FINITE ELEMENT METHOD

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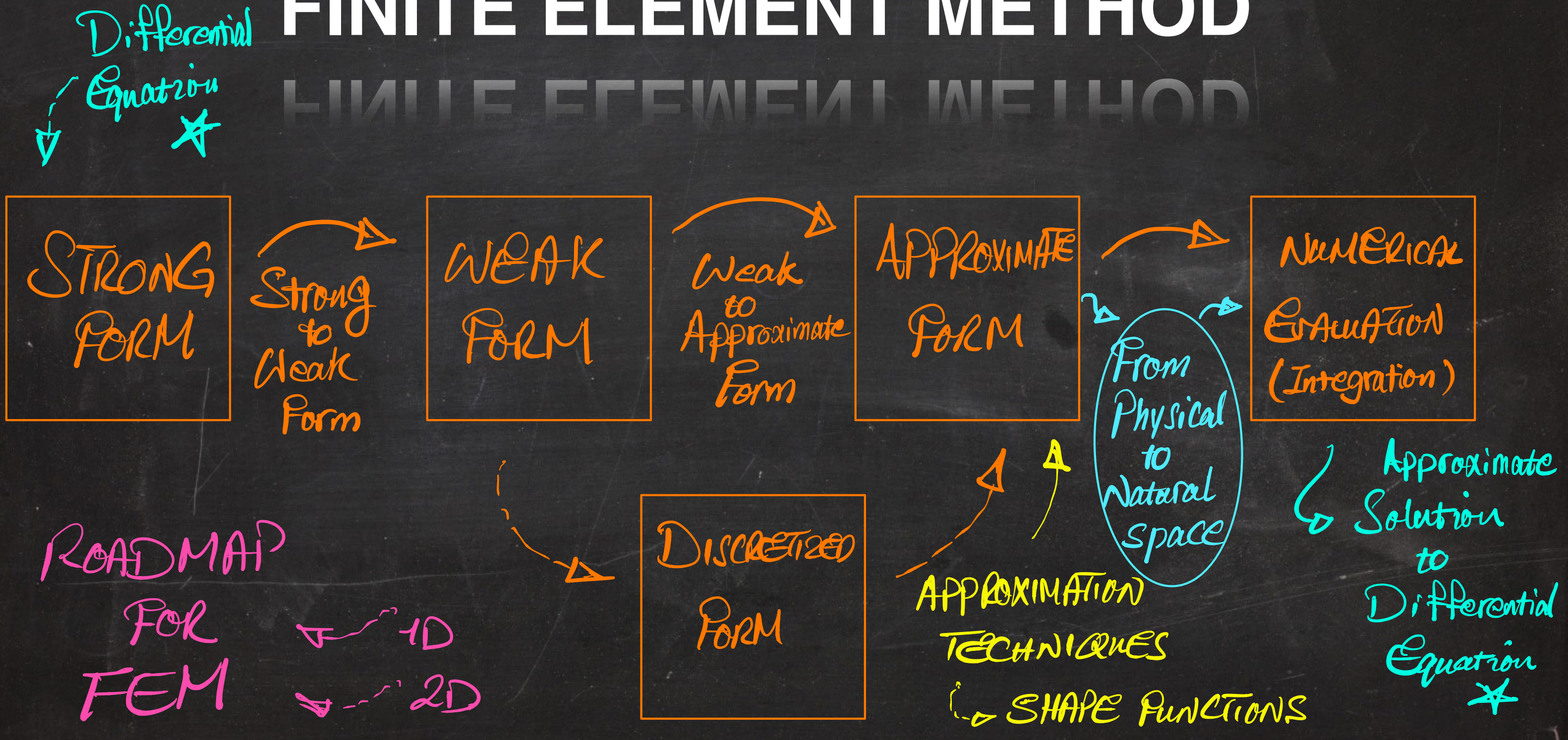
FINITE ELEMENT METHOD



UNDERSTANDING FEM VIA AN ANALOGY (A BRUTAL SIMPLIFICATION)

Approximate Solution

$\sin 45^\circ$ → Calculator ⇝ 0.707 106 7812

→ Taylor Expansion ⇝ $x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!}$

Approximate Equation

$$\sin x = \sum_{n}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!}$$

0.707 143 0458

Solution approximation

UNDERSTANDING FEM VIA AN ANALOGY (A BRUTAL SIMPLIFICATION)

