FINITE ELEMENT METHOD

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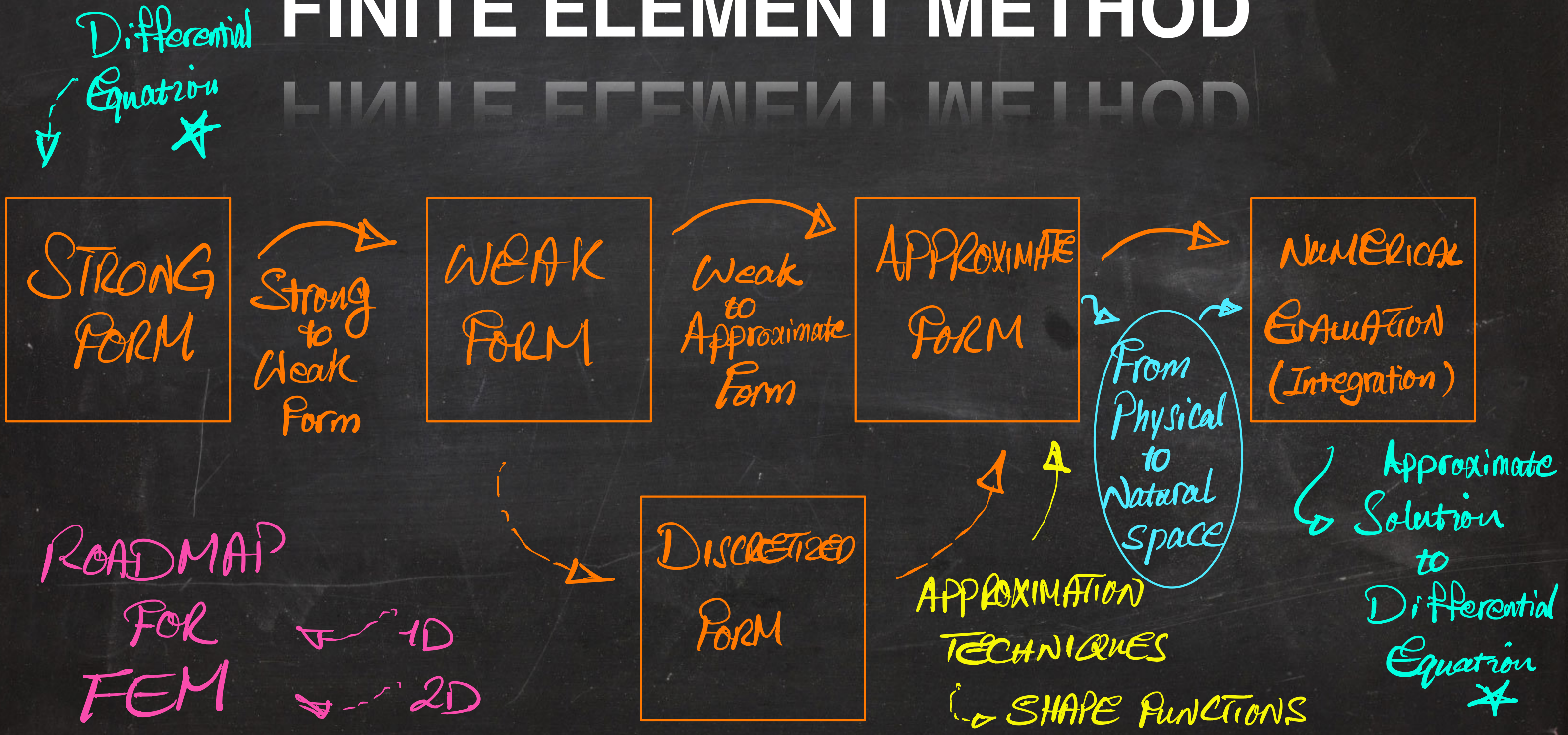
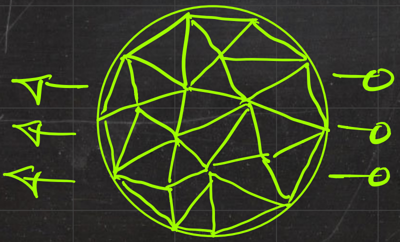
FINITE ELEMENT METHOD



UNDERSTANDING FEM VIA AN ANALOGY (A BRUTAL SIMPLIFICATION)

Approximations in FEM

- → Solution Approximation → inherent to numerical techniques using computers
- → Equation Approximation → diff equation is transformed to weak form + space approximation
- → Input Approximation → space discretization



Discretization (Approximation)
→ DOMAIN (X)
→ SOLUTION (u)
→ TEST (w)

diff. Eq.
→ STRONG FORM
TO
integral → WEAK FORM

$$\frac{d}{dx}\left(EA\frac{du}{dx}\right)+bA=0 \quad \text{Subject to BCs}$$

$\hookrightarrow$ E, A: Const. $\leadsto$ $EAu'' + bA = 0$ $\quad f := \frac{b}{E}$

STRONG FORM

$$\leadsto \boxed{\begin{aligned} & u'' + f = 0 \qquad 0 \leqslant x \leqslant 1 \\ & D: \; u(0) = u_0 \\ & N: \; u'(1) = t \end{aligned}}$$

prescribed

FROM STRONG TO WEAK FORM

$$u'' + f = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = u_0$ ← prescribed

N: $u'(1) = t$ ←

STRONG FORM ← Differential Eq.

(I) MULTIPLY BY TEST FUNCTION w

(II) INTEGRATE OVER THE DOMAIN

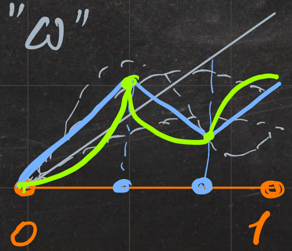
w : ARBITRARY, CONTINUOUS, $w|_D = 0$ ← ZERO @ DIRICHLET BOUNDARY CONDITIONS

Integral form ← WEAK FORM

→ BECAUSE LOWER ORDER DIFFERENTIATION OF DISPLACEMENT u

STRONG : u''

WEAK : u'



FROM STRONG TO WEAK FORM

$$u'' + f = 0 \qquad 0 \leq x \leq 1$$

D: $u(0) = u_0$ ← prescribed

N: $u'(1) = t$

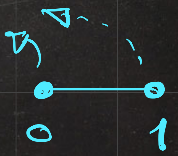
I) $[u'' + f = 0] \times w \Rightarrow w u'' + w f = 0$

II) $\int_0^1 [w u'' + w f]\, dx = 0$ $\qquad w u'' = (w u')' - w' u'$

↙ WEAK FORM

$$\underbrace{\int_0^1 w' u' \, dx}_{\text{INTERNAL CONTRIBUTIONS OVER THE DOMAIN}} = \underbrace{\int_0^1 w f \, dx}_{\text{EXTERNAL CONTRIBUTIONS OVER THE DOMAIN}} + \underbrace{w(1)\, u'(1) - w(0)\, u'(0)}_{\text{EXTERNAL CONTRIBUTIONS OVER THE BOUNDARY OF THE DOMAIN}}$$

$$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$$



FROM STRONG TO WEAK FORM

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$

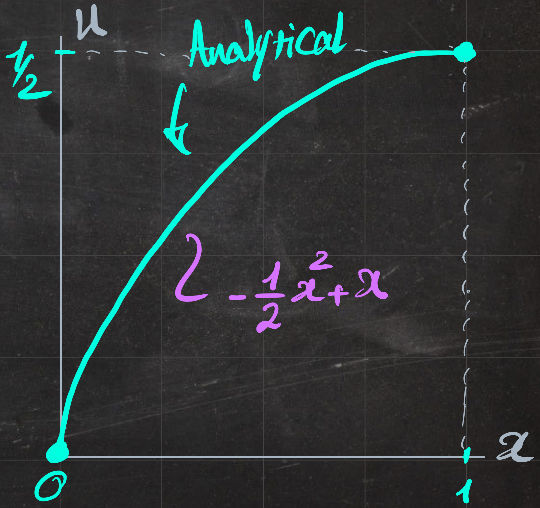
$$u'' = -1 \Rightarrow u' = -x + C_1$$

$$\Rightarrow u = -\frac{1}{2}x^2 + C_1 x + C_2$$

$u(0)=0 \Rightarrow C_2 = 0$

$u'(1)=0 \Rightarrow C_1 = 1$

Analytical Solution $\Rightarrow u = -\frac{1}{2}x^2 + x$



FROM STRONG TO WEAK FORM

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

$$D: \; u(0) = 0 \qquad \text{prescribed}$$

$$N: \; u'(1) = 0$$

$$\int_0^1 w' u' \, dx = \int_0^1 w f \, dx + w(1) u'(1) - w(0) u'(0)$$

$$\Rightarrow \int_0^1 w' u' \, dx = \int_0^1 w \, dx$$

WEAK FORM

w : ARBITRARY, CONTINUOUS, $w|_D = 0$

Compute approximate solution

from different spaces

$\Downarrow$

EXERCISE $\mapsto$ ...

UNDERSTANDING THE SIGNIFICANCE OF WEAK FORM BY EXAMPLE

$$\int_0^1 w' u' \, dx = \int_0^1 w \, dx$$

$$u'' + 1 = 0 \qquad 0 \leq x \leq 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

$$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$$

→ 1-PIECE LINEAR APPROXIMATION

→ 2-PIECE LINEAR (UNIFORM) APPROXIMATION

→ 2-PIECE LINEAR (NON-UNIFORM) APPROXIMATION (I)

→ 2-PIECE LINEAR (NON-UNIFORM) APPROXIMATION (II)

→ 2-PIECE LINEAR (GENERAL) APPROXIMATION

UNDERSTANDING THE SIGNIFICANCE OF WEAK FORM BY EXAMPLE

$$\int_0^1 w' u' \, dx = \int_0^1 w \, dx$$

$u'' + 1 = 0 \qquad 0 \leq x \leq 1$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$

$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$

→ 3-PIECE LINEAR (UNIFORM) APPROXIMATION

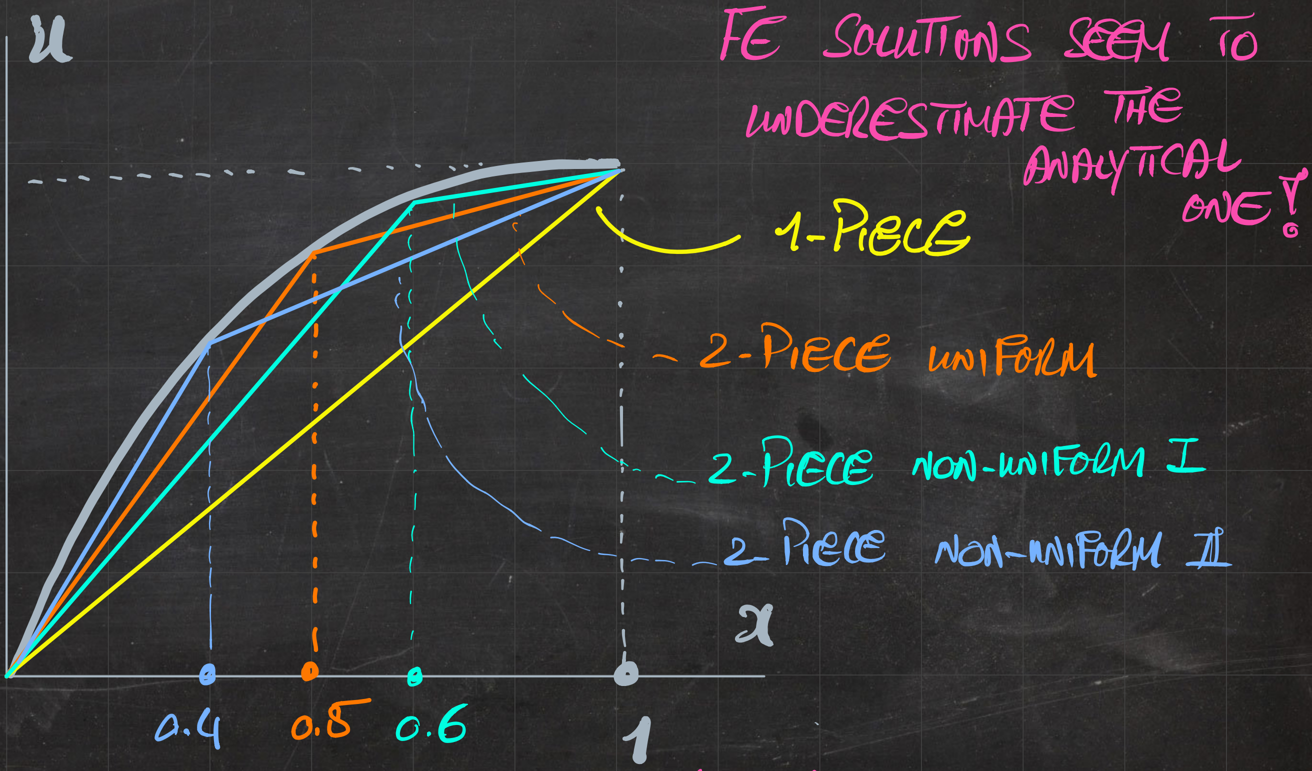
→ 3-PIECE LINEAR (GENERIC) APPROXIMATION

