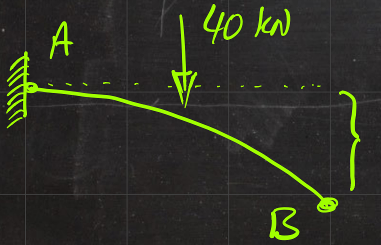
ALTERNATIVE APPROACH USING SUPERPOSITION



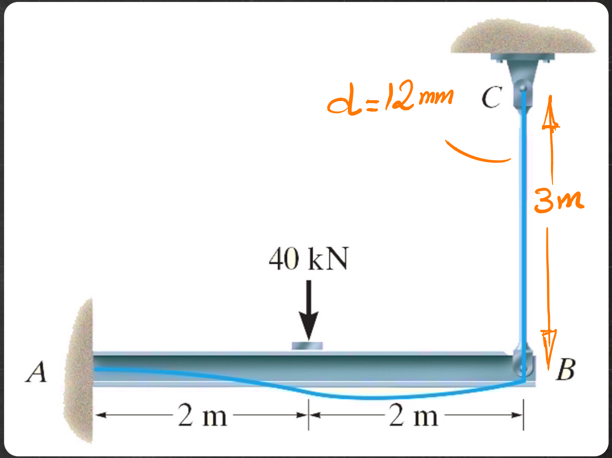
ALTERNATIVE APPROACH USING SUPERPOSITION



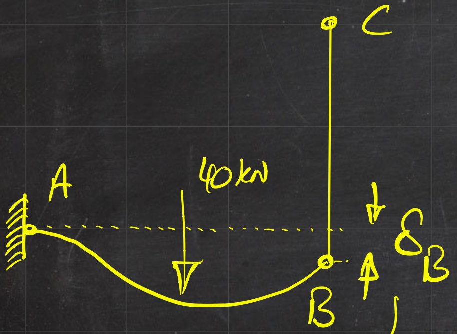
$$- \frac{F_B L}{EA}$$



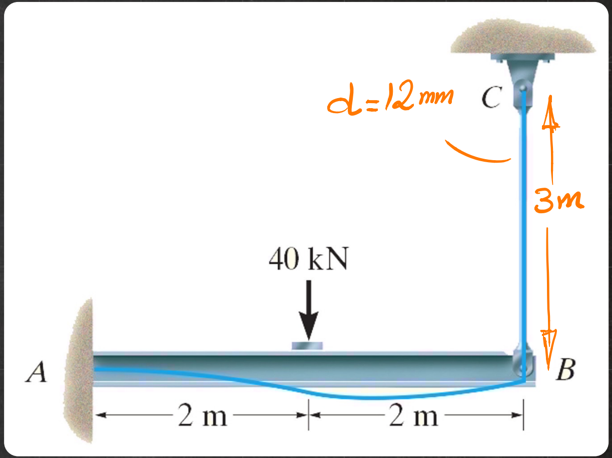
$$\delta_B^{I} \Rightarrow - \frac{200 L^3}{48EI}$$

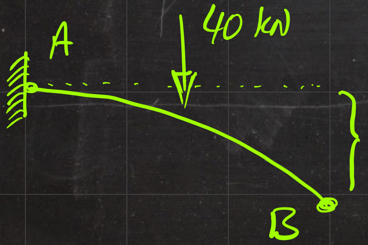


ALTERNATIVE APPROACH USING SUPERPOSITION



$$\delta_B \rightarrow -\frac{F_B L}{EA}$$

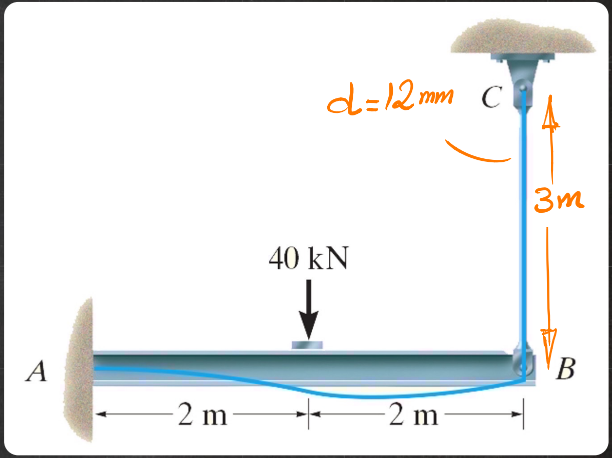


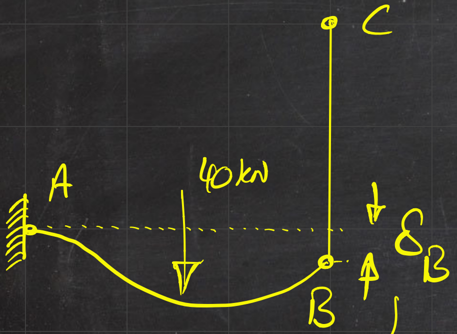


$$\delta_B^{I} \rightarrow -\frac{200 L^3}{48EI}$$

$$\delta_B^{II} \rightarrow \frac{F_B L^3}{3EI}$$

ALTERNATIVE APPROACH USING SUPERPOSITION



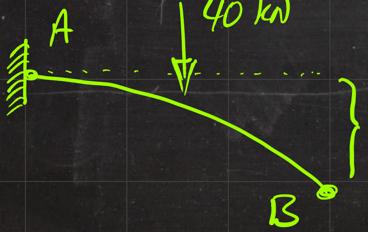


$$\delta_B = \delta_B^{I} + \delta_B^{II}$$

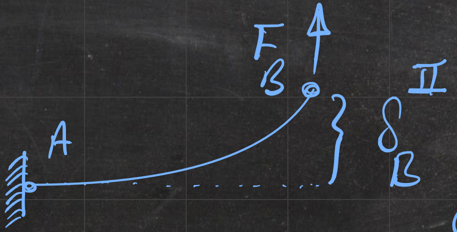
$$\Downarrow$$

$$F_B = 10.15 \text{ kN}$$

$$\delta_B \rightarrow -\frac{F_B L}{EA}$$



$$\delta_B^{I} \rightarrow -\frac{200 L^3}{48EI}$$



$$\delta_B^{II} \rightarrow \frac{F_B L^3}{3EI}$$

MECHANICS AND MATERIALS I

Deflection of beams and shafts

Sect. ... 12.1 – 12.3 ... 12.5 – 12.7 ... 12.9

Chap. 12

[Hibbeler 9th edition]

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MECHANICS AND MATERIALS I

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