Sin 45°

→ Calculator → 0.707 106 7812 | — Approximate Solution

→ Taylor Expansion → $x - \frac{x^3}{3!} + \frac{x^5}{5!}$ | — Approximate Equation

→ Input Approximation

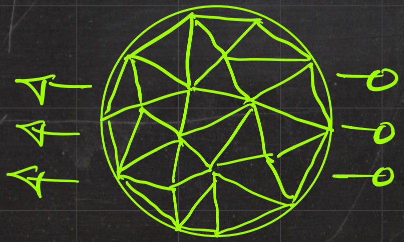
0.707 143 0458 | — Solution approximation

$45^\circ = \frac{\pi}{4} = 0.785\,398\,1634$ | —

UNDERSTANDING FEM VIA AN ANALOGY (A BRUTAL SIMPLIFICATION)

Approximations in FEM

- → Solution Approximation → inherent to numerical techniques using computers
- → Equation Approximation → diff equation is transformed to weak form + space approximation
- → Input Approximation → space discretization



→ DOMAIN (X)

Discretization (Approximation)

→ SOLUTION (u)

→ TEST (w)

diff. Eq.

→ STRONG FORM

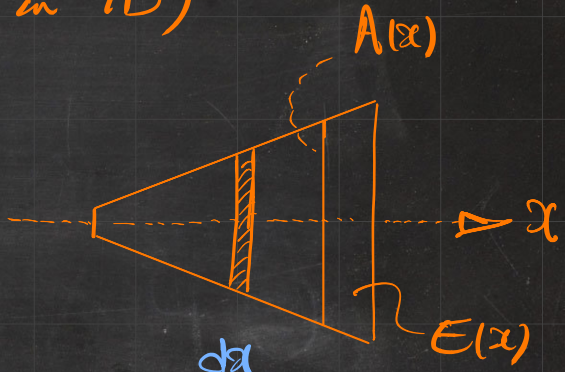
TO

integral

→ WEAK FORM

STRONG FORM (Differential Equation in 1D)

$$F + dF - F + b\,\overbrace{dV}^{A\,dx} = 0$$

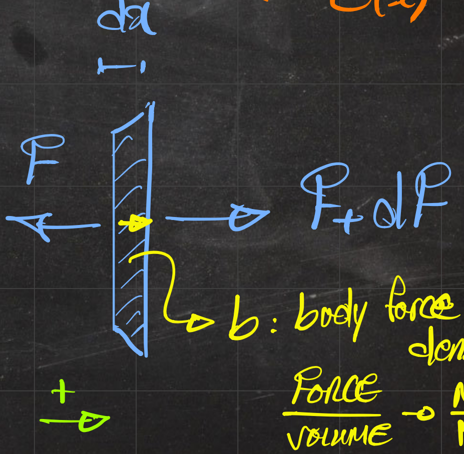


$$\frac{dF}{dx} + \underbrace{bA}_{w} = 0$$

↳ force density per length

$[N/M]$

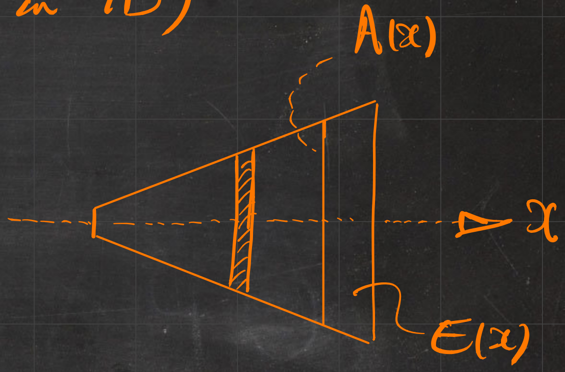
1D force density



↳ b: body force density

$\frac{\text{FORCE}}{\text{VOLUME}} \rightarrow \frac{N}{M^3}$

STRONG FORM (Differential Equation in 1D)



$$\frac{dF}{dx} + bA = 0$$

$$F = \sigma A$$

$$\sigma = E \varepsilon$$

$$\frac{d}{dx}(\sigma A) + bA = 0$$

$$\varepsilon = du/dx = u'$$

1D-Problem

E, A : CONST.