→ 4-PIECE LINEAR (UNIFORM) APPROXIMATION

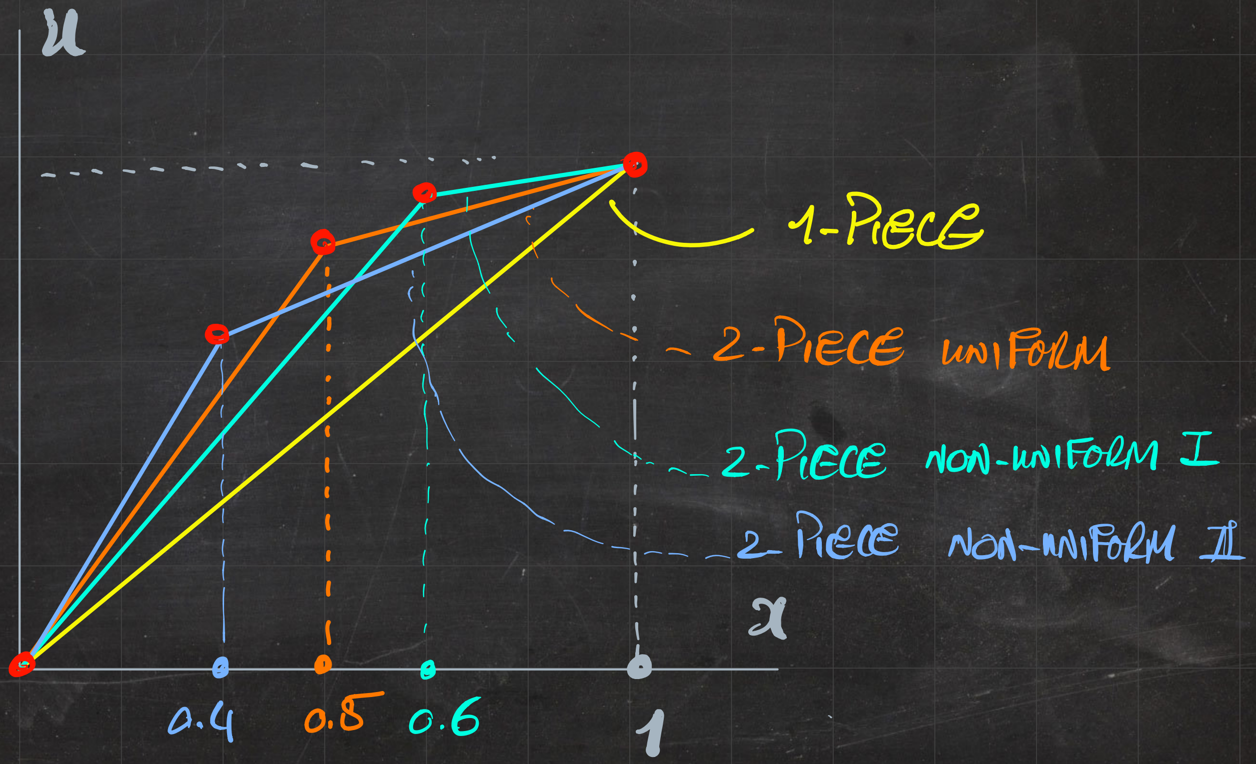
→ 4-PIECE LINEAR (GENERIC) APPROXIMATION

→ 1-PIECE QUADRATIC

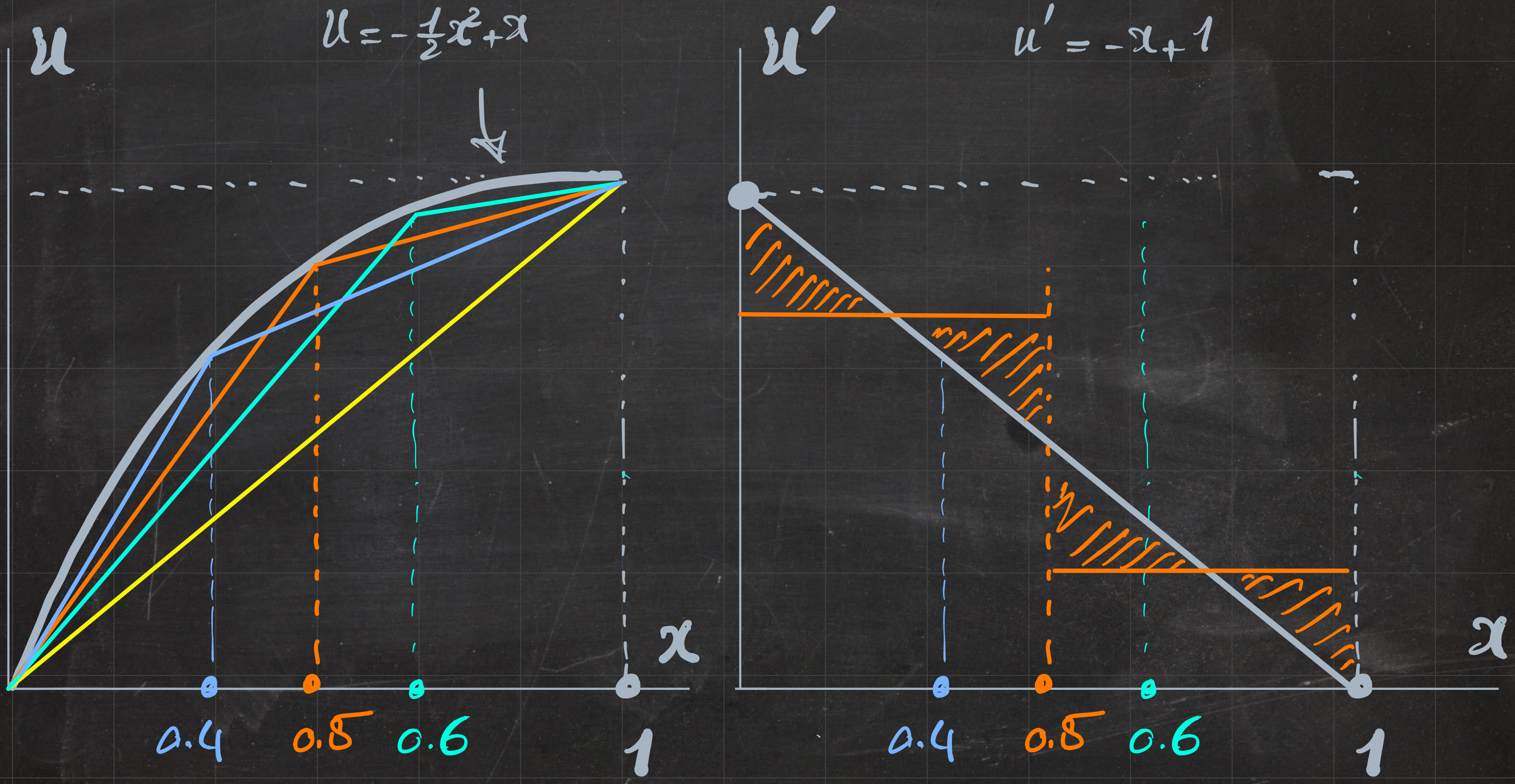
→ 1-PIECE CUBIC



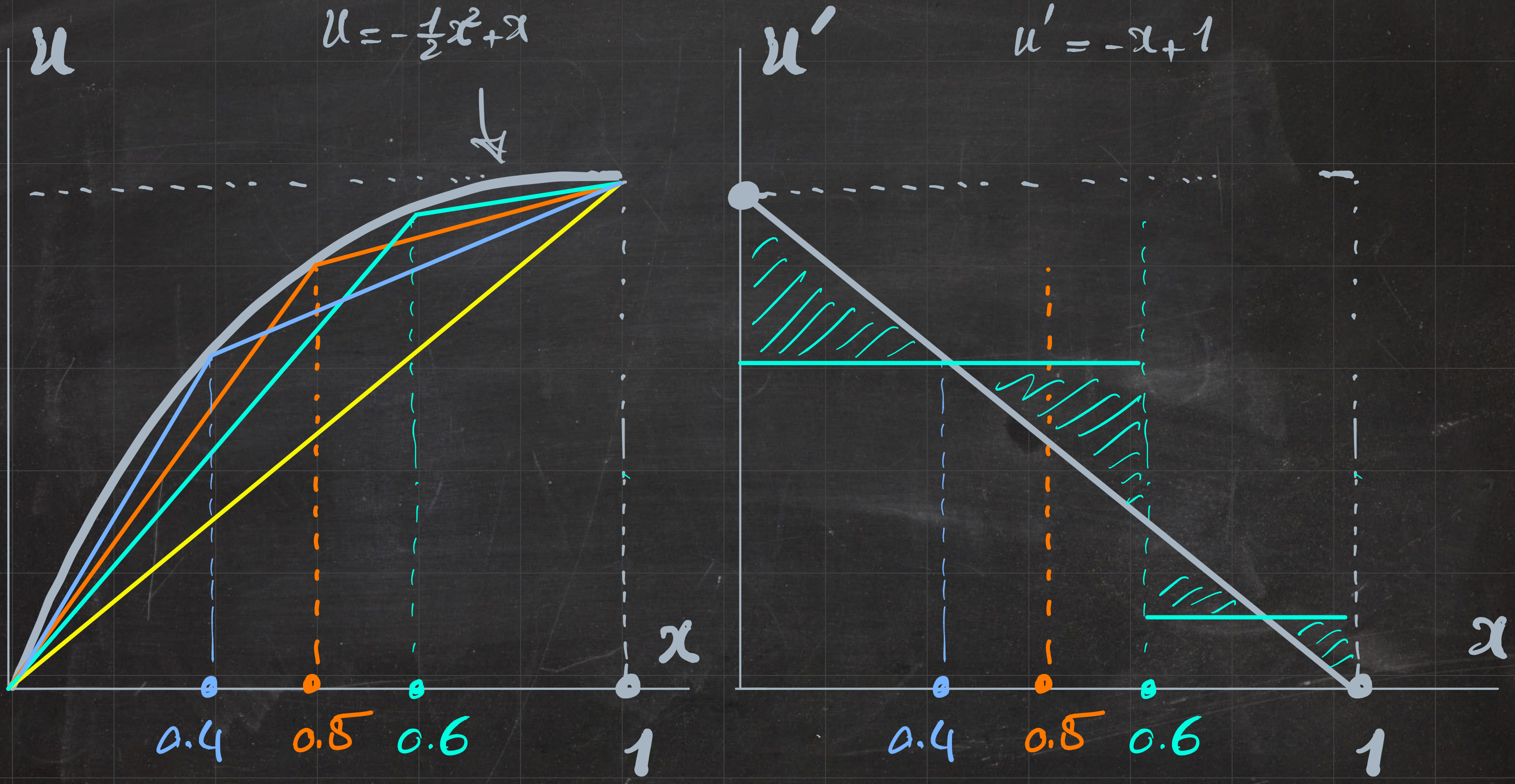
FE Solution approaches analytical one from below!



u
1-PIECE
2-PIECE UNIFORM
2-PIECE NON-UNIFORM I
2-PIECE NON-UNIFORM II
x
0.4
0.5
0.6
1



u
u = -½x² + x
x
0.4
0.5
0.6
1
u′
u′ = -x + 1
x
0.4
0.5
0.6
1



u
$u = -\frac{1}{2}x^2 + x$
x
0.4
0.5
0.6
1
u'
$u' = -x + 1$
x
0.4
0.5
0.6
1

$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \qquad x \in [0,1]$$

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = 0$ ← prescribed

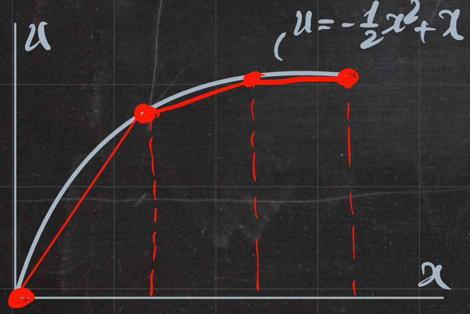
N: $u'(1) = 0$

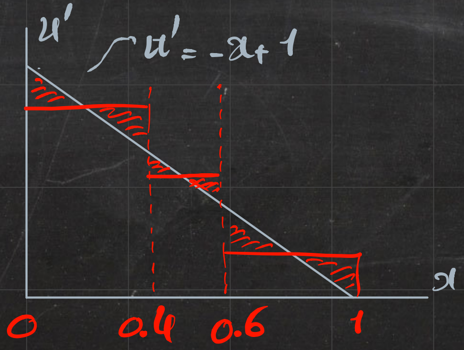
→ 3 - PIECE LINEAR (GENERIC) APPROXIMATION

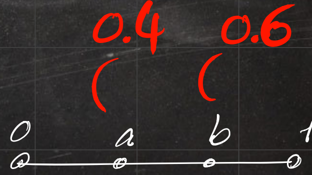
$$\begin{cases} u = 0.8x & x \in [0, 0.4] \\ u = 0.5x + 0.12 & x \in [0.4, 0.6] \\ u = 0.2x + 0.3 & x \in [0.6, 1] \end{cases}$$

$$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$$

ANALYTICAL







$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \qquad x \in [0,1]$$

$\rightarrow$ 4-PIECE LINEAR (GENERIC) APPROXIMATION

$$\begin{cases} u = \frac{7}{8}x & x \in [0, 0.25] \\ u = \frac{5}{8}x + \frac{1}{16} & x \in [0.25, 0.50] \\ u = \frac{3}{8}x + \frac{3}{16} & x \in [0.50, 0.75] \\ u = \frac{1}{8}x + \frac{6}{16} & x \in [0.75, 1.00] \end{cases}$$

$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

$$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$$

ANALYTICAL

$0 < a < b < c < 1$

$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \qquad x \in [0,1]$$

$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$

→ 1-PIECE QUADRATIC APPROXIMATION

$w|_D = 0$

$x \in [0,1] \qquad w = C_1 x^2 + C_2 x + \cancel{C_3}^{\,0}$

$u = D_1 x^2 + D_2 x + \cancel{D_3}^{\,0}$ ← $u(0)=0$

w : ARBITRARY, CONTINUOUS, $w|_D = 0$

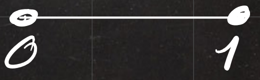
ANALYTICAL

IF THE APPROXIMATION SPACE IS LARGE ENOUGH, IT CAN INCLUDE THE EXACT SOLUTION!

$$u = -\frac{1}{2}x^2 + x$$

approximation that has zero error

IDENTICAL TO ANALYTICAL SOLUTION



$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \qquad x \in [0,1]$$

$$\boxed{\begin{array}{ll} u'' + 1 = 0 & 0 \leq x \leq 1 \\ D:\ u(0) = 0 & \leftarrow \text{prescribed} \\ N:\ u'(1) = 0 & \leftarrow \end{array}}$$

→ 1-PIECE CUBIC APPROXIMATION

$$x \in [0,1] \qquad u = D_1 x^3 + D_2 x^2 + D_3 x + \cancelto{0}{D_4}$$

$$w: \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$$

ANALYTICAL

$$\ldots \Rightarrow c_1\left[\tfrac{9}{5} D_1 + \tfrac{6}{4} D_2 + D_3 - \tfrac{1}{4}\right] + c_2\left[\tfrac{6}{4} D_1 + \tfrac{4}{3} D_2 + D_3 - \tfrac{1}{3}\right] + c_3\left[D_1 + D_2 + D_3 - \tfrac{1}{2}\right] = 0 \qquad \checkmark\ c_1, c_2, c_3$$

$$\begin{cases} \left[\tfrac{9}{5} D_1 + \tfrac{6}{4} D_2 + D_3 - \tfrac{1}{4}\right] = 0 \\ \left[\tfrac{6}{4} D_1 + \tfrac{4}{3} D_2 + D_3 - \tfrac{1}{3}\right] = 0 \\ \left[D_1 + D_2 + D_3 - \tfrac{1}{2}\right] = 0 \end{cases} \Rightarrow \begin{cases} D_1 = 0 \\ D_2 = -\tfrac{1}{2} \\ D_3 = 1 \end{cases} \Rightarrow u = -\tfrac{1}{2}x^2 + x$$

if approximation space is large enough, we recover the exact solution!

