FINITE ELEMENT METHOD

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FINITE ELEMENT METHOD

Differential Equation ✱

STRONG FORM

Strong to Weak Form

WEAK FORM

Weak to Approximate Form

APPROXIMATE FORM

From Physical to Natural space

NUMERICAL EVALUATION (Integration)

Approximate Solution to Differential Equation ✱

DISCRETIZED FORM

APPROXIMATION TECHNIQUES

SHAPE FUNCTIONS

ROADMAP FOR FEM

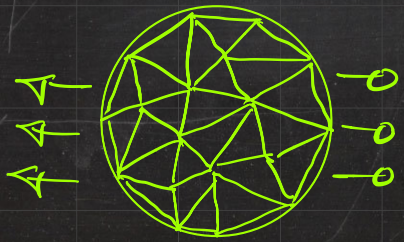
1D

2D

UNDERSTANDING FEM VIA AN ANALOGY (A BRUTAL SIMPLIFICATION)

Approximations in FEM

- Solution Approximation → inherent to numerical techniques using computers
- Equation Approximation → diff equation is transformed to weak form + space approximation
- Input Approximation → space discretization



Discretization (Approximation)

- → DOMAIN (X)
- → SOLUTION (u)
- → TEST (w)

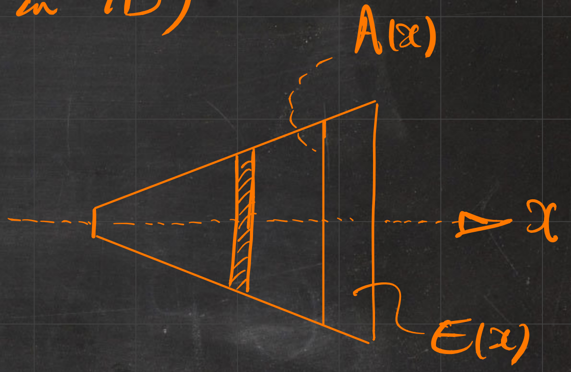
diff. Eq.

→ STRONG FORM

TO

→ WEAK FORM (integral)

STRONG FORM (Differential Equation in 1D)



$$\frac{dF}{dx} + bA = 0 \qquad F = \sigma A$$

$$\sigma = E\varepsilon$$

$$\frac{d}{dx}(\sigma A) + bA = 0 \qquad \varepsilon = du/dx = u'$$

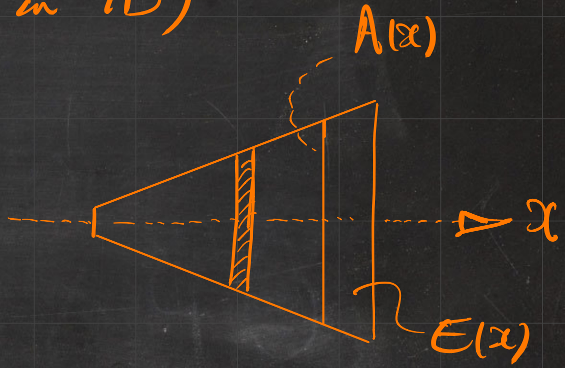
1D-Problem

E, A : CONST.

$$\frac{d}{dx}(EA\varepsilon) + bA = 0 \Rightarrow \frac{d}{dx}\left(EA\frac{du}{dx}\right) + bA = 0$$

$$Eu'' + b = 0$$

Strong Form (Differential Equation in 1D)



$$\frac{dF}{dx} + bA = 0$$

$$F = \sigma A$$

$$\sigma = E\varepsilon$$

$$\varepsilon = u'$$

ODE of 2nd order

$$\Rightarrow E u'' + b = 0$$

& 2BCs on boundary $\begin{cases} x = 0 \\ x = L \end{cases}$

L: Length of the domain

STRONG FORM

Bar under its own weight

$$Eu'' + b = 0$$

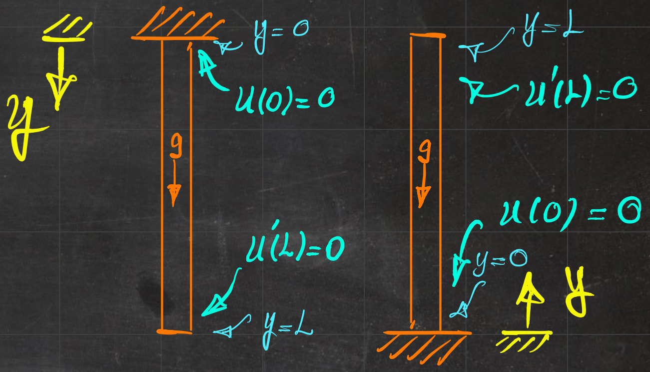
2 ends ← boundary condition

boundary

$F = \sigma A = E\varepsilon A = EAu'$

Dirichlet ← Disp. ← u

Neumann ← Force ← u'



Hanging bar → Elongated

Standing bar → Compressed

$$\frac{d}{dx}\left(EA\frac{du}{dx}\right) + bA = 0 \quad \text{Subject to BCs}$$

$\hookrightarrow$ E, A: Const. $\rightsquigarrow$ $EA\,u'' + bA = 0$ $\leftarrow$ $f := \frac{b}{E}$

STRONG FORM

$$\boxed{\begin{aligned} &\rightsquigarrow\ u'' + f = 0 \qquad 0 \leqslant x \leqslant 1 \\ &D:\ u(0) = u_0 \\ &N:\ u'(1) = t \end{aligned}}$$

prescribed

FROM STRONG TO WEAK FORM

$$u'' + f = 0 \qquad 0 \leq x \leq 1$$

D: $u(0) = u_0$ prescribed

N: $u'(1) = t$

STRONG FORM Differential Eq.