$$\frac{d}{dx}(EA\varepsilon) + bA = 0 \Rightarrow \frac{d}{dx}\left(EA\frac{du}{dx}\right) + bA = 0$$

$$Eu'' + b = 0$$

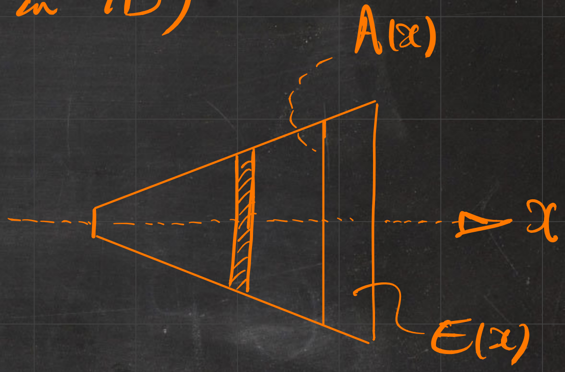
STRONG FORM (Differential Equation in 1D)

$$\frac{dF}{dx} + bA = 0$$

$F = \sigma A$

$\sigma = E\varepsilon$

$\varepsilon = u'$



ODE of 2nd order

$\Rightarrow \quad E u'' + b = 0$ & 2BCs on boundary $\begin{cases} x = 0 \\ x = L \end{cases}$

Length of the domain

Strong Form

$$E u'' + b = 0$$

2 ends ← boundary condition

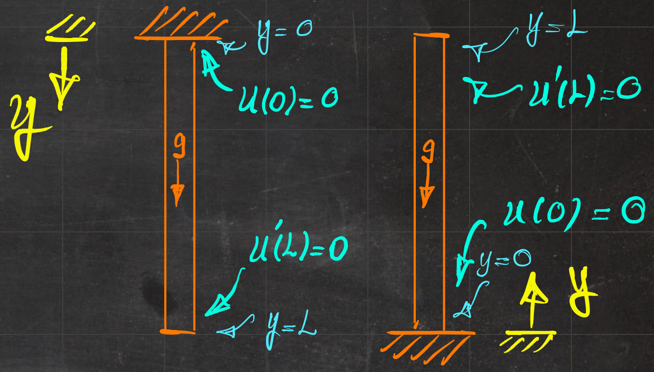
boundary

Dirichlet ← Disp. ← u

Neumann ← Force ← u'

$$F = \sigma A = E\varepsilon A = EAu'$$

Bar under its own weight



Hanging bar → Elongated

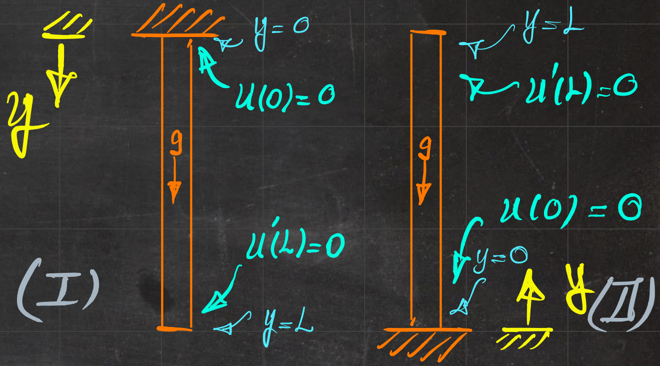
Standing bar → Compressed

STRONG FORM

Bar under its own weight

$$E u'' + b = 0 \qquad + \rho g$$

(I) Hanging bar $\leftarrow E u'' + b = 0$



$$E u'' + \rho g = 0 \quad \text{subject to} \quad u(0) = 0\,, \quad u'(L) = 0$$

$$u = -\frac{1}{2}\frac{\rho g}{E} y^2 + C_1 y + C_2 \quad \overset{\text{BCs}}{\Longrightarrow} \quad u = -\frac{1}{2}\frac{\rho g}{E} y^2 + \frac{\rho g L}{E} y$$

$u(0) = 0$

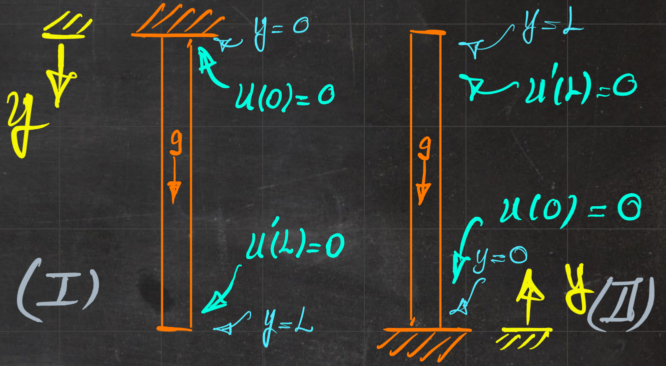
$u(L) = \frac{1}{2}\frac{\rho g}{E} L^2$