FINITE ELEMENT METHOD IN A NUTSHELL

2ND. O.D.E.

Diff. Eq.

STRONG FORM

$$(EAu')' + b = 0$$

(I) MULTIPLY BY w (test function)

(II) INTEGRATE

WEAK FORM

INTEGRAL FORM

$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx + w(1)\,u'(1) - w(0)\,u'(0)$$

PIECEWISE

DISCRETIZED FORM

Approximation

Approximate Discretized Weak Form

APPROXIMATE FORM

NUMERICAL INTEGRATION

another source of approx...

ELEMENT-WISE QUANTITIES

PIECEWISE INTEGRALS (SOLUTIONS)

GLOBAL SYSTEM

$$[K][u] = [F]$$

ASSEMBLY

SOLVE

PostProcess

FROM GLOBAL TO ELEMENTS

FROM INTEGRAL OVER THE DOMAIN TO SUBINTEGRALS

$$\int_0^1 \ldots\,dx = \int_0^a \ldots\,dx + \int_a^b \ldots\,dx + \ldots$$

APPROXIMATION: UNDERSTANDING VIA EXAMPLES

(I)

$u(0) = 0$

$u(1) = 1$

$u(0.5) = ? \approx 0.5$

$u(0.8) = ? \approx 0.8$

(II)

$u(0) = 1$

$u(1.6) = 5$

$u(0.8) = ? \approx 3$

$u(1) = ? \approx 3.5$

(III)

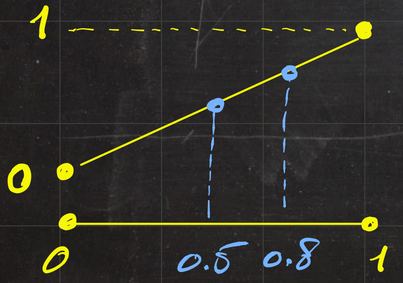
$u(0) = 1$

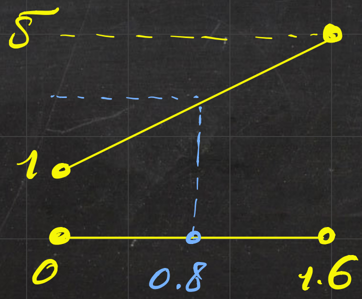
$u(0.5) = 2$

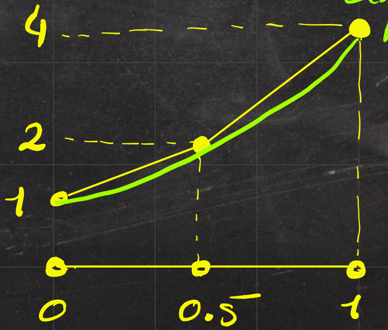
$u(1) = 4$

$u(0.8) = ? \approx 3.08$ QUADRATIC APPROXIMATION

LINEAR APPROXIMATION 3.2







APPROXIMATION: UNDERSTANDING VIA EXAMPLES

QUADRATIC APPROXIMATION?

(III) $u(0) = 1$

$u(0.5) = 2$

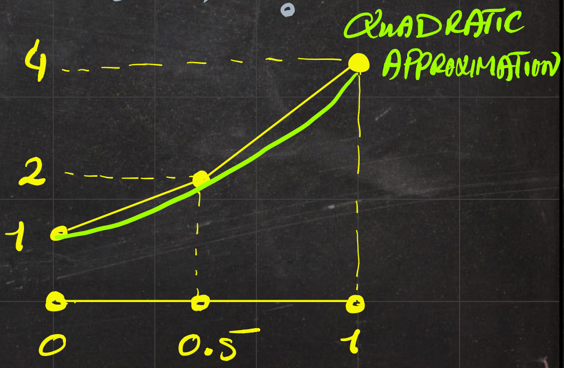
$u(1) = 4$

$u(0.8) = ? \approx 3.08$

LINEAR APPROXIMATION 3.2

$$f(x) = ax^2 + bx + c \Rightarrow f(x) = 2x^2 + x + 1$$

$$\left.\begin{aligned} f(0) &= 1 \\ f(0.5) &= 2 \\ f(1) &= 4 \end{aligned}\right\} \Rightarrow \begin{matrix} 3 \text{ EQNS} \\ 3 \text{ UNKNOWNS} \end{matrix} \Rightarrow \begin{aligned} a &= 2 \\ b &= 1 \\ c &= 1 \end{aligned}$$



APPROXIMATION: UNDERSTANDING VIA EXAMPLES

(IV)

$$\left.\begin{aligned} u(0)&=1 \\ u(0.2)&=2 \\ u(0.6)&=4 \\ u(1)&=8 \end{aligned}\right\} \Rightarrow u(0.8) = ?$$

$$f(x) = ax^3 + bx^2 + cx + d$$

$$\left.\begin{aligned} &4 \text{ Equations} \\ &4 \text{ Unknowns} \end{aligned}\right\} \Rightarrow a, b, c, d$$

$\checkmark$ $f(x)$

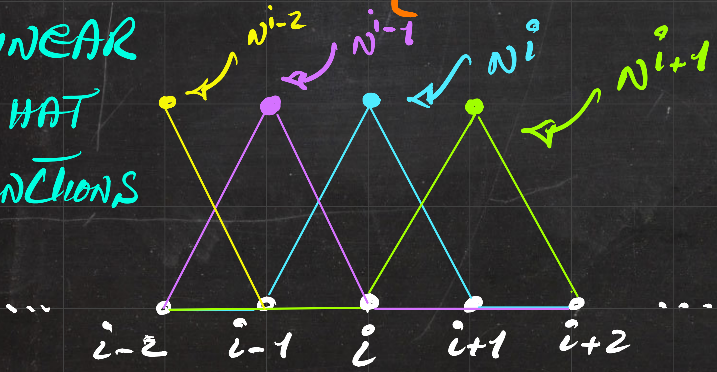
SHAPE FUNCTIONS (HAT FUNCTIONS, TENT FUNCTIONS)

↳ A powerful tool for approximations ⇝ SYSTEMATIC

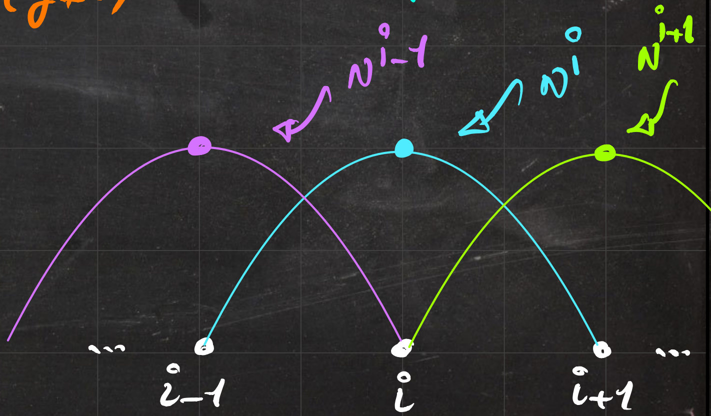
⇝ NEARLY IDENTICAL FOR 2D / 3D

$$N^i(x) \rightsquigarrow \begin{cases} N^i = 1 \ @ \ x^j \ (j = i) \\ N^i = 0 \ @ \ x^j \ (j \neq i) \end{cases}$$

LINEAR HAT FUNCTIONS



QUADRATIC HAT FUNCTIONS



SHAPE FUNCTIONS (HAT FUNCTIONS, TENT FUNCTIONS)

↳ A powerful tool for approximations ⇝ SYSTEMATIC

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$$N^i(x) \rightsquigarrow \begin{cases} N^i = 1 \ @ \ x^j \ (j=i) \\ N^i = 0 \ @ \ x^j \ (j \neq i) \end{cases}$$

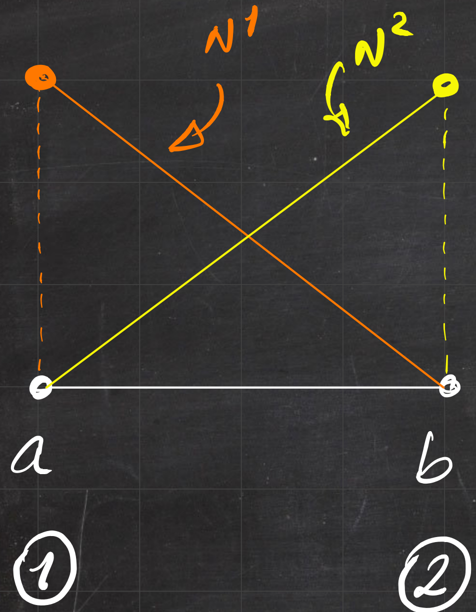
NODES PER ELEMENT → NPE

$$u \simeq \sum_{i=1}^{NPE} N^i u^i \to \begin{cases} u = N^1u^1 + N^2u^2 \\ u = N^1u^1 + N^2u^2 + N^3u^3 \\ u = N^1u^1 + N^2u^2 + N^3u^3 + N^4u^4 \end{cases}$$