(I) MULTIPLY BY TEST FUNCTION w

(II) INTEGRATE OVER THE DOMAIN

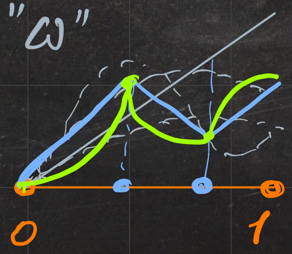
w :
- ARBITRARY
- CONTINUOUS
- $w|_D = 0$ ZERO @ DIRICHLET BOUNDARY CONDITIONS

Integral form WEAK FORM

BECAUSE LOWER ORDER DIFFERENTIATION OF DISPLACEMENT u

STRONG : u''

WEAK : u'



FROM STRONG TO WEAK FORM

$$u'' + f = 0 \qquad 0 \leq x \leq 1$$

D: $u(0) = u_0$ ← prescribed

N: $u'(1) = t$

STRONG FORM

(I) MULTIPLY BY TEST FUNCTION w

(II) INTEGRATE OVER THE DOMAIN

w : ARBITRARY, CONTINUOUS, $w|_D = 0$

I) $[u'' + f = 0] \times w \Rightarrow w u'' + w f = 0$

II) $\displaystyle\int_0^1 [w u'' + w f]\, dx = 0$

$$w u'' = (w u')' - w' u'$$

$$\int_0^1 (w u')'\, dx - \int_0^1 w' u'\, dx + \int_0^1 w f\, dx = 0$$

FROM STRONG TO WEAK FORM

$u'' + f = 0 \qquad 0 \leqslant x \leqslant 1$

D: $u(0) = u_0$ ← prescribed

N: $u'(1) = t$

I) $[u'' + f = 0] \times \omega \Rightarrow \omega u'' + \omega f = 0$

II) $\int_0^1 [\omega u'' + \omega f]\, dx = 0 \qquad \omega u'' = (\omega u')' - \omega' u'$

$$\int_0^1 (\omega u')'\, dx - \int_0^1 \omega' u'\, dx + \int_0^1 \omega f\, dx = 0$$

ω : ARBITRARY, CONTINUOUS, $\omega|_D = 0$

$$\int_0^1 \omega' u'\, dx = \int_0^1 \omega f\, dx + \omega u' \Big]_0^1$$

$$\int_0^1 \omega' u'\, dx = \int_0^1 \omega f\, dx + \omega(1)\, u'(1) - \omega(0)\, u'(0)$$

FROM STRONG TO WEAK FORM

$$u'' + f = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = u_0$ ← prescribed

N: $u'(1) = t$ ←

I) $[u'' + f = 0] \times \omega \Rightarrow \omega u'' + \omega f = 0$

II) $\displaystyle\int_0^1 [\omega u'' + \omega f]\, dx = 0 \qquad \omega u'' = (\omega u')' - \omega' u'$

$$\int_0^1 \omega' u' \, dx = \int_0^1 \omega f \, dx + \overbrace{\omega(1) u'(1) - \omega(0) u'(0)}$$

TEST FUNCTION @ 1 TEST FUNCTION @ 0

ω : ARBITRARY, CONTINUOUS, $\omega|_D = 0$

BC: DIRICHLET u ✓ u' ?

NEUMANN u ? u' ✓

FROM STRONG TO WEAK FORM

$$u'' + f = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = u_0$ ← prescribed

N: $u'(1) = t$ ←

I) $[u'' + f = 0] \times \omega \Rightarrow \omega u'' + \omega f = 0$

II) $\int_0^1 [\omega u'' + \omega f]\, dx = 0 \qquad \omega u'' = (\omega u')' - \omega' u'$

↙ WEAK FORM

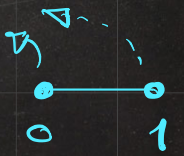
$$\underbrace{\int_0^1 \omega' u' \, dx}_{} = \underbrace{\int_0^1 \omega f \, dx}_{} + \underbrace{\omega(1)\, u'(1) - \omega(0)\, u'(0)}_{}$$

$\omega :$ $\begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$

INTERNAL CONTRIBUTIONS OVER THE DOMAIN

EXTERNAL CONTRIBUTIONS OVER THE DOMAIN

EXTERNAL CONTRIBUTIONS OVER THE BOUNDARY OF THE DOMAIN



THOUGHT EXPERIMENT

BUBNOV-GALERKIN

PETROV-GALERKIN

Approximation

ISOPARAMETRIC CONCEPTS

Prepare a salad made of 3 ingredients

Lettuce

Tomato

Olive Oil

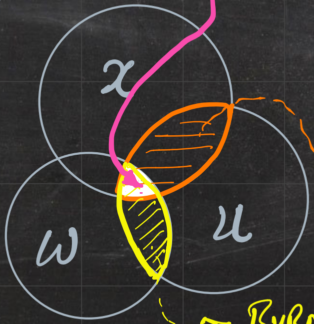
geometry

x ← coordinate

u ← displacement

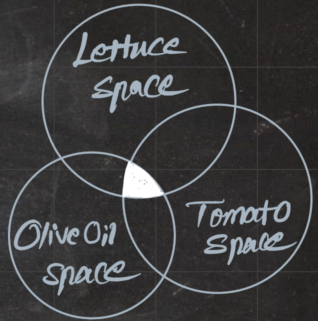
w ← test function

WE ALWAYS PICK FROM HERE



isoparametric concept

BUBNOV-GALERKIN APPROXIMA..