STRONG FORM

Bar under its own weight

$$Eu'' + b = 0 \qquad -\rho g$$

(II) Standing bar $Eu'' + b = 0$



$$Eu'' - \rho g = 0 \quad \text{subject to} \quad u(0) = 0 \,, \quad u'(L) = 0$$

$$u = +\frac{1}{2}\frac{\rho g}{E} y^2 + C_1 y + C_2 \xrightarrow{BCs} u = \frac{1}{2}\frac{\rho g}{E} y^2 - \frac{\rho g L}{E} y$$

$u(0) = 0$

$u(L) = -\frac{1}{2}\frac{\rho g L^2}{E}$

STRONG FORM $\quad Eu'' + b = 0$

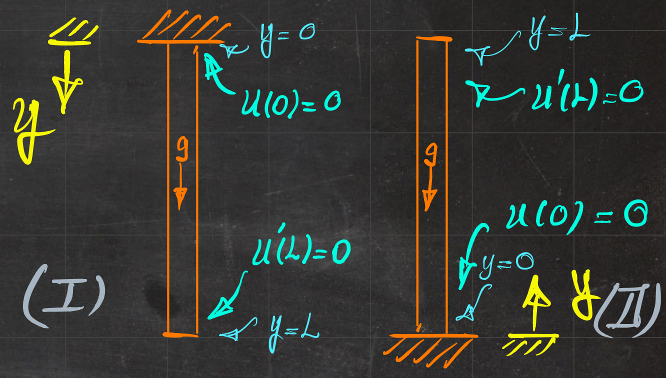
Bar under its own weight

(I) Hanging bar $\rightarrow$ $Eu'' + \rho g = 0$

$$u = -\frac{1}{2}\frac{\rho g}{E}y^2 + \frac{\rho g L}{E}y$$

(II) Standing bar $\rightarrow$ $Eu'' - \rho g = 0$

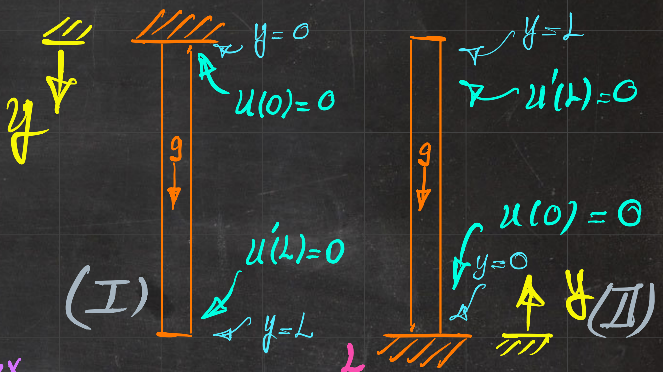
$$u = +\frac{1}{2}\frac{\rho g}{E}y^2 - \frac{\rho g L}{E}y$$