linear approximation

quadratic approximation

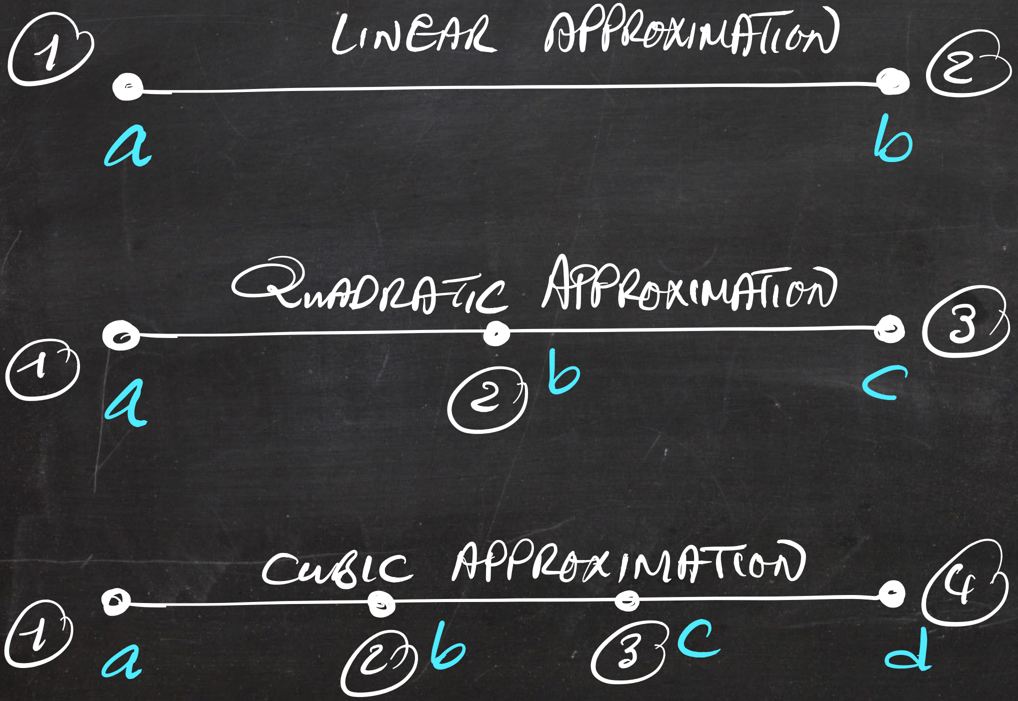
cubic approximation



$$N^1 = \frac{x-b}{a-b}$$

$$N^2 = \frac{x-a}{b-a}$$

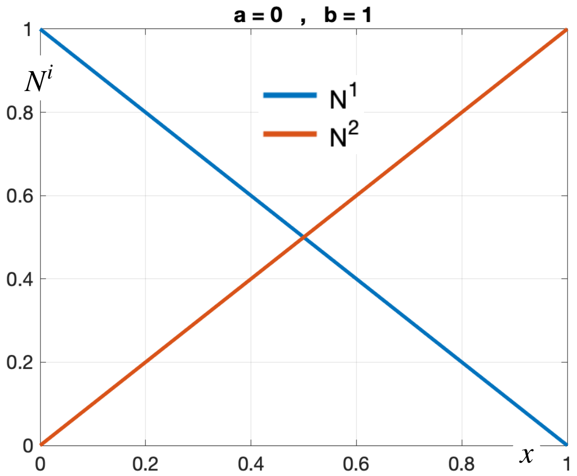
LINEAR SHAPE FUNCTIONS

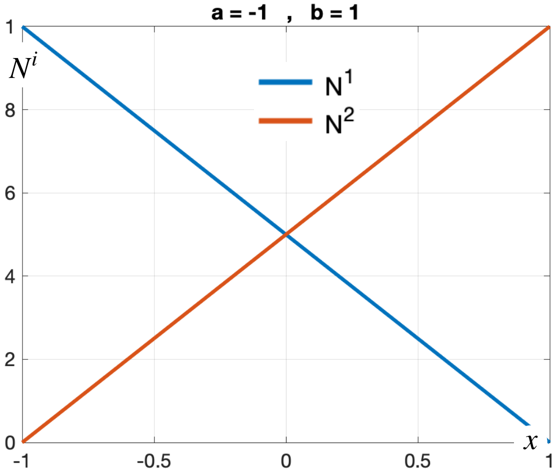

LINEAR APPROXIMATION
1
2
a
b
QUADRATIC APPROXIMATION
1
2
3
a
b
c
CUBIC APPROXIMATION
1
2
3
4
a
b
c
d

1D Linear Shape Functions

$$N^1 = \frac{[x - b]}{[a - b]}$$

$$N^2 = \frac{[x - a]}{[b - a]}$$



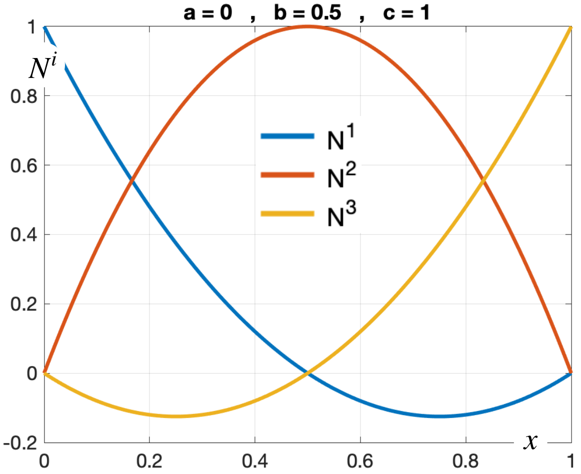


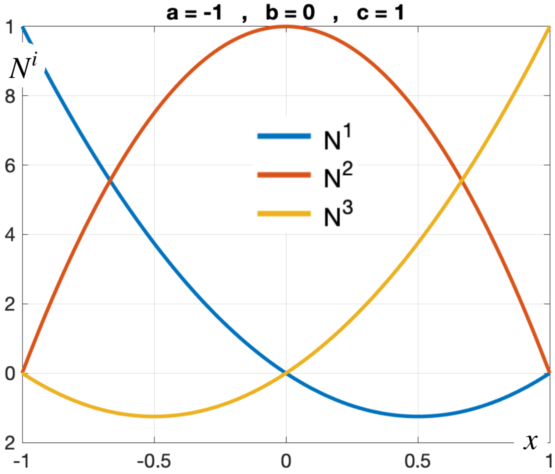
1D Quadratic Shape Functions

$$N^1 = \frac{[x-b][x-c]}{[a-b][a-c]}$$

$$N^2 = \frac{[x-a][x-c]}{[b-a][b-c]}$$

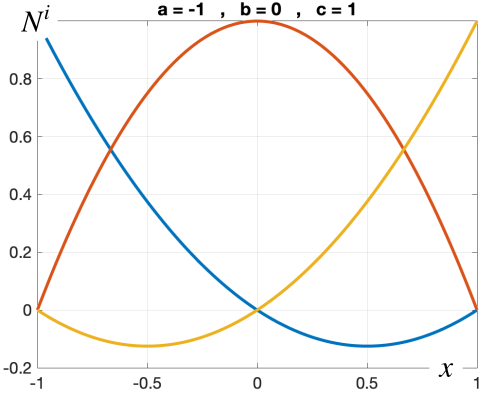
$$N^3 = \frac{[x-a][x-b]}{[c-a][c-b]}$$

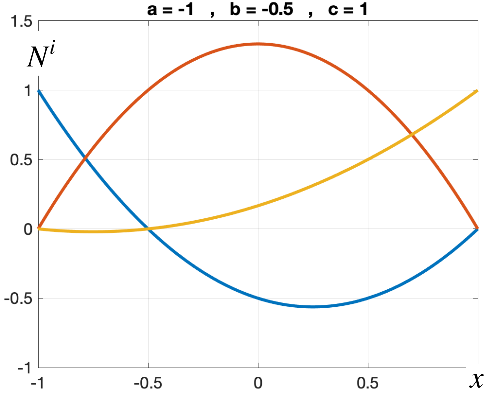


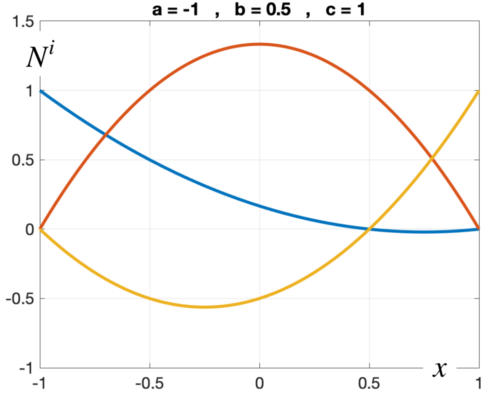


1D Quadratic Shape Functions

Numerical Analysis and High-level Applications Lab
NAHAL



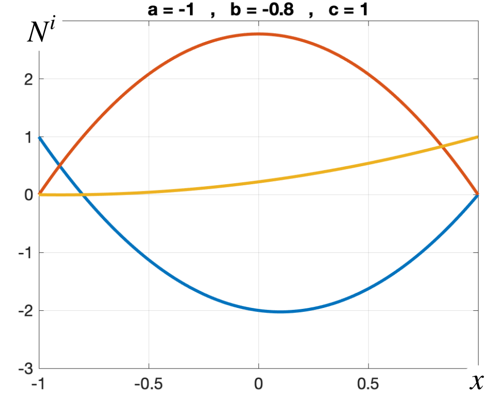


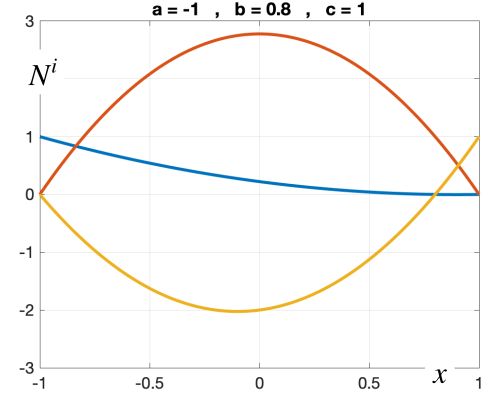


$$N^1 = \frac{[x-b][x-c]}{[a-b][a-c]}$$

$$N^2 = \frac{[x-a][x-c]}{[b-a][b-c]}$$

$$N^3 = \frac{[x-a][x-b]}{[c-a][c-b]}$$





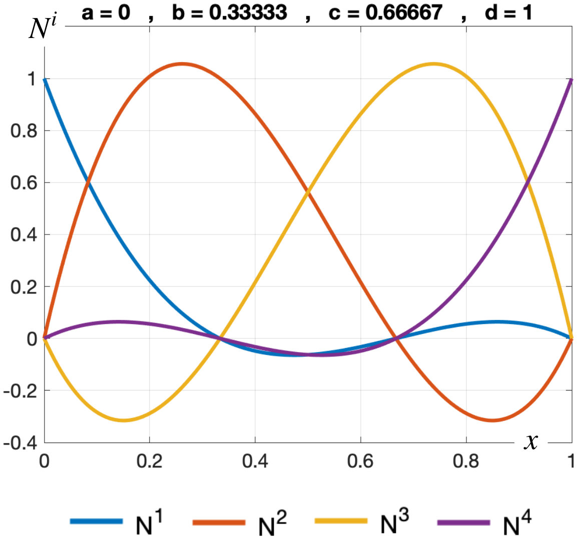
1D Cubic Shape Functions

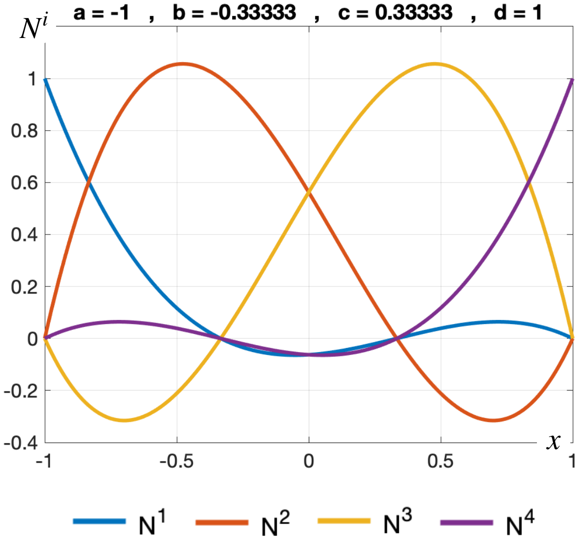
$$N^1 = \frac{[x-b][x-c][x-d]}{[a-b][a-c][a-d]}$$

$$N^2 = \frac{[x-a][x-c][x-d]}{[b-a][b-c][b-d]}$$

$$N^3 = \frac{[x-a][x-b][x-d]}{[c-a][c-b][c-d]}$$

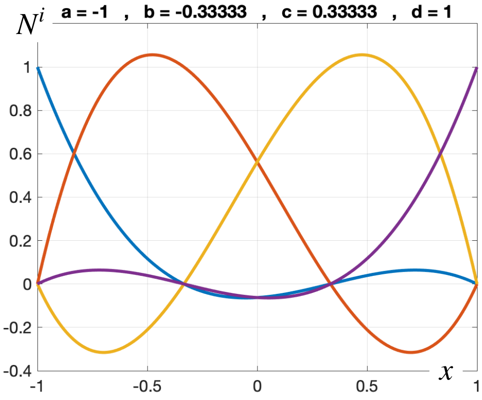
$$N^4 = \frac{[x-a][x-b][x-c]}{[d-a][d-b][d-c]}$$

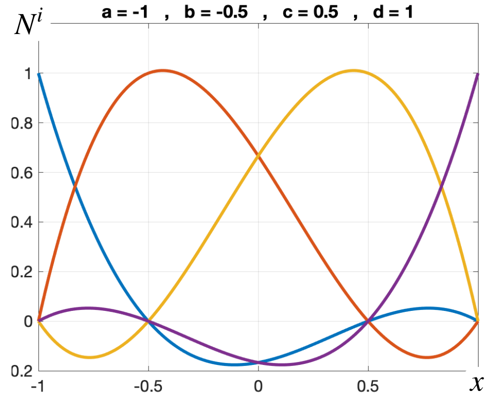


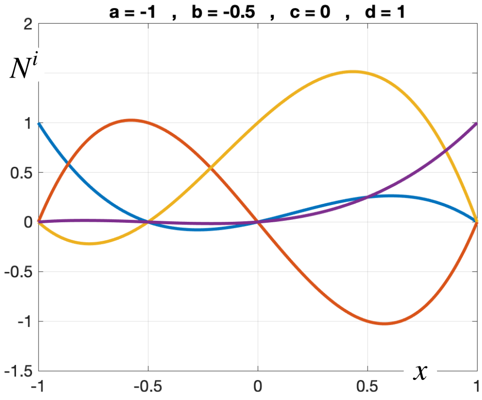


1D Cubic Shape Functions

Numerical Analysis and High-level Applications Lab NAHAL







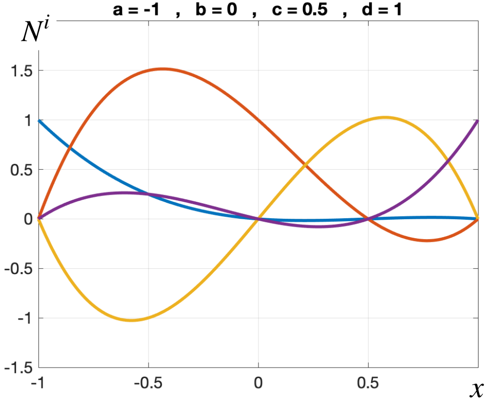
$$N^1 = \frac{[x-b][x-c][x-d]}{[a-b][a-c][a-d]}$$

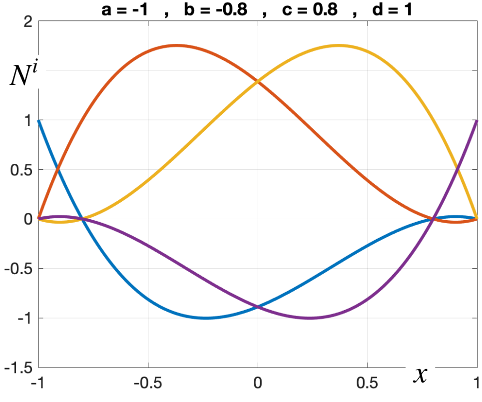
$$N^2 = \frac{[x-a][x-c][x-d]}{[b-a][b-c][b-d]}$$

$$N^3 = \frac{[x-a][x-b][x-d]}{[c-a][c-b][c-d]}$$

$$N^4 = \frac{[x-a][x-b][x-c]}{[d-a][d-b][d-c]}$$

N^1
N^2
N^3
N^4





APPROXIMATION: UNDERSTANDING VIA EXAMPLES

(I) $u(0) = 0 \rightarrow u^1$

$u(1) = 1 \rightarrow u^2$

$u(0.5) = ? \approx 0.5$

$u(0.8) = ? \approx 0.8$

$u = N^1 u^1 + N^2 u^2$

$= [1-x]\, u^1 + x\, u^2$

$= x \Rightarrow u(x) = x$ ✓

(II) $u(0) = 1 \rightarrow u^1$

$u(1.6) = 5 \rightarrow u^0$

$u(0.8) = ? \approx 3$

$u(1) = ? \approx 3.5$

$u = N^1 u^1 + N^2 u^2$

$= \dfrac{[x-1.6]}{-1.6}\, u^1 + \dfrac{[x-0]}{1.6}\, u^2$

$\Rightarrow u(x) = 2.5x + 1$

(III) $u(0) = 1 \rightarrow u^1$

$u(0.5) = 2 \rightarrow u^2$

$u(1) = 4 \rightarrow u^3$

$u(0.8) = ?$

$N^1 = \dfrac{[x-0.5][x-1]}{0.5}$

$N^2 = \dfrac{[x-0][x-1]}{-0.25}$

$N^3 = \dfrac{[x-0][x-0.5]}{0.5}$

$u = N^1 u^1 + N^2 u^2 + N^3 u^3$

$= 2[x^2 - 1.5x + 0.5]$

$- 8[x^2 - x] + 8[x^2 - 0.5x]$

$\Rightarrow u(x) = 2x^2 + x + 1$

APPROXIMATION: UNDERSTANDING VIA EXAMPLES

(IV)

$$\left.\begin{aligned} u(0) &= 1 \\ u(0.2) &= 2 \\ u(0.6) &= 4 \\ u(1) &= 8 \end{aligned}\right\} \Rightarrow u(0.8) = ?$$

$$f(x) = ax^3 + bx^2 + cx + d$$

$$\Downarrow$$

$$\left.\begin{aligned} &4 \text{ Equations} \\ &4 \text{ Unknowns} \end{aligned}\right\} \Rightarrow \begin{aligned} a \\ b \\ c \\ d \end{aligned}$$

$$\checkmark\ f(x)$$

APPROXIMATION:

UNDERSTANDING VIA EXAMPLES

(IV)

$$\overbrace{\underbrace{-0.2}\quad\underbrace{-0.6}\quad\underbrace{-1}}^{-0.12}$$

① → $u(0)=1$ $\quad N^1=[x-0.2][x-0.6][x-1]\,/\,[-0.12]$

② → $u(0.2)=2$ $\quad N^2=[x-0][x-0.6][x-1]\,/\,[0.064]$

③ → $u(0.6)=4$ $\quad N^3=[x-0][x-0.2][x-1]\,/\,[-0.096]$

④ → $u(1)=8$ $\quad N^4=[x-0][x-0.2][x-0.6]\,/\,[0.32]$

FROM COMPUTER PERSPECTIVE → UNNECESSARY

ADDITIONAL STEP

$$\Rightarrow u=N^1u^1+N^2u^2+N^3u^3+N^4u^4 \;\ldots\; \Rightarrow \underbrace{u(x)=\alpha x^3+\beta x^2+\gamma x+\xi}$$

UNDERSTANDING THE SIGNIFICANCE OF WEAK FORM BY EXAMPLE

$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx$$

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

$$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$$

- → 1-PIECE LINEAR APPROXIMATION
- → 2-PIECE LINEAR (UNIFORM) APPROXIMATION
- → 1-PIECE QUADRATIC APPROXIMATION

REMEMBER HOW WE DID BEFORE

$w = C_1 x + C_2 \qquad u = D_1 x + D_2$

$\ldots \int_0^1 w'u'\,dx = \int_0^1 w\,dx \ldots \quad \forall w$

$\ldots \Rightarrow D_1 \,\&\, D_2 \checkmark$

UNDERSTANDING THE SIGNIFICANCE OF WEAK FORM BY EXAMPLE

$$\int_0^1 w'u'dx = \int_0^1 w\,dx$$

$u'' + 1 = 0 \quad 0 \leqslant x \leqslant 1$

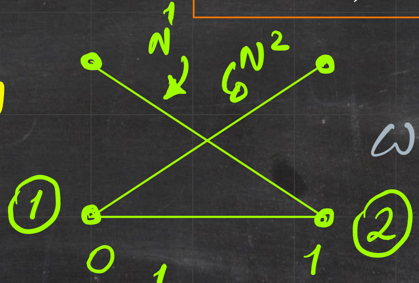
D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

→ 1-PIECE LINEAR APPROXIMATION

$w = N^1 w^1 + N^2 w^2 \qquad u = N^1 u^1 + N^2 u^2$

w @ node 1 $\quad w$ @ node 2 $\quad u$ @ node 1 $\quad u$ @ node 2



$N^1 = 1 - x$

$N^2 = x$

$$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$$

REMEMBER HOW WE DID BEFORE

$w = C_1 x + C_2 \quad u = D_1 x + D_2$

$\ldots \int_0^1 w'u'dx = \int_0^1 w\,dx \ldots \;\forall w$

$\ldots \Rightarrow D_1 \,\&\, D_2$ ✓

UNDERSTANDING THE SIGNIFICANCE OF WEAK FORM BY EXAMPLE

$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx$$

$u'' + 1 = 0 \qquad 0 \leq x \leq 1$

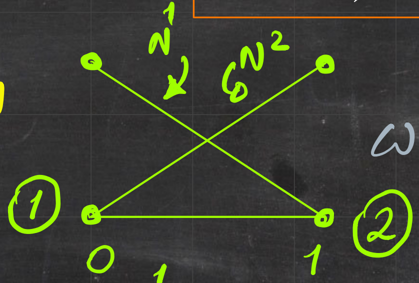
D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

→ 1-PIECE LINEAR APPROXIMATION

$w = N^1 w^1 + N^2 w^2 \qquad u = N^1 u^1 + N^2 u^2$

$w^1 = 0$ ← $w|_D = 0 \qquad u^1 = 0$ ← $u(0) = 0$



$N^1 = 1 - x$

$N^2 = x$

w : ARBITRARY, CONTINUOUS, $w|_D = 0$

$w = N^2 w^2 = x w^2 \qquad u = N^2 u^2 = x u^2$

$$\int_0^1 w^2 u^2\,dx = \int_0^1 x w^2\,dx \to w^2 u^2 = w^2 \left[\tfrac{1}{2} x^2\right]_0^1$$ ← $\frac{1}{2}$

REMEMBER HOW WE DID BEFORE

$w = C_1 x + C_2 \qquad u = D_1 x + D_2$

$\ldots \int_0^1 w'u'\,dx = \int_0^1 w\,dx \ldots \forall w$

$\ldots \Rightarrow D_1$ & D_2 ✓

UNDERSTANDING THE SIGNIFICANCE OF WEAK FORM BY EXAMPLE

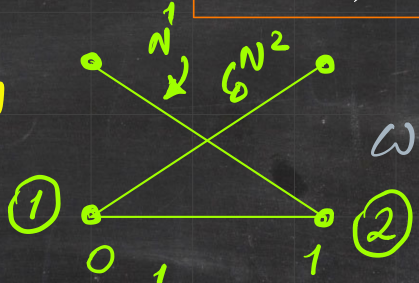
$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx$$

$$u'' + 1 = 0 \qquad 0 \leq x \leq 1$$

$D: \; u(0) = 0$ ← prescribed

$N: \; u'(1) = 0$ ←

→ 1-PIECE LINEAR APPROXIMATION



$w :$ { ARBITRARY, CONTINUOUS, $w|_D = 0$ }

$$w = N^1 w^1 + N^2 w^2 \qquad u = N^1 u^1 + N^2 u^2$$

$$w^1 = 0 \leftarrow w|_D = 0 \qquad u^1 = 0 \leftarrow u(0) = 0 \qquad N^1 = 1 - x$$

$$w = N^2 w^2 = x w^2 \qquad u = N^2 u^2 = x u^2 \qquad N^2 = x$$

$$\int_0^1 w^2 u^2\,dx = \int_0^1 x w^2\,dx \Rightarrow w^2 u^2 = w^2 \tfrac{1}{2} \qquad \forall w^2 \Rightarrow u^2 = \tfrac{1}{2}$$

REMEMBER HOW WE DID BEFORE

$$w = C_1 x + C_2 \qquad u = D_1 x + D_2$$

$$\ldots \int_0^1 w'u'\,dx = \int_0^1 w\,dx \ldots \quad \forall w$$

$$\ldots \Rightarrow D_1 \,\&\, D_2 \checkmark$$

UNDERSTANDING THE SIGNIFICANCE OF WEAK FORM BY EXAMPLE

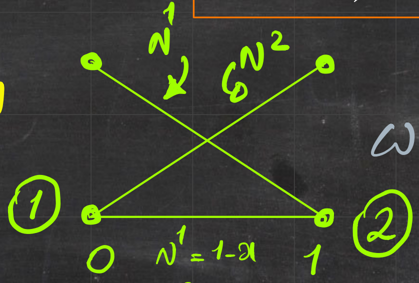
$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx$$

$$u'' + 1 = 0 \qquad 0 \leq x \leq 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

→ 1-PIECE LINEAR APPROXIMATION



$N^1 = 1 - x$

$N^2 = x$

w : ARBITRARY, CONTINUOUS, $w|_D = 0$

$w = N^1 w^1 + N^2 w^2 \qquad u = N^1 u^1 + N^2 u^2$

$w^1 = 0 \leftarrow w|_D = 0 \qquad u^1 = 0 \leftarrow u(0) = 0$

$w = N^2 w^2 = x w^2 \qquad u = N^2 u^2 = x u^2 \Rightarrow u = \frac{1}{2} x$ ✓

$$\int_0^1 w^2 u^2\,dx = \int_0^1 x w^2\,dx \rightarrow w^2 u^2 = w^2 \frac{1}{2} \quad \forall w^2 \Rightarrow u^2 = \frac{1}{2}$$

REMEMBER HOW WE DID BEFORE

$w = C_1 x + C_2 \qquad u = D_1 x + D_2$

$\ldots \int_0^1 w'u'\,dx = \int_0^1 w\,dx \ldots \forall w$

$\ldots \Rightarrow D_1 \,\&\, D_2$ ✓

2- PIECE LINEAR UNIFORM APPROXIMATION

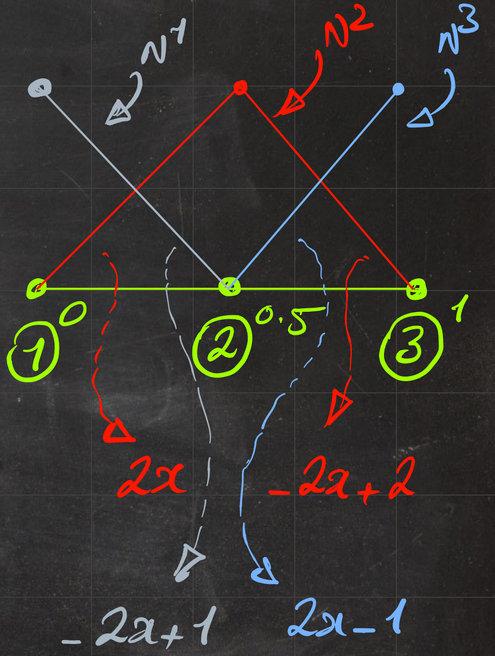
$$w = \underbrace{\cancel{N^1 w^1}}_{0} + N^2 w^2 + N^3 w^3 \qquad u = \underbrace{\cancel{N^1 u^1}}_{0} + N^2 u^2 + N^3 u^3$$

$$\Rightarrow w = N^2 w^2 + N^3 w^3 \qquad \Rightarrow u = N^2 u^2 + N^3 u^3$$

$$\int_0^1 w' u' \, dx = \int_0^1 w \, dx \Rightarrow \int_0^{1/2} \cdots + \int_{1/2}^1 \cdots = \cdots$$

$$\int_0^{1/2} w' u' \, dx + \int_{1/2}^1 w' u' \, dx = \int_0^{1/2} w \, dx + \int_{1/2}^1 w \, dx$$

$$w' = N^{2\prime} w^2 + N^{3\prime} w^3 \qquad u' = N^{2\prime} u^2 + N^{3\prime} u^3$$



$u'' + 1 = 0 \qquad 0 \leq x \leq 1$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$

2. Piece linear uniform approximation

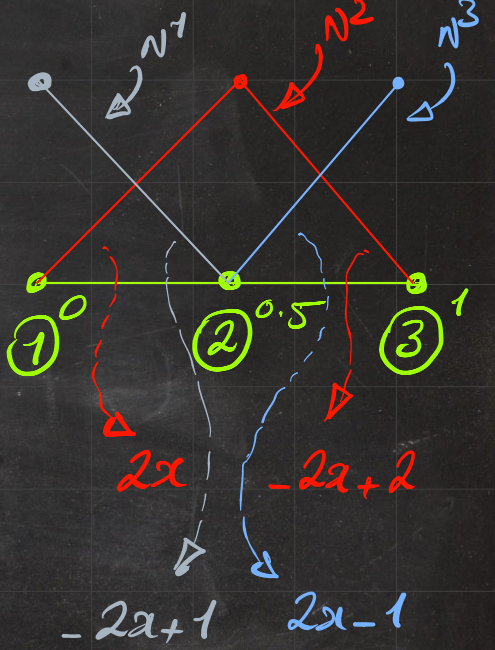
$$w = \cancel{N^1 w^1}^{\,0} + N^2 w^2 + N^3 w^3 \qquad u = \cancel{N^1 u^1}^{\,0} + N^2 u^2 + N^3 u^3$$

$$\Rightarrow w = N^2 w^2 + N^3 w^3 \qquad \Rightarrow u = N^2 u^2 + N^3 u^3$$

$$\int_0^{1/2} [N^{2\prime} w^2 + N^{3\prime} w^3][N^{2\prime} u^2 + N^{3\prime} u^3]\, dx$$

$$+ \int_{1/2}^{1} [N^{2\prime} w^2 + N^{3\prime} w^3][N^{2\prime} u^2 + N^{3\prime} u^3]\, dx$$

$$= \int_0^{1/2} [N^2 w^2 + N^3 w^3]\, dx + \int_{1/2}^{1} [N^2 w^2 + N^3 w^3]\, dx$$



$u'' + 1 = 0 \qquad 0 \leq x \leq 1$

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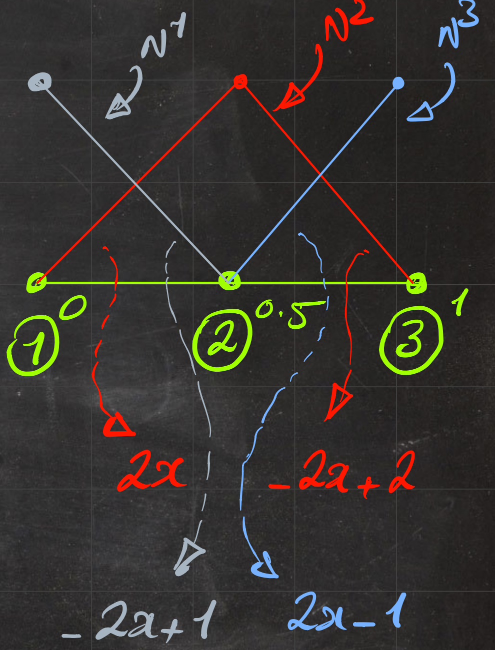
2- Piece linear uniform approximation

$$w = N^1 \underbrace{\cancel{w^1}}_{0} + N^2 w^2 + N^3 w^3 \qquad u = N^1 \underbrace{\cancel{u^1}}_{0} + N^2 u^2 + N^3 u^3$$

$$\Rightarrow w = N^2 w^2 + N^3 w^3 \qquad \Rightarrow u = N^2 u^2 + N^3 u^3$$



$$\int_0^{1/2} [\underbrace{N^{2\prime}}_{2} w^2 + \underbrace{N^{3\prime}}_{0} w^3][\underbrace{N^{2\prime}}_{2} u^2 + \underbrace{N^{3\prime}}_{0} u^3]\, dx$$

$$+ \int_{1/2}^{1} [\underbrace{N^{2\prime}}_{-2} w^2 + \underbrace{N^{3\prime}}_{2} w^3][\underbrace{N^{2\prime}}_{-2} u^2 + \underbrace{N^{3\prime}}_{2} u^3]\, dx$$

$$= \int_0^{1/2} [\underbrace{N^2}_{2x} w^2 + \overbrace{N^3}^{0} w^3]\, dx + \int_{1/2}^{1} [\underbrace{N^2}_{2 \to -2x+2} w^2 + \overbrace{N^3}^{2x-1} w^3]\, dx$$