From strong to weak form

$$u'' + 1 = 0 \qquad 0 \leq x \leq 1$$

D: $u(0) = 0$ ← prescribed
N: $u'(1) = 0$

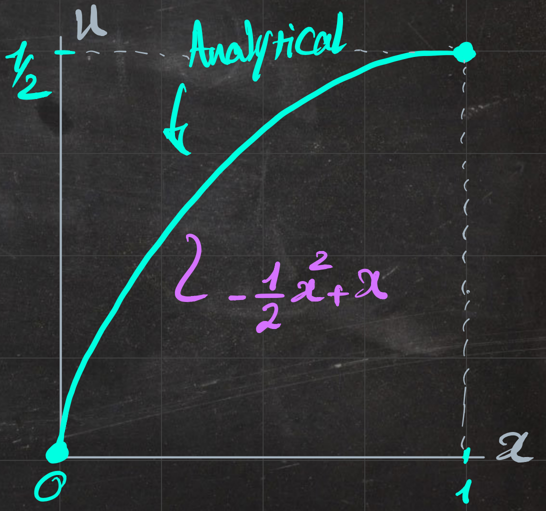
$$u'' = -1 \Rightarrow u' = -x + C_1$$

$$\Rightarrow u = -\frac{1}{2}x^2 + C_1 x + C_2$$

$u(0) = 0 \Rightarrow C_2 = 0$

$u'(1) = 0 \Rightarrow C_1 = 1$

Analytical Solution $\Rightarrow u = -\frac{1}{2}x^2 + x$



FROM STRONG TO WEAK FORM

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$

$$\int_0^1 w' u' \, dx = \int_0^1 w f \, dx + w(1) u'(1) - w(0) u'(0)$$

$$\Rightarrow \int_0^1 w' u' \, dx = \int_0^1 w \, dx$$ ← WEAK FORM

w : $\begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$

→ Compute approximate solution

→ from different spaces

⇓

EXERCISE ↦ ...

UNDERSTANDING THE SIGNIFICANCE OF WEAK FORM BY EXAMPLE

$$\int_0^1 w' u' \, dx = \int_0^1 w \, dx$$

$$u'' + 1 = 0 \qquad 0 \leq x \leq 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

$$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$$

→ 1-PIECE LINEAR APPROXIMATION

→ 2-PIECE LINEAR (UNIFORM) APPROXIMATION

→ 2-PIECE LINEAR (NON-UNIFORM) APPROXIMATION (I)

→ 2-PIECE LINEAR (NON-UNIFORM) APPROXIMATION (II)

→ 2-PIECE LINEAR (GENERAL) APPROXIMATION

UNDERSTANDING THE SIGNIFICANCE OF WEAK FORM BY EXAMPLE

$$\int_0^1 w'u'dx = \int_0^1 w\,dx$$

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$

$$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$$

→ 3-PIECE LINEAR (UNIFORM) APPROXIMATION

→ 3-PIECE LINEAR (GENERIC) APPROXIMATION

→ 4-PIECE LINEAR (UNIFORM) APPROXIMATION

→ 4-PIECE LINEAR (GENERIC) APPROXIMATION

→ 1-PIECE QUADRATIC

→ 1-PIECE CUBIC

$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \qquad x \in [0,1]$$

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$

→ 1-PIECE LINEAR APPROXIMATION

$$w = C_1 x + C_2 \rightarrow C_2 = 0 \qquad u = D_1 x + D_2 \rightarrow D_2 = 0$$

$w(0) = 0 \Rightarrow$ $\quad w|_D = 0 \Leftarrow u(0)$ is GIVEN $\qquad u(0) = 0$

w: ARBITRARY, CONTINUOUS, $w|_D = 0$

$$\int_0^1 C_1 D_1\,dx = \int_0^1 C_1 x\,dx \Rightarrow C_1 D_1 x\Big]_0^1 = \frac{1}{2} C_1 x^2\Big]_0^1$$

$$\Rightarrow D_1 = \frac{1}{2}$$

C_1: Cancels out

$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \qquad x \in [0,1]$$

$$u'' + 1 = 0 \qquad 0 \leq x \leq 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$

→ 1-PIECE LINEAR APPROXIMATION

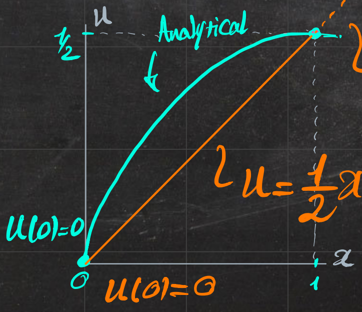
$w = C_1 x + C_2 \rightarrow C_2 = 0$ $\qquad u = D_1 x + D_2 \rightarrow D_2 = 0$

$w(0) = 0 \qquad w|_D = 0 \Leftarrow u(0)$ is GIVEN $\qquad u(0) = 0$

w : ARBITRARY, CONTINUOUS, $w|_D = 0$

$u'(1) = 0$

C_1 : Cancels out $\Rightarrow D_1 = \frac{1}{2}$



$u'(1) = \frac{1}{2}$

$u = \frac{1}{2}x$ ← Approximate Solution

$u(0) = 0$

APPROXIMATE SOLUTION FOR u $\Rightarrow u = \frac{1}{2}x$

$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \qquad x \in [0,1]$$

$$\boxed{\begin{array}{ll} u'' + 1 = 0 & 0 \leq x \leq 1 \\ D:\ u(0) = 0 & \leftarrow \text{prescribed} \\ N:\ u'(1) = 0 & \end{array}}$$