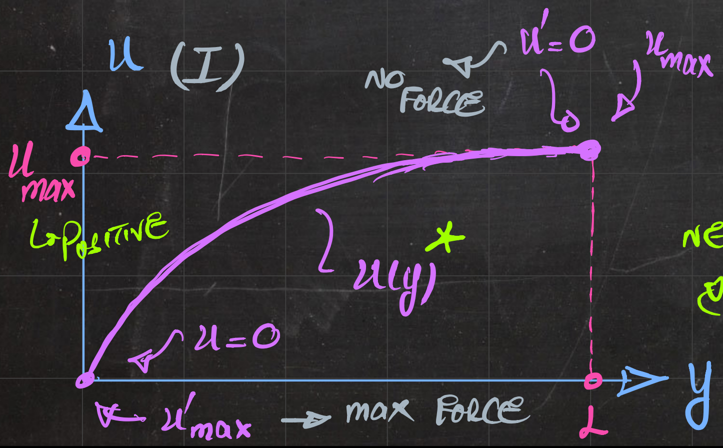


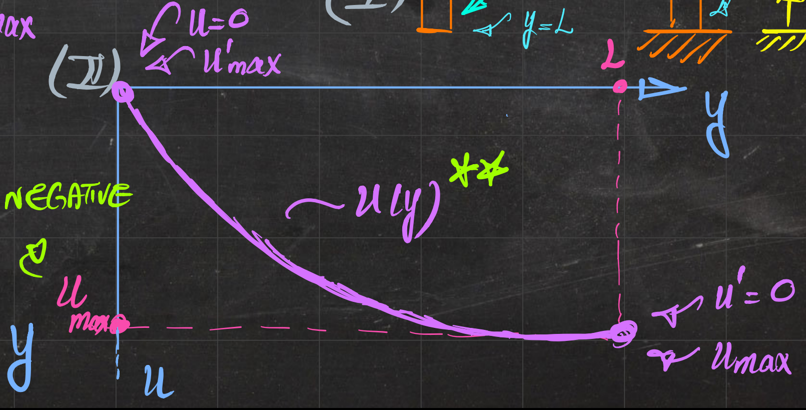
STRONG FORM $E u'' + b = 0$ Bar under its own weight

(I) Hanging bar $u = -\frac{1}{2}\frac{\rho g}{E} y^2 + \frac{\rho g L}{E} y$ *

(II) Standing bar $u = +\frac{1}{2}\frac{\rho g}{E} y^2 - \frac{\rho g L}{E} y$ **







$$\frac{d}{dx}\left(EA\frac{du}{dx}\right) + bA = 0 \quad \text{Subject to BCs}$$

$\hookrightarrow$ E, A: Const. $\rightsquigarrow$ $EA\,u'' + bA = 0$ $\leftarrow$ $f := \frac{b}{E}$

STRONG FORM

$\rightsquigarrow$ $\boxed{\begin{array}{ll} u'' + f = 0 & 0 \leqslant x \leqslant 1 \\ D:\; u(0) = u_0 & \\ N:\; u'(1) = t & \end{array}}$ prescribed

FROM STRONG TO WEAK FORM

STRONG FORM ← Differential Eq.

(I) MULTIPLY BY TEST FUNCTION w

(II) INTEGRATE OVER THE DOMAIN

Integral form ← WEAK FORM

→ BECAUSE LOWER ORDER DIFFERENTIATION OF DISPLACEMENT u

$$u'' + f = 0 \qquad 0 \leqslant x \leqslant 1$$

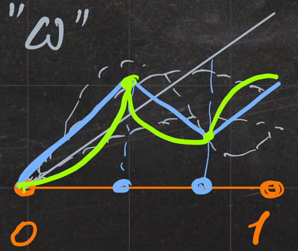
D: $u(0) = u_0$ ← prescribed

N: $u'(1) = t$

w: ARBITRARY

CONTINUOUS

$w|_D = 0$ ← ZERO @ DIRICHLET BOUNDARY CONDITIONS



STRONG : u''

WEAK : u'

FROM STRONG TO WEAK FORM

$$u'' + f = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = u_0$ ← prescribed

N: $u'(1) = t$

STRONG FORM

(I) MULTIPLY BY TEST FUNCTION w

(II) INTEGRATE OVER THE DOMAIN

w : ARBITRARY, CONTINUOUS, $w|_D = 0$

I) $[u'' + f = 0] \times w \Rightarrow w u'' + w f = 0$

II) $\displaystyle\int_0^1 [w u'' + w f]\, dx = 0$ $\qquad w u'' = (w u')' - w' u'$

$$\int_0^1 (w u')'\, dx - \int_0^1 w' u'\, dx + \int_0^1 w f\, dx = 0$$

FROM STRONG TO WEAK FORM

$$u'' + f = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = u_0$ ← prescribed

N: $u'(1) = t$

I) $[u'' + f = 0] \times \omega \Rightarrow \omega u'' + \omega f = 0$

II) $\int_0^1 [\omega u'' + \omega f]\, dx = 0$ $\qquad \omega u'' = (\omega u')' - \omega' u'$

$$\int_0^1 (\omega u')'\, dx - \int_0^1 \omega' u'\, dx + \int_0^1 \omega f\, dx = 0$$

$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$

$$\int_0^1 \omega' u'\, dx = \int_0^1 \omega f\, dx + \omega u' \Big]_0^1$$

$$\int_0^1 \omega' u'\, dx = \int_0^1 \omega f\, dx + \omega(1)\, u'(1) - \omega(0)\, u'(0)$$

FROM STRONG TO WEAK FORM

$$u'' + f = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = u_0$ ← prescribed