$u'' + 1 = 0 \qquad 0 \le x \le 1$

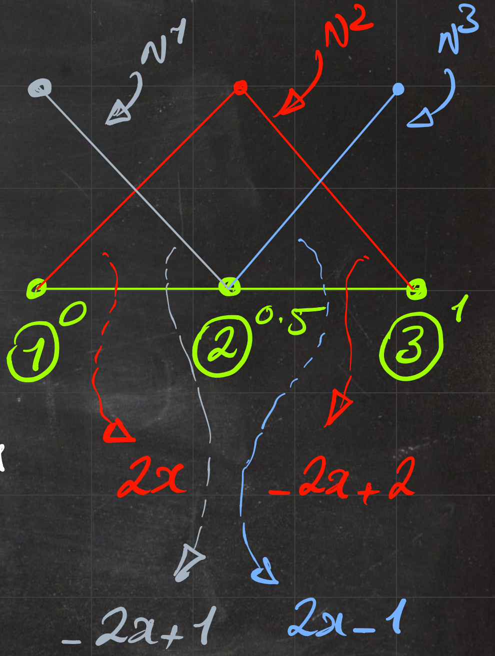
D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

2. Piece linear uniform approximation

$$w = N^1 \underbrace{\cancel{w^1}}_{0} + N^2 w^2 + N^3 w^3 \qquad u = N^1 \underbrace{\cancel{u^1}}_{0} + N^2 u^2 + N^3 u^3$$

$$\Rightarrow w = N^2 w^2 + N^3 w^3 \qquad \Rightarrow u = N^2 u^2 + N^3 u^3$$



$$\int_0^{1/2} 2w^2 \times 2u^2 \, dx + \int_{1/2}^{1} [-2w^2 + 2w^3][-2u^2 + 2u^3] \, dx$$

$$= \int_0^{1/2} 2x w^2 \, dx + \int_{1/2}^{1} [-2x+2] w^2 \, dx$$

$$+ \int_{1/2}^{1} [2x-1] w^3 \, dx$$

$u'' + 1 = 0 \qquad 0 \leq x \leq 1$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$

2. PIECE LINEAR UNIFORM APPROXIMATION

$$w = N^1 \cancel{w^1}^{\,0} + N^2 w^2 + N^3 w^3 \qquad u = N^1 \cancel{u^1}^{\,0} + N^2 u^2 + N^3 u^3$$

$$\Rightarrow w = N^2 w^2 + N^3 w^3 \qquad \Rightarrow u = N^2 u^2 + N^3 u^3$$

INTEGRATION

$$\dots \Longrightarrow 2w^2u^2 + 2w^2u^2 - 2w^2u^3 - 2w^3u^2 + 2w^3u^3 = \frac{1}{2}w^2 + \frac{1}{4}w^3 \quad \forall w^2, w^3$$

$$w^2\left[4u^2 - 2u^3 - \frac{1}{2}\right] + w^3\left[2u^3 - 2u^2 - \frac{1}{4}\right] = 0 \quad \forall w^2, w^3$$

$u'' + 1 = 0 \qquad 0 \leq x \leq 1$

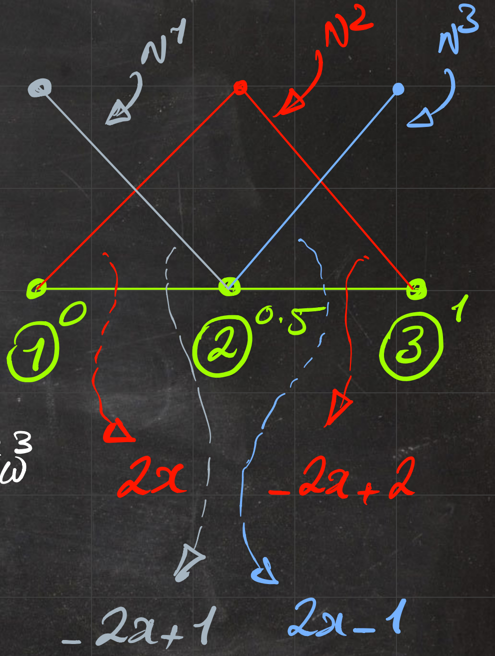
D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

2- PIECE LINEAR UNIFORM APPROXIMATION

$$w = N^1 \cancel{w^1}^{\to 0} + N^2 w^2 + N^3 w^3 \qquad u = N^1 \cancel{u^1}^{\to 0} + N^2 u^2 + N^3 u^3$$

$$\Rightarrow w = N^2 w^2 + N^3 w^3 \qquad \Rightarrow u = N^2 u^2 + N^3 u^3$$



$$w^2\left[4u^2 - 2u^3 - \frac{1}{2}\right] + w^3\left[2u^3 - 2u^2 - \frac{1}{4}\right] = 0 \quad \forall w^2, w^3$$

$$\begin{cases} 4u^2 - 2u^3 - \dfrac{1}{2} = 0 \\ 2u^3 - 2u^2 - \dfrac{1}{4} = 0 \end{cases} \Rightarrow \begin{cases} u^2 = \dfrac{3}{8} \\ u^3 = \dfrac{1}{2} \end{cases} \Rightarrow u = u(x) \checkmark$$

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

Summary:

in the previous approach $\rightsquigarrow$ we had $\begin{cases} u = \alpha_1 x + \beta_1 & 0 \leq x \leq \frac{1}{2} \\ u = \alpha_2 x + \beta_2 & \frac{1}{2} \leq x \leq 1 \end{cases}$

... $\Rightarrow$ WE CALCULATED $\alpha_1, \alpha_2, \beta_1, \beta_2$ $\rightsquigarrow$ THEN COMPUTE NODAL VALUES

NECESSARY

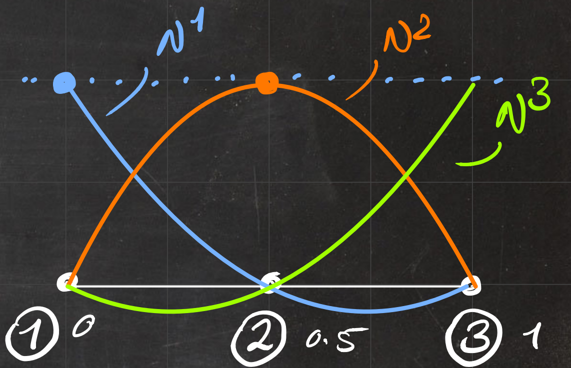
in the current approach $\rightsquigarrow$ we have $u = N^1 u^1 + N^2 u^2 + N^3 u^3$ $\begin{cases} 0 \leq x \leq \frac{1}{2} \\ \frac{1}{2} \leq x \leq 1 \end{cases}$

UNNECESSARY

... $\Rightarrow$ WE CALCULATE u^1, u^2, u^3 $\rightsquigarrow$ THEN COMPUTE $\begin{cases} u = u(x) & 0 \leq x \leq \frac{1}{2} \\ u = u(x) & \frac{1}{2} \leq x \leq 1 \end{cases}$

1 - Piece Quadratic Approximation

$$w = \cancel{N^1 w^1}^{\,0} + N^2 w^2 + N^3 w^3 \qquad u = \cancel{N^1 u^1}^{\,0} + N^2 u^2 + N^3 u^3$$



$$\int_0^1 w' u' \, dx = \int_0^1 w \, dx \qquad \forall w^2, w^3$$

$$\int_0^1 [N^{2\prime} w^2 + N^{3\prime} w^3][N^{2\prime} u^2 + N^{3\prime} u^3] \, dx$$

$$= \int_0^1 [N^2 w^2 + N^3 w^3] \, dx$$

$$\Downarrow$$

$$\ldots \rightarrow \text{2 EQNS} \Rightarrow u^2, u^3 \checkmark$$

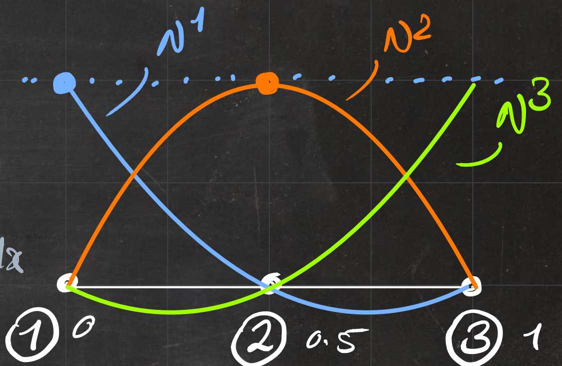
$$\begin{cases} N^1 = 2[x-0.5][x-1] = 2x^2 - 3x + 1 \\ N^2 = -4[x-0][x-1] = -4x^2 + 4x \\ N^3 = 2[x-0][x-0.5] = 2x^2 - x \end{cases}$$

$$\begin{cases} N^{1\prime} = 4x - 3 \\ N^{2\prime} = -8x + 4 \\ N^{3\prime} = 4x - 1 \end{cases}$$

$$\boxed{\begin{array}{ll} u'' + 1 = 0 & 0 \leq x \leq 1 \\ D:\ u(0) = 0 & \leftarrow \text{prescribed} \\ N:\ u'(1) = 0 & \checkmark \end{array}}$$

1 - Piece Quadratic Approximation

$$w = \cancel{N^1 w^1}^{\,0} + N^2 w^2 + N^3 w^3 \qquad u = \cancel{N^1 u^1}^{\,0} + N^2 u^2 + N^3 u^3$$



$$\int_0^1 [N^{2\prime} w^2 + N^{3\prime} w^3][N^{2\prime} u^2 + N^{3\prime} u^3]\, dx = \int_0^1 [N^2 w^2 + N^3 w^3]\, dx$$

$$\checkmark\, w^2, w^3$$

$$\int_0^1 [N^{2\prime} N^{2\prime} w^2 u^2 + N^{2\prime} N^{3\prime} w^3 u^2 + N^{3\prime} N^{2\prime} w^2 u^3 + N^{3\prime} N^{3\prime} w^3 u^3]\, dx = \dots$$

$$N^1 = 2[x - 0.5][x - 1] = 2x^2 - 3x + 1$$

$$N^2 = -4[x - 0][x - 1] = -4x^2 + 4x$$

$$N^3 = 2[x - 0][x - 0.5] = 2x^2 - x$$

$$\int_0^1 w^2 \left[[N^{2\prime}]^2 u^2 + N^{2\prime} N^{3\prime} u^3 - N^2\right] \qquad \checkmark\, w^2, w^3$$

$$+ w^3 [N^{3\prime} N^{2\prime} u^2 + [N^{3\prime}]^2 u^3 - N^3]\, dx = 0$$

$$N^{1\prime} = 4x - 3$$

$$N^{2\prime} = -8x + 4$$

$$N^{3\prime} = 4x - 1$$

$$\boxed{\begin{array}{ll} u'' + 1 = 0 & 0 \leq x \leq 1 \\ D:\ u(0) = 0 & \leftarrow \text{prescribed} \\ N:\ u'(1) = 0 & \leftarrow \end{array}}$$

1 - Piece Quadratic Approximation

$$w = N^1 \cancelto{0}{w^1} + N^2 w^2 + N^3 w^3 \qquad u = N^1 \cancelto{0}{u^1} + N^2 u^2 + N^3 u^3$$

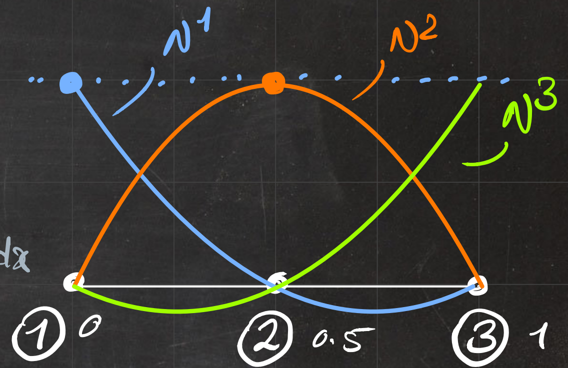
$$\int_0^1 [N^{2\prime} w^2 + N^{3\prime} w^3][N^{2\prime} u^2 + N^{3\prime} u^3]\, dx = \int_0^1 [N^2 w^2 + N^3 w^3]\, dx$$

$$\int_0^1 w^2 \left[[N^{2\prime}]^2 u^2 + N^{2\prime} N^{3\prime} u^3 - N^2\right] \qquad \forall w^2, w^3$$

$$+ w^3 \left[N^{3\prime} N^{2\prime} u^2 + [N^{3\prime}]^2 u^3 - N^3\right] dx = 0$$

$$\int_0^1 \left[[N^{2\prime}]^2 u^2 + N^{2\prime} N^{3\prime} u^3 - N^2\right] dx = 0$$

$$\int_0^1 \left[N^{3\prime} N^{2\prime} u^2 + [N^{3\prime}]^2 u^3 - N^3\right] dx = 0$$



$$N^1 = 2[x-0.5][x-1] = 2x^2 - 3x + 1$$

$$N^2 = -4[x-0][x-1] = -4x^2 + 4x$$

$$N^3 = 2[x-0][x-0.5] = 2x^2 - x$$

$$N^{1\prime} = 4x - 3$$

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$$\boxed{\begin{array}{ll} u'' + 1 = 0 & 0 \leq x \leq 1 \\ D:\ u(0) = 0 & \leftarrow \text{prescribed} \\ N:\ u'(1) = 0 & \end{array}}$$