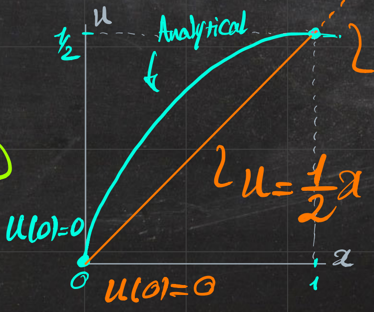
→ 1-PIECE LINEAR APPROXIMATION

$$w = C_1 x + C_2 \rightarrow C_2 = 0 \qquad u = D_1 x + D_2 \rightarrow D_2 = 0$$

$w(0) = 0$ $\Rightarrow$ $w|_D = 0 \Leftarrow u(0)$ is GIVEN $\qquad u(0) = 0$

$$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$$

$$\Rightarrow u = \frac{1}{2}x$$



$u'(1) = \frac{1}{2}$

Approximate Solution

DIRICHLET BCs ARE STRONGLY SATISFIED

NEUMANN BCs ARE WEAKLY SATISFIED

$$\int_0^1 \omega' u' \, dx = \int_0^1 \omega \, dx \qquad x \in [0,1]$$

$$u'' + 1 = 0 \qquad 0 \leq x \leq 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$

→ 2-PIECE LINEAR (UNIFORM) APPROXIMATION

$0 \leftarrow \omega|_D = 0$; $0 \leftarrow u(0) = 0$

$x \in [0, 0.5] \qquad \omega = C_1 x + \cancel{C_2} \qquad u = E_1 x + \cancel{E_2}$

$x \in [0.5, 1] \qquad \omega = D_1 x + D_2 \qquad u = F_1 x + F_2$

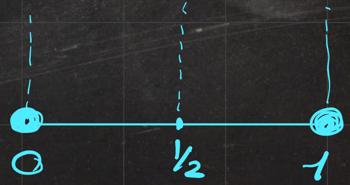
$\omega :$ $\begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$

$\Rightarrow \frac{1}{2} C_1 + C_2 = \frac{1}{2} D_1 + D_2 \qquad \Rightarrow \frac{1}{2} E_1 + E_2 = \frac{1}{2} F_1 + F_2$

↳ Employ BCs and Continuity Conditions

↳ ω Continuous @ 0.5

↳ u Continuous @ 0.5



$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \qquad x\in[0,1]$$

$$u'' + 1 = 0 \qquad 0 \le x \le 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

→ 2-PIECE LINEAR (UNIFORM) APPROXIMATION

$0 \leftarrow w|_D = 0$

$x \in [0, 0.5] \qquad w = C_1 x + \cancel{C_2} \qquad u = E_1 x + \cancel{E_2}\ 0 \leftarrow u(0)=0$

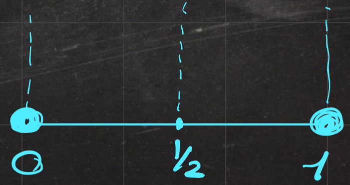
$x \in [0.5, 1] \qquad w = D_1 x + D_2 \qquad u = F_1 x + F_2$

w : ARBITRARY, CONTINUOUS, $w|_D = 0$

$\Rightarrow \frac{1}{2}C_1 + C_2 = \frac{1}{2}D_1 + D_2 \qquad \Rightarrow \frac{1}{2}E_1 + E_2 = \frac{1}{2}F_1 + F_2$

$\Downarrow \qquad \Downarrow$

$D_2 = \frac{1}{2}[C_1 - D_1] \qquad F_2 = \frac{1}{2}[E_1 - F_1]$



$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \qquad x\in[0,1]$$

$$\boxed{\begin{array}{ll} u''+1=0 & 0\leq x\leq 1 \\ D:\ u(0)=0 & \leftarrow \text{prescribed} \\ N:\ u'(1)=0 & \end{array}}$$

→ 2-PIECE LINEAR (UNIFORM) APPROXIMATION

$0 \leftarrow w|_D=0$

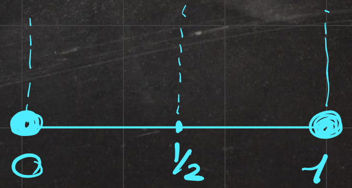
$x\in[0,0.5] \qquad w = C_1 x + \cancel{C_2} \qquad u = E_1 x + \cancel{E_2} \qquad 0 \leftarrow u(0)=0$

$x\in[0.5,1] \qquad w = D_1 x + D_2 \qquad u = F_1 x + F_2$

$w:\ \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D=0 \end{cases}$

$D_2 = \frac{1}{2}[C_1 - D_1] \qquad F_2 = \frac{1}{2}[E_1 - F_1]$

$$\int_0^{0.5} w'u'\,dx + \int_{0.5}^1 w'u'\,dx = \int_0^{0.5} w\,dx + \int_{0.5}^1 w\,dx$$



$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \qquad x\in[0,1]$$

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$

→ 2-PIECE LINEAR (UNIFORM) APPROXIMATION

$0 \leftarrow w|_D = 0$

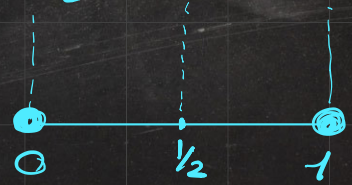
$x \in [0, 0.5]$ $\quad w = C_1 x + \cancel{C_2} \quad u = E_1 x + \cancel{E_2}$ $0 \leftarrow u(0) = 0$