N: $u'(1) = t$ ←

I) $[u'' + f = 0] \times \omega \Rightarrow \omega u'' + \omega f = 0$

II) $\int_0^1 [\omega u'' + \omega f]\, dx = 0$ $\qquad \omega u'' = (\omega u')' - \omega' u'$

$$\int_0^1 \omega' u' \, dx = \int_0^1 \omega f \, dx + \overbrace{\omega(1)\, u'(1) - \omega(0)\, u'(0)}$$

TEST FUNCTION @ 1 $\qquad$ TEST FUNCTION @ 0

ω : ARBITRARY, CONTINUOUS, $\omega|_D = 0$

BC:

DIRICHLET	u ✓	u' ?
NEUMANN	u ?	u' ✓

FROM STRONG TO WEAK FORM

$$u'' + f = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = u_0$ ← prescribed

N: $u'(1) = t$

I) $[u'' + f = 0] \times w \Rightarrow w u'' + w f = 0$

II) $\int_0^1 [w u'' + w f]\, dx = 0$ $\qquad w u'' = (w u')' - w' u'$

WEAK FORM

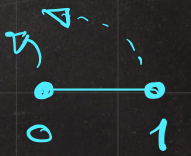
$$\int_0^1 w' u' \, dx = \int_0^1 w f \, dx + w(1) u'(1) - w(0) u'(0)$$

w : ARBITRARY, CONTINUOUS, $w|_D = 0$

INTERNAL CONTRIBUTIONS OVER THE DOMAIN

EXTERNAL CONTRIBUTIONS OVER THE DOMAIN

EXTERNAL CONTRIBUTIONS OVER THE BOUNDARY OF THE DOMAIN



THOUGHT EXPERIMENT

Prepare a salad made of 3 ingredients

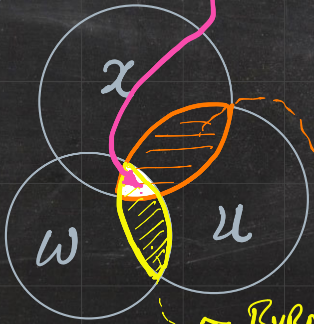
Lettuce
Tomato
Olive oil

x ← coordinate ← geometry
u ← displacement
w ← test function

BUBNOV-GALERKIN
PETROV-GALERKIN
Approximation

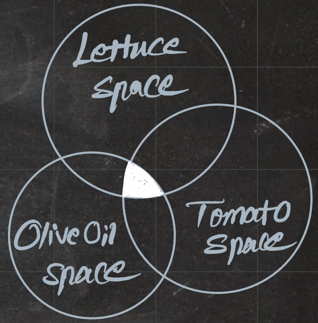
ISOPARAMETRIC CONCEPTS

WE ALWAYS PICK FROM HERE



isoparametric concept

BUBNOV-GALERKIN APPROXIMA...



From strong to weak form

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

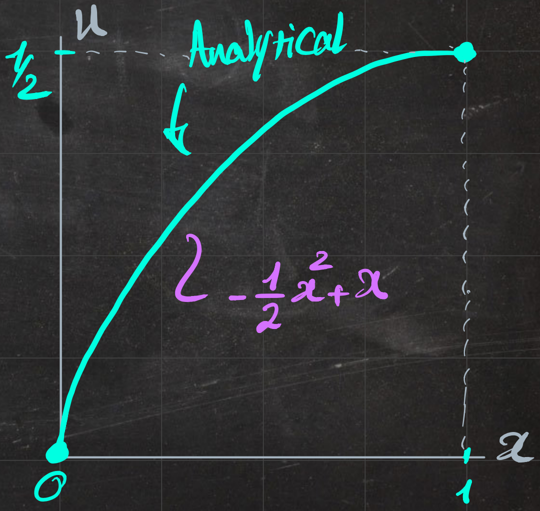
$$u'' = -1 \Rightarrow u' = -x + C_1$$

$$\Rightarrow u = -\frac{1}{2}x^2 + C_1 x + C_2$$

$\hookrightarrow u(0)=0 \Rightarrow C_2 = 0$

$\hookrightarrow u'(1)=0 \Rightarrow C_1 = 1$

Analytical Solution $\Rightarrow u = -\frac{1}{2}x^2 + x$



FROM STRONG TO WEAK FORM

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

$D:\ u(0) = 0$ ← prescribed

$N:\ u'(1) = 0$

$$\int_0^1 w' u' \, dx = \int_0^1 w f \, dx + w(1)\, u'(1) - w(0)\, u'(0)$$

$$\Rightarrow \int_0^1 w' u' \, dx = \int_0^1 w \, dx$$ ← WEAK FORM

$w:$ $\begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$

→ Compute approximate solution → from different spaces

⇓

EXERCISE ⟼ ...