1 - Piece Quadratic Approximation

$$w = \cancel{N^1 w^1}^{\,0} + N^2 w^2 + N^3 w^3 \qquad u = \cancel{N^1 u^1}^{\,0} + N^2 u^2 + N^3 u^3$$

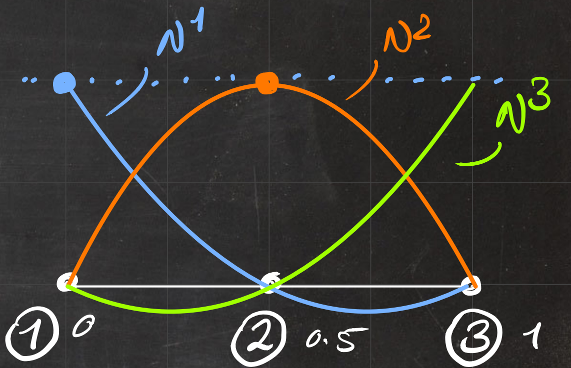
$$\int_0^1 \left[[N^{2\prime}]^2 u^2 + N^{2\prime} N^{3\prime} u^3 - N^2 \right] dx = 0$$

$$\int_0^1 \left[N^{3\prime} N^{2\prime} u^2 + [N^{3\prime}]^2 u^3 - N^3 \right] dx = 0$$

$$Au + Bv - C = 0$$

$$Du + Ev - F = 0$$

$$\Rightarrow \begin{bmatrix} A & B \\ D & E \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} C \\ F \end{bmatrix}$$



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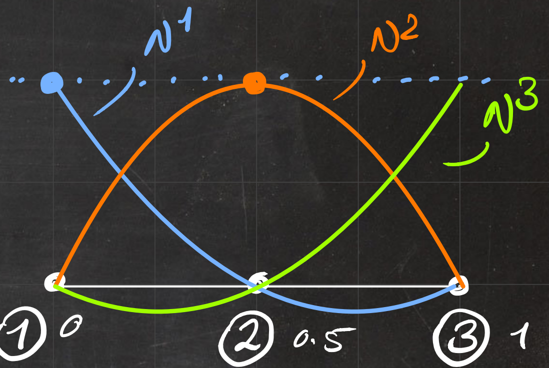
1 - Piece Quadratic Approximation



$$w = \cancelto{0}{N^1 w^1} + N^2 w^2 + N^3 w^3 \qquad u = \cancelto{0}{N^1 u^1} + N^2 u^2 + N^3 u^3$$

$$\int_0^1 \left[[N^{2\prime}]^2 u^2 + N^{2\prime} N^{3\prime} u^3 - N^2 \right] dx = 0$$

$$\int_0^1 \left[N^{3\prime} N^{2\prime} u^2 + [N^{3\prime}]^2 u^3 - N^3 \right] dx = 0$$

$$\begin{bmatrix} \int_0^1 N^{2\prime} N^{2\prime} dx & \int_0^1 N^{2\prime} N^{3\prime} dx \\ \int_0^1 N^{2\prime} N^{3\prime} dx & \int_0^1 N^{3\prime} N^{3\prime} dx \end{bmatrix} \begin{bmatrix} u^2 \\ u^3 \end{bmatrix} = \begin{bmatrix} \int_0^1 N^2 dx \\ \int_0^1 N^3 dx \end{bmatrix}$$

$$N^1 = 2[x - 0.5][x - 1] = 2x^2 - 3x + 1$$

$$N^2 = -4[x - 0][x - 1] = -4x^2 + 4x$$

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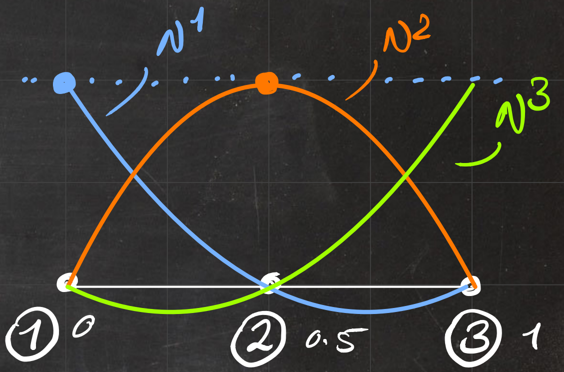
$$\boxed{\begin{array}{ll} u'' + 1 = 0 & 0 \leq x \leq 1 \\ \text{D: } u(0) = 0 & \leftarrow \text{prescribed} \\ \text{N: } u'(1) = 0 & \leftarrow \end{array}}$$

1-Piece Quadratic Approximation

$$w = N^1 \cancelto{0}{w^1} + N^2 w^2 + N^3 w^3 \qquad u = N^1 \cancelto{0}{u^1} + N^2 u^2 + N^3 u^3$$



$$\begin{bmatrix} \int_0^1 N^{2\prime} N^{2\prime}\, dx & \int_0^1 N^{2\prime} N^{3\prime}\, dx \\ \int_0^1 N^{2\prime} N^{3\prime}\, dx & \int_0^1 N^{3\prime} N^{3\prime}\, dx \end{bmatrix} \begin{bmatrix} u^2 \\ u^3 \end{bmatrix} = \begin{bmatrix} \int_0^1 N^2\, dx \\ \int_0^1 N^3\, dx \end{bmatrix}$$

$$N^1 = 2[x-0.5][x-1] = 2x^2 - 3x + 1$$

$$N^2 = -4[x-0][x-1] = -4x^2 + 4x$$

$$N^3 = 2[x-0][x-0.5] = 2x^2 - x$$

$$\int_0^1 \underbrace{N^{2\prime}}_{-8x+4}\,\underbrace{N^{2\prime}}_{-8x+4}\, dx = \int_0^1 64x^2 - 64x + 16\, dx$$

$$= 64\frac{x^3}{3} - 32x^2 + 16x \Big|_0^1$$

$$= \frac{64}{3} - 32 + 16 = \frac{16}{3}$$

$$N^{1\prime} = 4x - 3$$

$$N^{2\prime} = -8x + 4$$

$$N^{3\prime} = 4x - 1$$

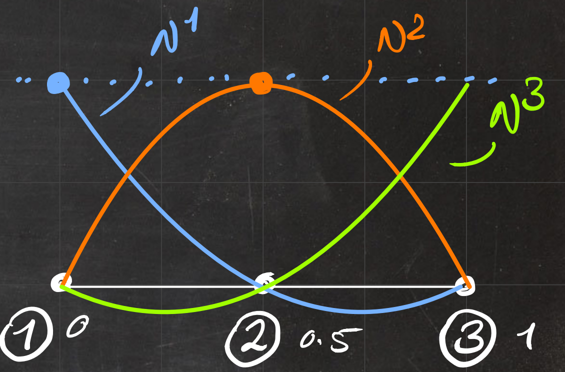
$u'' + 1 = 0 \qquad 0 \leq x \leq 1$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

1-Piece Quadratic Approximation

$$w = \cancelto{0}{N^1 w^1} + N^2 w^2 + N^3 w^3 \qquad u = \cancelto{0}{N^1 u^1} + N^2 u^2 + N^3 u^3$$



$$\begin{bmatrix} \int_0^1 N^{2\prime}N^{2\prime}\,dx & \int_0^1 N^{2\prime}N^{3\prime}\,dx \\ \int_0^1 N^{2\prime}N^{3\prime}\,dx & \int_0^1 N^{3\prime}N^{3\prime}\,dx \end{bmatrix} \begin{bmatrix} u^2 \\ u^3 \end{bmatrix} = \begin{bmatrix} \int_0^1 N^2\,dx \\ \int_0^1 N^3\,dx \end{bmatrix}$$

$$\int_0^1 N^2\,dx = \frac{2}{3} \qquad \int_0^1 N^3\,dx = \frac{1}{6}$$

$$N^1 = 2[x-0.5][x-1] = 2x^2 - 3x + 1$$

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$$N^3 = 2[x-0][x-0.5] = 2x^2 - x$$

$$\int_0^1 N^{2\prime}N^{2\prime}\,dx = \frac{16}{3} \qquad \int_0^1 N^{2\prime}N^{3\prime}\,dx = -\frac{8}{3}$$

$$\int_0^1 N^{2\prime}N^{3\prime}\,dx = -\frac{8}{3} \qquad \int_0^1 N^{3\prime}N^{3\prime}\,dx = \frac{7}{3}$$

$$N^{1\prime} = 4x - 3$$

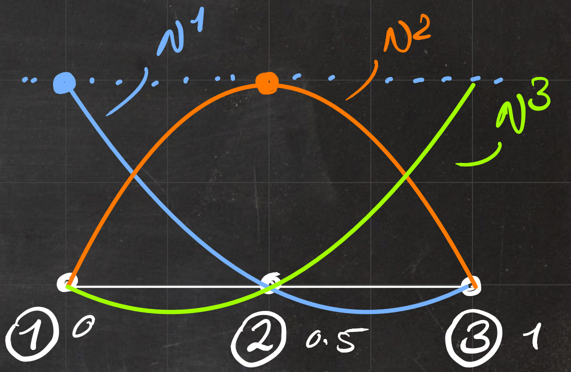
$$N^{2\prime} = -8x + 4$$

$$N^{3\prime} = 4x - 1$$

$$\boxed{\begin{array}{ll} u'' + 1 = 0 & 0 \leq x \leq 1 \\ \text{D: } u(0) = 0 & \leftarrow \text{prescribed} \\ \text{N: } u'(1) = 0 & \leftarrow \end{array}}$$

1-Piece Quadratic Approximation

$$w = N^1 \cancel{w^1}^{\,0} + N^2 w^2 + N^3 w^3 \qquad u = N^1 \cancel{u^1}^{\,0} + N^2 u^2 + N^3 u^3$$



$$\begin{bmatrix} 16/3 & -8/3 \\ -8/3 & 16/3 \end{bmatrix} \begin{bmatrix} u^2 \\ u^3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/6 \end{bmatrix}$$

$$\int_0^1 N^2 \, dx = \frac{2}{3} \qquad \int_0^1 N^3 \, dx = \frac{1}{6}$$

$$N^1 = 2[x-0.5][x-1] = 2x^2 - 3x + 1$$

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$$\int_0^1 N^{2\prime} N^{2\prime} \, dx = \frac{16}{3} \qquad \int_0^1 N^{2\prime} N^{3\prime} \, dx = -\frac{8}{3}$$

$$\int_0^1 N^{2\prime} N^{3\prime} \, dx = -\frac{8}{3} \qquad \int_0^1 N^{3\prime} N^{3\prime} \, dx = \frac{7}{3}$$

$$N^{1\prime} = 4x - 3$$

$$N^{2\prime} = -8x + 4$$

$$N^{3\prime} = 4x - 1$$

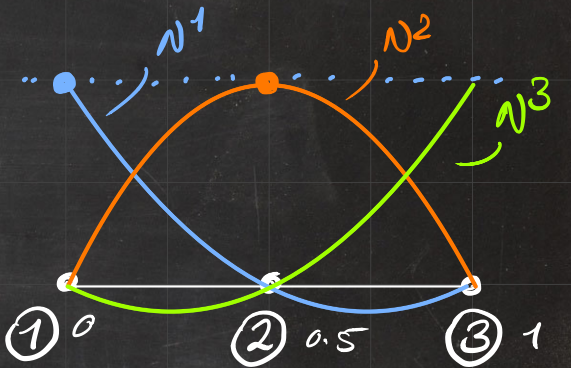
$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

1 - Piece Quadratic Approximation

$$w = N^1 \cancel{w^1}^{\,0} + N^2 w^2 + N^3 w^3 \qquad u = N^1 \cancel{u^1}^{\,0} + N^2 u^2 + N^3 u^3$$



$$\begin{bmatrix} 16/3 & -8/3 \\ -8/3 & 16/3 \end{bmatrix} \begin{bmatrix} u^2 \\ u^3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/6 \end{bmatrix}$$

$$\Rightarrow \begin{cases} u^2 = 3/8 \\ u^3 = 1/2 \end{cases}$$

$$u = N^2 u^2 + N^3 u^3$$
$$= [-4x^2 + 4x]\, 3/8$$
$$+ [2x^2 - x]\, 1/2$$

this was analytical solution

$$\Rightarrow u = -\frac{1}{2}x^2 + x$$

$$N^1 = 2[x - 0.5][x - 1] = 2x^2 - 3x + 1$$
$$N^2 = -4[x - 0][x - 1] = -4x^2 + 4x$$
$$N^3 = 2[x - 0][x - 0.5] = 2x^2 - x$$

$$N^{1\prime} = 4x - 3$$
$$N^{2\prime} = -8x + 4$$
$$N^{3\prime} = 4x - 1$$

$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$

D: $u(0) = 0$ ← prescribed

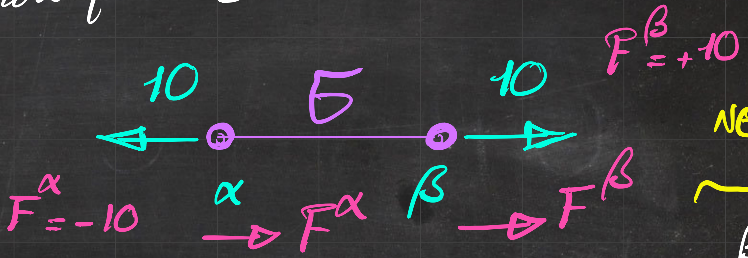
N: $u'(1) = 0$ ←

Consider the generic strong form $(EA\,u')' + b = 0 \quad \alpha \leqslant x \leqslant \beta$

Subject to BCs at α, β

↳ ... ↤ weak from derivation process

Corresponding weak form

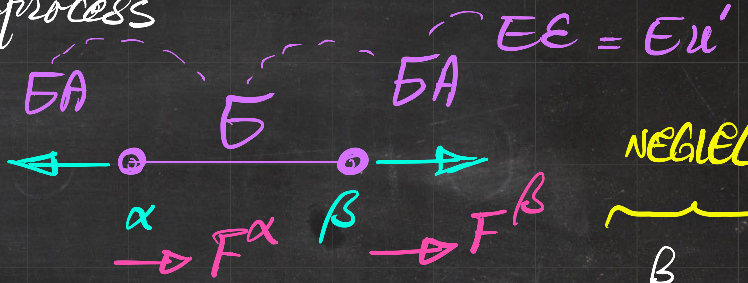


$$\int_\alpha^\beta EA\, w' u'\,dx = EA\,u'(\beta)\,w(\beta) - EA\,u'(\alpha)\,w(\alpha) + \overbrace{\underbrace{\int_\alpha^\beta b\,w\,dx}_{\text{body forces over the domain}}}^{\text{NEGLECT!}}$$