$x \in [0.5, 1]$ $\quad w = D_1 x + D_2 \quad u = F_1 x + F_2$

w : { ARBITRARY, CONTINUOUS, $w|_D = 0$

$D_2 = \frac{1}{2}[C_1 - D_1] \qquad F_2 = \frac{1}{2}[E_1 - F_1]$

$$\int_0^{0.5} \overset{C_1}{w'}\,\underset{E_1}{u'}\,dx + \int_{0.5}^{1} \overset{D_1}{w'}\,\underset{F_1}{u'}\,dx = \int_0^{0.5} \underset{C_1 x}{w}\,dx + \int_{0.5}^{1} \overset{D_1 x + \frac{1}{2}[C_1 - D_1]}{w}\,dx$$



$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \qquad x\in[0,1]$$

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$

→ 2-PIECE LINEAR (UNIFORM) APPROXIMATION

$$\int_0^{0.5} C_1 E_1\,dx + \int_{0.5}^{1} D_1 F_1\,dx$$

$$= \int_0^{0.5} C_1 x\,dx + \int_{0.5}^{1} \left[D_1 x + \tfrac{1}{2}[C_1 - D_1]\right] dx$$

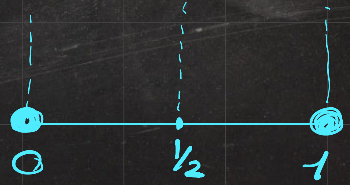
$$u = E_1 x + \cancel{E_2}^{\,0}$$

$$u = F_1 x + F_2$$

$$F_2 = \tfrac{1}{2}[E_1 - F_1]$$

w : ARBITRARY, CONTINUOUS, $w|_D = 0$

$$\frac{1}{2} C_1 E_1 + \frac{1}{2} D_1 F_1 = \frac{1}{8} C_1 + \frac{3}{8} D_1 + \frac{1}{4}[C_1 - D_1]$$



$$\int_0^1 \omega' u' \, dx = \int_0^1 \omega \, dx \qquad x \in [0,1]$$

$$u'' + 1 = 0 \qquad 0 \leq x \leq 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$ ←

→ 2-PIECE LINEAR (UNIFORM) APPROXIMATION

$$\frac{1}{2} C_1 E_1 + \frac{1}{2} D_1 F_1$$

$$= \frac{1}{8} C_1 + \frac{3}{8} D_1 + \frac{1}{4} [C_1 - D_1]$$

$$u = E_1 x + \cancel{E_2}^{\,0}$$

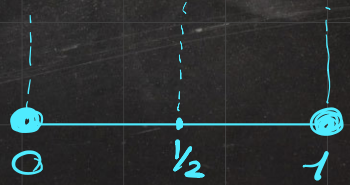
$$u = F_1 x + F_2$$

$$F_2 = \frac{1}{2} [E_1 - F_1]$$

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$$

THIS SEEMS LIKE 1 EQUATION

BUT IT IS NOT !



$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \qquad x \in [0,1]$$

$$u'' + 1 = 0 \qquad 0 \leq x \leq 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$

→ 2-PIECE LINEAR (UNIFORM) APPROXIMATION

$$\frac{1}{2} C_1 E_1 + \frac{1}{2} D_1 F_1 \qquad \forall\, C_1, D_1$$

$$= \frac{1}{8} C_1 + \frac{3}{8} D_1 + \frac{1}{4}[C_1 - D_1]$$

$$u = E_1 x + \cancel{E_2}^{\,0}$$

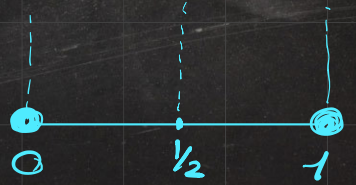
$$u = F_1 x + F_2$$

$$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$$

$$F_2 = \frac{1}{2}[E_1 - F_1]$$

$$C_1 = 1,\ D_1 = 0 \Rightarrow \begin{cases} \frac{1}{2} E_1 - \frac{3}{8} = 0 \\ \frac{1}{2} F_1 - \frac{1}{8} = 0 \end{cases} \Rightarrow \begin{aligned} E_1 &= 3/4 \\ F_1 &= 1/4 \end{aligned}$$

$$C_1 = 0,\ D_1 = 1 \Rightarrow$$



$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \qquad x\in[0,1]$$

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$

→ 2-PIECE LINEAR (UNIFORM) APPROXIMATION

$$\frac{1}{2} C_1 E_1 + \frac{1}{2} D_1 F_1 \qquad \forall\, C_1, D_1$$

$$= \frac{1}{8} C_1 + \frac{3}{8} D_1 + \frac{1}{4}\left[C_1 - D_1\right]$$

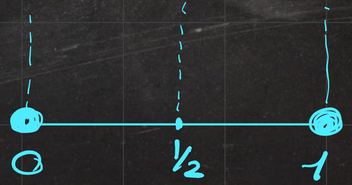
$$u = E_1 x + \cancel{E_2}^{\,0}$$

$$u = F_1 x + F_2$$

$$F_2 = \frac{1}{2}\left[E_1 - F_1\right]$$

w : ARBITRARY, CONTINUOUS, $w|_D = 0$

w : ARBITRARY $\Rightarrow$ C_1 & D_1 : ARBITRARY



$$\int_0^1 \omega' u' \, dx = \int_0^1 \omega \, dx \qquad x \in [0,1]$$

$$u'' + 1 = 0 \qquad 0 \leq x \leq 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$