UNDERSTANDING THE SIGNIFICANCE OF WEAK FORM BY EXAMPLE

$$\int_0^1 w' u' \, dx = \int_0^1 w \, dx$$

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$

$$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$$

- → 1-PIECE LINEAR APPROXIMATION
- → 2-PIECE LINEAR (UNIFORM) APPROXIMATION
- → 2-PIECE LINEAR (NON-UNIFORM) APPROXIMATION (I)
- → 2-PIECE LINEAR (NON-UNIFORM) APPROXIMATION (II)
- → 2-PIECE LINEAR (GENERAL) APPROXIMATION

UNDERSTANDING THE SIGNIFICANCE OF WEAK FORM BY EXAMPLE

$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx$$

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$

w : ARBITRARY, CONTINUOUS, $w|_D = 0$

→ 3-PIECE LINEAR (UNIFORM) APPROXIMATION

→ 3-PIECE LINEAR (GENERIC) APPROXIMATION

→ 4-PIECE LINEAR (UNIFORM) APPROXIMATION

→ 4-PIECE LINEAR (GENERIC) APPROXIMATION

→ 1-PIECE QUADRATIC

→ 1-PIECE CUBIC

$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \qquad x \in [0,1]$$

$$u'' + 1 = 0 \qquad 0 \leq x \leq 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$

→ 1-PIECE LINEAR APPROXIMATION

$w = C_1 x + C_2 \to C_2 = 0 \qquad u = D_1 x + D_2 \to D_2 = 0$

$w(0) = 0 \Rightarrow \qquad w|_D = 0 \Leftarrow u(0)$ is GIVEN $\qquad u(0) = 0$

$$w: \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$$

$$\int_0^1 C_1 D_1\,dx = \int_0^1 C_1 x\,dx \Rightarrow C_1 D_1 x\Big]_0^1 = \frac{1}{2} C_1 x^2\Big]_0^1$$

$$\Rightarrow D_1 = \frac{1}{2} \qquad C_1 : \text{Cancels out}$$

$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \qquad x\in[0,1]$$

$$u'' + 1 = 0 \qquad 0 \leq x \leq 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$

→ 1- PIECE LINEAR APPROXIMATION

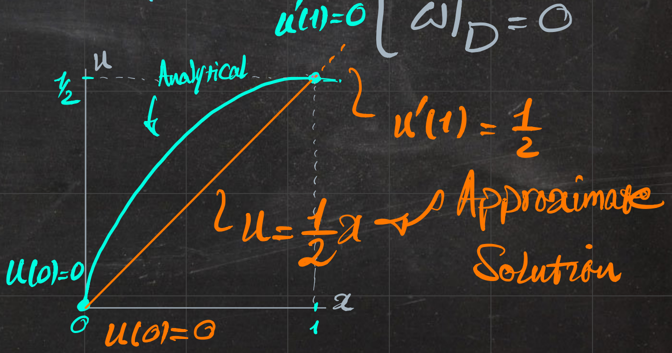
$w = C_1 x + C_2 \rightarrow C_2 = 0 \qquad u = D_1 x + D_2 \rightarrow D_2 = 0$

$w(0) = 0 \qquad w|_D = 0 \Leftarrow u(0)$ is GIVEN $\qquad u(0) = 0$

w : ARBITRARY, CONTINUOUS, $w|_D = 0$

C_1 : Cancels out $\qquad \Rightarrow D_1 = \frac{1}{2}$

APPROXIMATE SOLUTION FOR u $\Rightarrow u = \frac{1}{2}x$



$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \qquad x \in [0,1]$$

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$

→ 1-PIECE LINEAR APPROXIMATION

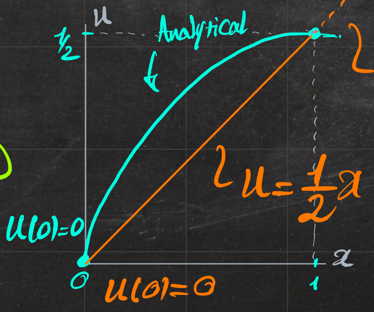
$$w = C_1 x + C_2 \rightarrow C_2 = 0 \qquad u = D_1 x + D_2 \rightarrow D_2 = 0$$