Consider the generic strong form $(EAu')' + b = 0 \quad \alpha \leq x \leq \beta$

Subject to BCs at α, β

↳ ... ← weak form derivation process



Corresponding weak form

$$\int_\alpha^\beta EA\, w' u' \, dx = \underbrace{EA\, u'(\beta)\, w(\beta)}_{F^\beta} - \underbrace{EA\, u'(\alpha)\, w(\alpha)}_{F^\alpha} + \overbrace{\underbrace{\int_\alpha^\beta b\, w \, dx}_{\text{body forces over the domain}}}^{\text{NEGLECT!}}$$

$F^{\alpha,\beta}$: EXTERNAL FORCES AT NODES α, β

Consider the generic strong form $(EA\,u')' + b = 0 \quad \alpha \leqslant x \leqslant \beta$

Subject to BCs at α, β

↳ ... ← weak from derivation process

Corresponding weak form

$F^{\alpha,\beta}$: EXTERNAL FORCES AT NODES α, β

$$\int_{\alpha}^{\beta} EA\, w' u' \, dx = w(\beta) F^{\beta} + w(\alpha) F^{\alpha}$$

Consider the generic strong form $(EAu')' + b = 0 \quad \alpha \leqslant x \leqslant \beta$

Subject to BCs at α, β

$\hookrightarrow$ ... $\leftarrow$ weak form derivation process

Corresponding weak form

$F^{\alpha,\beta}$: EXTERNAL FORCES AT NODES α, β

$$\boxed{EA \int_{\alpha}^{\beta} w' u' \, dx = w(\beta) F^{\beta} + w(\alpha) F^{\alpha}}$$

Approximate using

1- PIECE LINEAR

1- PIECE QUADRATIC

(?) 2- PIECE LINEAR

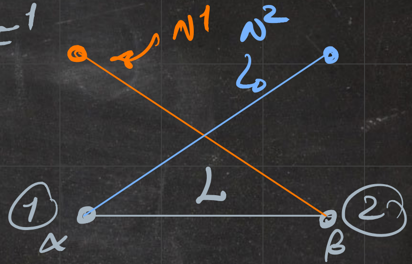
$$\boxed{EA\int_{\alpha}^{\beta} w'u'\,dx = w(\beta)F^{\beta} + w(\alpha)F^{\alpha}}$$

1- Piece Linear Approximation

$$w = N^1 w^1 + N^2 w^2 \qquad u = N^1 u^1 + N^2 u^2$$

$$\boxed{EA\int_\alpha^\beta w' u' \, dx = w(\beta)F^\beta + w(\alpha)F^\alpha}$$

$$EA\int_\alpha^\beta [N^{1\prime} w^1 + N^{2\prime} w^2][N^{1\prime} u^1 + N^{2\prime} u^2] \, dx = w^2 F^2 + w^1 F^1$$



$$EA\int_\alpha^\beta w^1 [N^{1\prime} N^{1\prime} u^1 + N^{1\prime} N^{2\prime} u^2] + w^2 [N^{1\prime} N^{2\prime} u^1 + N^{2\prime} N^{2\prime} u^2] \, dx$$

$$= w^2 F^2 + w^1 F^1 \qquad \forall w^1, w^2$$

$$N^1 = \frac{x-\beta}{\alpha-\beta} = \frac{1}{L}[\beta - x]$$

$$N^2 = \frac{x-\alpha}{\beta-\alpha} = \frac{1}{L}[x-\alpha]$$

1- PIECE LINEAR APPROXIMATION

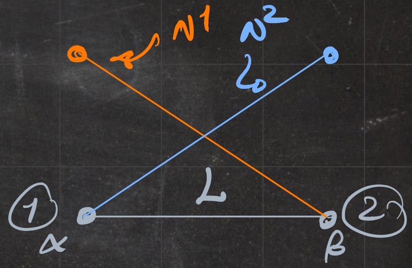
$$w = N^1 w^1 + N^2 w^2 \qquad u = N^1 u^1 + N^2 u^2$$

$$\boxed{EA \int_\alpha^\beta w' u' \, dx = w(\beta) F^\beta + w(\alpha) F^\alpha}$$

$$EA \int_\alpha^\beta w^1 [N^{1\prime} N^{1\prime} u^1 + N^{1\prime} N^{2\prime} u^2] + w^2 [N^{1\prime} N^{2\prime} u^1 + N^{2\prime} N^{2\prime} u^2] \, dx$$

$$= w^2 F^2 + w^1 F^1 \quad \forall w^1, w^2$$

$$\begin{cases} EA \int_\alpha^\beta [N^{1\prime} N^{1\prime} u^1 + N^{1\prime} N^{2\prime} u^2] \, dx = F^1 \\ EA \int_\alpha^\beta [N^{1\prime} N^{2\prime} u^1 + N^{2\prime} N^{2\prime} u^2] \, dx = F^2 \end{cases}$$



$$N^1 = \frac{x - \beta}{\alpha - \beta} = \frac{1}{L}[\beta - x]$$

$$N^2 = \frac{x - \alpha}{\beta - \alpha} = \frac{1}{L}[x - \alpha]$$

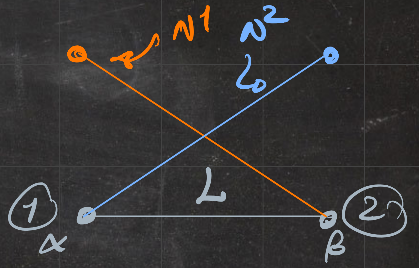
1- PIECE LINEAR APPROXIMATION

$$w = N^1 w^1 + N^2 w^2 \qquad u = N^1 u^1 + N^2 u^2$$

$$\boxed{EA\int_\alpha^\beta w' u' \, dx = w(\beta) F^\beta + w(\alpha) F^\alpha}$$

$$\begin{cases} EA\int_\alpha^\beta [N^{1\prime}N^{1\prime}u^1 + N^{1\prime}N^{2\prime}u^2]\, dx = F^1 \\ EA\int_\alpha^\beta [N^{1\prime}N^{2\prime}u^1 + N^{2\prime}N^{2\prime}u^2]\, dx = F^2 \end{cases}$$

$$\begin{bmatrix} EA\int_\alpha^\beta N^{1\prime}N^{1\prime}\, dx & EA\int_\alpha^\beta N^{1\prime}N^{2\prime}\, dx \\ EA\int_\alpha^\beta N^{2\prime}N^{1\prime}\, dx & EA\int_\alpha^\beta N^{2\prime}N^{2\prime}\, dx \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} = \begin{bmatrix} F^1 \\ F^2 \end{bmatrix}$$



$$N^1 = \frac{x-\beta}{\alpha-\beta} = \frac{1}{L}[\beta - x]$$

$$N^2 = \frac{x-\alpha}{\beta-\alpha} = \frac{1}{L}[x - \alpha]$$

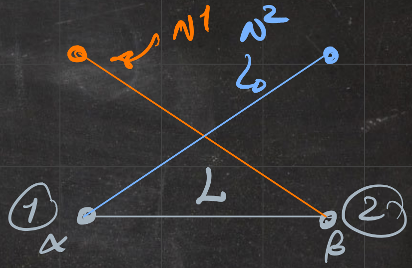
1- PIECE LINEAR APPROXIMATION

$$w = N^1 w^1 + N^2 w^2 \qquad u = N^1 u^1 + N^2 u^2$$

$$\boxed{EA\int_\alpha^\beta w' u' \, dx = w(\beta) F^\beta + w(\alpha) F^\alpha}$$

$$\begin{bmatrix} EA\int_\alpha^\beta N^{1\prime} N^{1\prime} dx & EA\int_\alpha^\beta N^{1\prime} N^{2\prime} dx \\ EA\int_\alpha^\beta N^{2\prime} N^{1\prime} dx & EA\int_\alpha^\beta N^{2\prime} N^{2\prime} dx \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} = \begin{bmatrix} F^1 \\ F^2 \end{bmatrix}$$

$$L = \beta - \alpha$$



$$\int_\alpha^\beta N^{1\prime} N^{1\prime} dx = \int_\alpha^\beta \frac{1}{L^2} dx$$

$$= \frac{1}{L^2} x \Big]_\alpha^\beta = \frac{\beta - \alpha}{L^2} = \frac{1}{L}$$

$$N^1 = \frac{x-\beta}{\alpha-\beta} = \frac{1}{L}[\beta - x]$$

$$N^2 = \frac{x-\alpha}{\beta-\alpha} = \frac{1}{L}[x - \alpha]$$

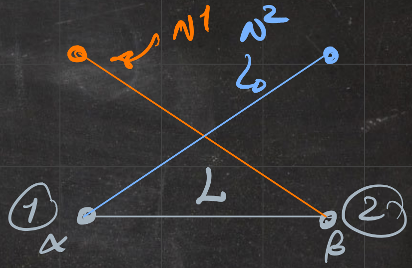
1- PIECE LINEAR APPROXIMATION

$$w = N^1 w^1 + N^2 w^2 \qquad u = N^1 u^1 + N^2 u^2$$

$$\boxed{EA \int_\alpha^\beta w' u' \, dx = w(\beta) F^\beta + w(\alpha) F^\alpha}$$

$$\begin{bmatrix} EA/L & -EA/L \\ -EA/L & EA/L \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} = \begin{bmatrix} F^1 \\ F^2 \end{bmatrix}$$

$$\Rightarrow \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} = \begin{bmatrix} F^1 \\ F^2 \end{bmatrix}$$



$$N^1 = \frac{x-\beta}{\alpha-\beta} = \frac{1}{L}[\beta - x]$$

$$N^2 = \frac{x-\alpha}{\beta-\alpha} = \frac{1}{L}[x-\alpha]$$

1- PIECE LINEAR APPROXIMATION

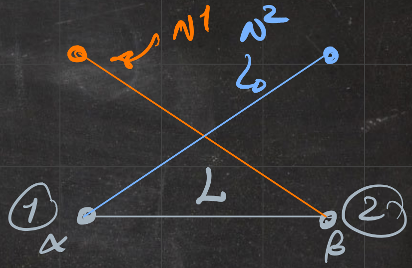
$$\boxed{EA\int_\alpha^\beta w'u'\,dx = w(\beta)F^\beta + w(\alpha)F^\alpha}$$

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u^1 \\ u^2 \end{bmatrix} = \begin{bmatrix} F^1 \\ F^2 \end{bmatrix}$$

$$\Downarrow$$

$$K = \frac{EA}{L}$$

$$\begin{bmatrix} K & -K \\ -K & K \end{bmatrix}\begin{bmatrix} u^1 \\ u^2 \end{bmatrix} = \begin{bmatrix} F^1 \\ F^2 \end{bmatrix}$$



$$N^1 = \frac{x-\beta}{\alpha-\beta} = \frac{1}{L}[\beta - x]$$

$$N^2 = \frac{x-\alpha}{\beta-\alpha} = \frac{1}{L}[x-\alpha]$$

Approximate using

1- PIECE LINEAR

1- PIECE QUADRATIC

(?) 2-PIECE LINEAR

↳ follows from assembly

$$\boxed{EA\int_\alpha^\beta w' u' \, dx = w(\beta) F^\beta + w(\alpha) F^\alpha}$$

$$[\mathbb{K}][u] = [F]$$

$$[u] = \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} \qquad [F] = \begin{bmatrix} F^1 \\ F^2 \end{bmatrix}$$

$$\mathbb{K} = EA \begin{bmatrix} \int_\alpha^\beta N^{1'} N^{1'} dx & \int_\alpha^\beta N^{1'} N^{2'} dx \\ \int_\alpha^\beta N^{2'} N^{1'} dx & \int_\alpha^\beta N^{2'} N^{2'} dx \end{bmatrix}$$

Approximate using

1- PIECE LINEAR

1- PIECE QUADRATIC

$$\boxed{EA\int_{\alpha}^{\beta} w'u'\,dx = w(\beta)F^{\beta} + w(\alpha)F^{\alpha}}$$

$$[\mathbb{K}][u] = [F]$$

$$[u] = \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} \qquad [F] = \begin{bmatrix} F^1 \\ F^2 \end{bmatrix}$$

$$\mathbb{K} = EA \begin{bmatrix} \int_{\alpha}^{\beta} N^{1\prime}N^{1\prime}\,dx & \int_{\alpha}^{\beta} N^{1\prime}N^{2\prime}\,dx \\ \int_{\alpha}^{\beta} N^{2\prime}N^{1\prime}\,dx & \int_{\alpha}^{\beta} N^{2\prime}N^{2\prime}\,dx \end{bmatrix}$$

$$\mathbb{K} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Approximate using

1- PIECE LINEAR

1- PIECE QUADRATIC

$$\mathbb{K} = \frac{EA}{3L}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}$$

$$[\mathbb{K}][\mathbb{u}] = [\mathbb{F}]$$

$$[\mathbb{u}] = \begin{bmatrix} u^1 \\ u^2 \\ u^3 \end{bmatrix} \qquad [\mathbb{F}] = \begin{bmatrix} F^1 \\ F^2 \\ F^3 \end{bmatrix}$$

$$\boxed{EA\int_\alpha^\beta w' u'\,dx = w(\beta)F^\beta + w(\alpha)F^\alpha}$$

$$\mathbb{K} = EA\begin{bmatrix} \int_\alpha^\beta N^{1'}N^{1'}dx & \int_\alpha^\beta N^{1'}N^{2'}dx & \int_\alpha^\beta N^{1'}N^{3'}dx \\ \int_\alpha^\beta N^{2'}N^{1'}dx & \int_\alpha^\beta N^{2'}N^{2'}dx & \int_\alpha^\beta N^{2'}N^{3'}dx \\ \int_\alpha^\beta N^{3'}N^{1'}dx & \int_\alpha^\beta N^{3'}N^{2'}dx & \int_\alpha^\beta N^{3'}N^{3'}dx \end{bmatrix}$$