→ 2-PIECE LINEAR (UNIFORM) APPROXIMATION

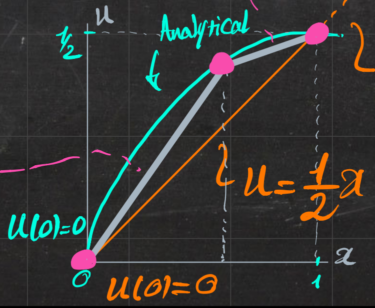
$x \in [0, 0.5] \qquad u = \frac{3}{4}x$

$x \in [0.5, 1] \qquad u = \frac{1}{4}x + \frac{1}{4}$

ω : ARBITRARY, CONTINUOUS, $\omega|_D = 0$

$\frac{1}{4}x + \frac{1}{4}$

$\frac{3}{4}x$



$u(0)=0$

$u'(1) = \frac{1}{2}$

$u = \frac{1}{2}x$ ← Approximate Solution

1-PIECE LINEAR

$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \qquad x \in [0,1]$$

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$

→ 2-PIECE LINEAR (NON-UNIFORM) APPROXIMATION

$x \in [0, 0.6]$ $\quad w = C_1 x + \cancel{C_2}^{\,0}$ $\quad u = E_1 x + \cancel{E_2}^{\,0}$

$x \in [0.6, 1]$ $\quad w = D_1 x + \cancel{D_2}$ $\quad u = F_1 x + \cancel{F_2}$

$0.6\,[C_1 - D_1]$ $\qquad 0.6\,[E_1 - F_1]$

$$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$$

$$\int_0^{0.6} \underbrace{w'}_{C_1}\underbrace{u'}_{E_1}\,dx + \int_{0.6}^{1} \underbrace{w'}_{D_1}\underbrace{u'}_{F_1}\,dx = \int_0^{0.6} \underbrace{w}_{C_1 x}\,dx + \int_{0.6}^{1} \underbrace{w}_{D_1 x + 0.6[C_1 - D_1]}\,dx$$

$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \qquad x \in [0,1]$$

$$u'' + 1 = 0 \qquad 0 \leq x \leq 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$

→ 2-PIECE LINEAR (NON-UNIFORM) APPROXIMATION

$x \in [0, 0.6]$ $\quad w = C_1 x + C_2 \quad u = E_1 x + E_2$

$x \in [0.6, 1]$ $\quad w = D_1 x + D_2 \quad u = F_1 x + F_2$

$C_2 = 0$, $E_2 = 0$

$D_2 = 0.6[C_1 - D_1]$ $\qquad F_2 = 0.6[E_1 - F_1]$

w : ARBITRARY, CONTINUOUS, $w|_D = 0$

$$\int_0^{0.6} C_1 E_1\,dx + \int_{0.6}^1 D_1 F_1\,dx = \int_0^{0.6} C_1 x\,dx + \int_{0.6}^1 [D_1 x + 0.6[C_1 - D_1]]\,dx$$

...

$$C_1[0.6E_1 - 0.42] + D_1[0.4F_1 - 0.08] = 0$$

$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \qquad x\in[0,1]$$

$$u'' + 1 = 0 \qquad 0 \leq x \leq 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$

→ 2-PIECE LINEAR (NON-UNIFORM) APPROXIMATION

$$x \in [0, 0.6] \qquad u = E_1 x + \cancel{E_2} \quad 0$$

$$x \in [0.6, 1] \qquad u = F_1 x + F_2$$

$$0.6\,[E_1 - F_1]$$

$$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$$

$$C_1\,[0.6 E_1 - 0.42] \qquad \checkmark C_1, D_1$$

$$+\, D_1\,[0.4 F_1 - 0.08] = 0$$

$$\Rightarrow E_1 = 0.7,\; F_1 = 0.2 \;\Rightarrow\; \begin{cases} u = 0.7x & 0 \leq x \leq 0.6 \\ u = 0.2x + 0.3 & 0.6 \leq x \leq 1 \end{cases}$$

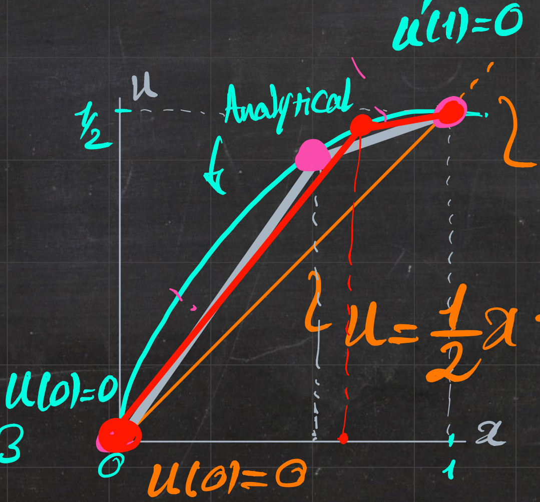
$$\int_0^1 w' u' \, dx = \int_0^1 w \, dx \qquad x \in [0,1]$$

$$u'' + 1 = 0 \qquad 0 \leq x \leq 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$

→ 2-PIECE LINEAR (NON-UNIFORM) APPROXIMATION



$$\begin{cases} u = 0.7\,x \\ u = \underline{0.2x} + 0.3 \end{cases}$$

$u'(1) = \frac{1}{2}$

$u = \frac{1}{2}x$ ← Approximate Solution

1-PIECE LINEAR

$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \qquad x \in [0,1]$$

$$u'' + 1 = 0 \qquad 0 \leq x \leq 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$