$w(0) = 0 \qquad w|_D = 0 \Leftarrow u(0)$ is GIVEN $\qquad u(0) = 0$

w : ARBITRARY, CONTINUOUS, $w|_D = 0$

$u'(1)=0$

$$\Rightarrow u = \frac{1}{2}x$$



$u'(1) = \frac{1}{2}$

$u = \frac{1}{2}x$ ← Approximate Solution

DIRICHLET BCs ARE STRONGLY SATISFIED

NEUMANN BCs ARE WEAKLY SATISFIED

$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \qquad x\in[0,1]$$

$$u'' + 1 = 0 \qquad 0 \le x \le 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

→ 2-PIECE LINEAR (UNIFORM) APPROXIMATION

$0 \leftarrow w|_D = 0$ $\qquad\qquad$ $0 \leftarrow u(0)=0$

$x\in[0,0.5] \qquad w = C_1 x + \cancel{C_2} \qquad u = E_1 x + \cancel{E_2}$

$x\in[0.5,1] \qquad w = D_1 x + D_2 \qquad u = F_1 x + F_2$

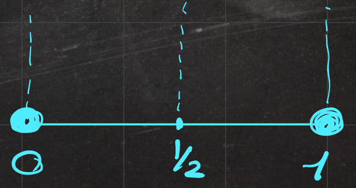
w : ARBITRARY, CONTINUOUS, $w|_D = 0$

$\Rightarrow \frac{1}{2}C_1 + C_2 = \frac{1}{2}D_1 + D_2 \qquad \Rightarrow \frac{1}{2}E_1 + E_2 = \frac{1}{2}F_1 + F_2$

↳ Employ BCs and Continuity Conditions

↳ w continuous @ 0.5

↳ u continuous @ 0.5



$$\int_0^1 w' u' \, dx = \int_0^1 w \, dx \qquad x \in [0,1]$$

$$u'' + 1 = 0 \qquad 0 \leq x \leq 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$

→ 2-PIECE LINEAR (UNIFORM) APPROXIMATION

$0 \leftarrow w|_D = 0$ $\qquad$ $0 \leftarrow u(0) = 0$

$x \in [0, 0.5]$ $\qquad w = C_1 x + \cancel{C_2}$ $\qquad u = E_1 x + \cancel{E_2}$

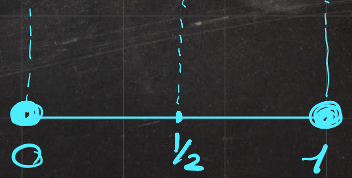
$x \in [0.5, 1]$ $\qquad w = D_1 x + D_2$ $\qquad u = F_1 x + F_2$

w : ARBITRARY, CONTINUOUS, $w|_D = 0$

$\Rightarrow \frac{1}{2} C_1 + C_2 = \frac{1}{2} D_1 + D_2$ $\qquad \Rightarrow \frac{1}{2} E_1 + E_2 = \frac{1}{2} F_1 + F_2$

$\Downarrow$ $\qquad \Downarrow$

$D_2 = \frac{1}{2} [C_1 - D_1]$ $\qquad F_2 = \frac{1}{2} [E_1 - F_1]$



$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \qquad x \in [0,1]$$

$$u'' + 1 = 0 \qquad 0 \leq x \leq 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

→ 2-PIECE LINEAR (UNIFORM) APPROXIMATION

$0 \leftarrow w|_D = 0$

$x \in [0, 0.5]$ $\quad w = C_1 x + \cancel{C_2}$ $\quad u = E_1 x + \cancel{E_2}$ $\quad 0 \leftarrow u(0) = 0$

$x \in [0.5, 1]$ $\quad w = D_1 x + D_2$ $\quad u = F_1 x + F_2$

w: ARBITRARY, CONTINUOUS, $w|_D = 0$

$D_2 = \frac{1}{2}[C_1 - D_1]$ $\qquad F_2 = \frac{1}{2}[E_1 - F_1]$

$$\int_0^{0.5} w'u'\,dx + \int_{0.5}^{1} w'u'\,dx = \int_0^{0.5} w\,dx + \int_{0.5}^{1} w\,dx$$