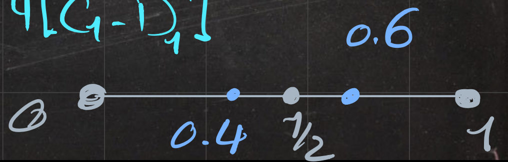
→ 2-PIECE LINEAR (NON-UNIFORM) APPROXIMATION

$x \in [0, 0.4]$ $\qquad w = C_1 x + \overset{0}{C_2} \qquad u = E_1 x + \overset{0}{E_2}$

$x \in [0.4, 1]$ $\qquad w = D_1 x + \underset{0.4[C_1 - D_1]}{D_2} \qquad u = F_1 x + \underset{0.4[E_1 - F_1]}{F_2}$

$w :$ ARBITRARY, CONTINUOUS, $w|_D = 0$

$$\int_0^{0.4} \underset{C_1}{w'}\,\underset{E_1}{u'}\,dx + \int_{0.4}^1 \underset{D_1}{w'}\,\underset{F_1}{u'}\,dx = \int_0^{0.4} \underset{C_1 x}{w}\,dx + \int_{0.4}^1 \underset{D_1 x + 0.4[C_1 - D_1]}{w}\,dx$$



$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \qquad x \in [0,1]$$

$$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$$

$$D: \; u(0) = 0$$

$$N: \; u'(1) = 0$$

prescribed

→ 2-PIECE LINEAR (NON-UNIFORM) APPROXIMATION

$$x \in [0, 0.4] \qquad w = C_1 x + C_2 \qquad u = E_1 x + E_2$$

$$C_2 = 0 \qquad E_2 = 0$$

$$x \in [0.4, 1] \qquad w = D_1 x + D_2 \qquad u = F_1 x + F_2$$

$$D_2 = 0.4[C_1 - D_1] \qquad F_2 = 0.4[E_1 - F_1]$$

$$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$$

$$\int_0^{0.4} C_1 E_1\,dx + \int_{0.4}^{1} D_1 F_1\,dx = \int_0^{0.4} C_1 x\,dx + \int_{0.4}^{1} [D_1 x + 0.4[C_1 - D_1]]\,dx$$

...

$$C_1\,[0.4E_1 - 0.32] + D_1\,[0.6F_1 - 0.18] = 0$$

$$\int_0^1 \omega' u' dx = \int_0^1 \omega \, dx \qquad x \in [0,1]$$

$\rightarrow$ 2-PIECE LINEAR (NON-UNIFORM) APPROXIMATION

$x \in [0, 0.4] \quad u = E_1 x + E_2$

$x \in [0.4, 1] \quad u = F_1 x + F_2$

$0.4[E_1 - F_1]$

$C_1 [0.4 E_1 - 0.32] \quad \checkmark C_1, D_1$

$+ D_1 [0.6 F_1 - 0.18] = 0$

$\Rightarrow E_1 = 0.8, \; F_1 = 0.3 \quad \Rightarrow \begin{cases} u = 0.8x & 0 \leqslant x \leqslant 0.4 \\ u = 0.3x + 0.2 & 0.4 \leqslant x \leqslant 1 \end{cases}$

$u'' + 1 = 0 \qquad 0 \leqslant x \leqslant 1$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$

$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$

$$\int_0^1 w' u' \, dx = \int_0^1 w \, dx \qquad x \in [0,1]$$

$$u'' + 1 = 0 \qquad 0 \leq x \leq 1$$

D: $u(0) = 0$ ← prescribed

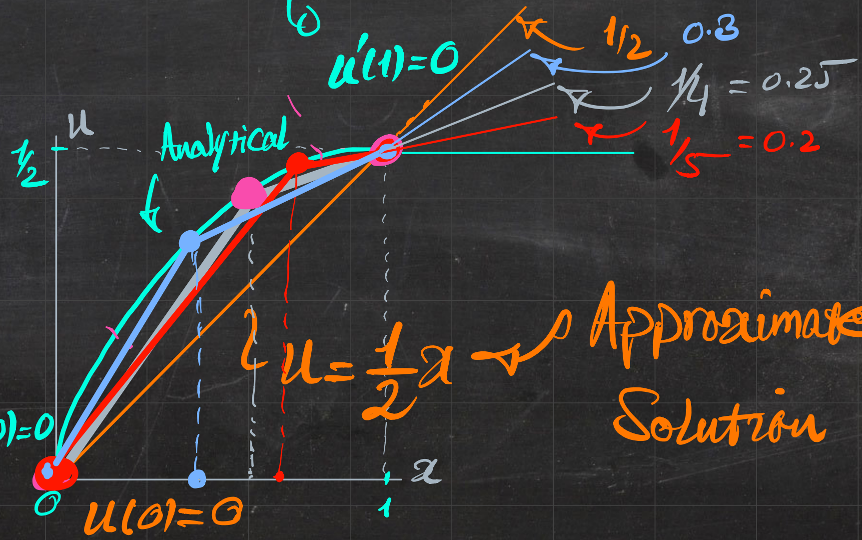
N: $u'(1) = 0$

→ 2-PIECE LINEAR (NON-UNIFORM) APPROXIMATION

ANALYTICAL

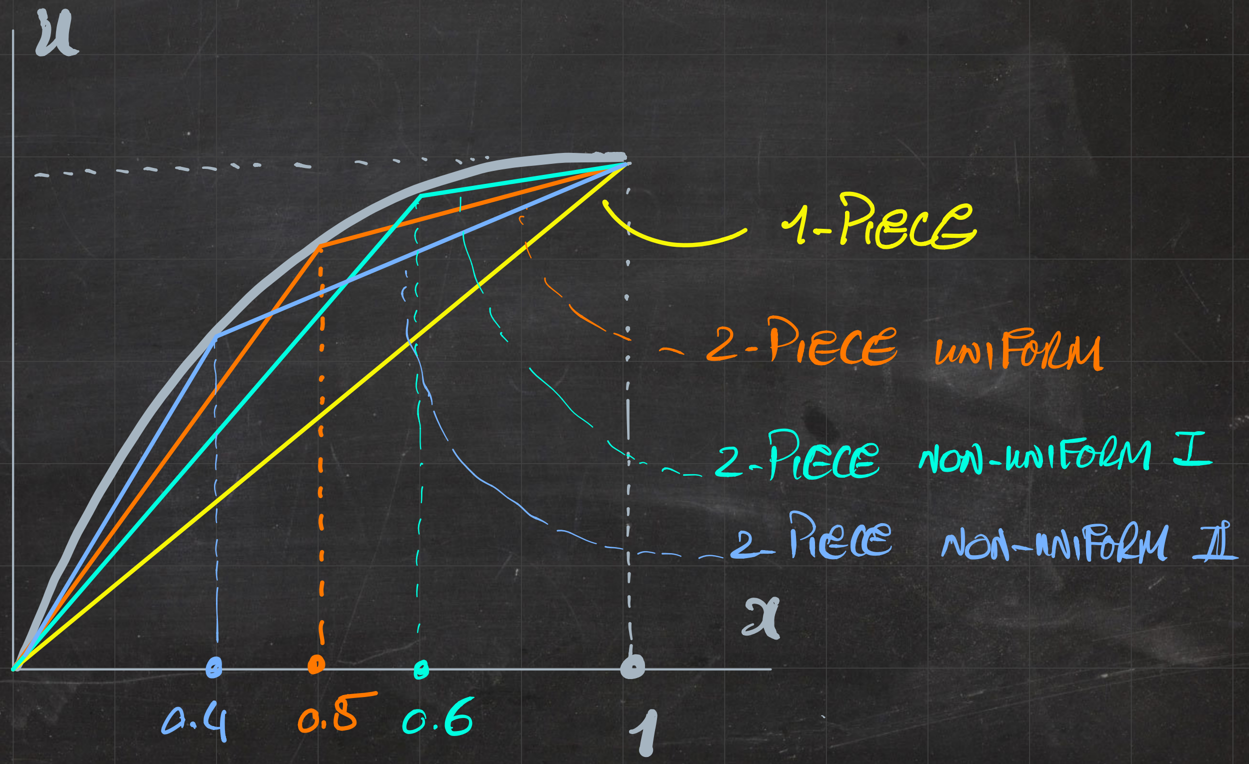
$$\begin{cases} u = 0.8x \\ u = 0.3x + 0.2 \end{cases}$$

$$\begin{cases} u = 0.7x \\ u = 0.2x + 0.3 \end{cases}$$



1-PIECE LINEAR

Approximate Solution


u
1-PIECE
2-PIECE UNIFORM
2-PIECE NON-UNIFORM I
2-PIECE NON-UNIFORM II
x
0.4
0.5
0.6
1

$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \qquad x \in [0,1]$$

$$u'' + 1 = 0 \qquad 0 \leq x \leq 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$

→ 2-PIECE LINEAR (GENERIC) APPROXIMATION

$x \in [0, a]$ $\quad w = C_1 x + \cancel{C_2}^{\,0} \quad u = E_1 x + \cancel{E_2}^{\,0}$

$x \in [a, 1]$ $\quad w = D_1 x + \cancel{D_2} \quad u = F_1 x + \cancel{F_2}$

$\quad a[C_1 - D_1] \qquad a[E_1 - F_1]$

$$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$$

$$\int_0^a w'u'\,dx + \int_a^1 w'u'\,dx = \int_0^a w\,dx + \int_a^1 w\,dx$$

$C_1 \quad E_1 \qquad D_1 \quad F_1 \qquad C_1 x \qquad D_1 x + a[C_1 - D_1]$

$0 \leq a \leq 1$

0.6

0 — 0.4 — 1/2 — a — 1

$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \qquad x\in[0,1]$$

$$u'' + 1 = 0 \qquad 0 \leq x \leq 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$

→ 2-PIECE LINEAR (GENERIC) APPROXIMATION

$x\in[0,a]$ $\quad w = C_1 x + \overset{0}{C_2}$ $\quad u = E_1 x + \overset{0}{E_2}$

$x\in[a,1]$ $\quad w = D_1 x + \underset{a[C_1-D_1]}{D_2}$ $\quad u = F_1 x + \underset{a[E_1-F_1]}{F_2}$

$$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$$

$$a_1 C_1 E_1 + [1-a]\, D_1 F_1 = \cdots + \cdots$$

$$= a\, C_1 \left[1 - \tfrac{1}{2}a\right] + D_1 [1-a]\left[\tfrac{1}{2} - \tfrac{1}{2}a\right]$$

$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \qquad x\in[0,1]$$

$$u'' + 1 = 0 \qquad 0 \leq x \leq 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$

→ 2-PIECE LINEAR (GENERIC) APPROXIMATION

$x \in [0, a]$ $\quad w = C_1 x + C_2 \qquad u = E_1 x + E_2 \qquad (C_2 = 0,\ E_2 = 0)$

$x \in [a, 1]$ $\quad w = D_1 x + D_2 \qquad u = F_1 x + F_2$

$a[C_1 - D_1] \qquad a[E_1 - F_1]$

w : $\begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$

$$a_1 C_1 E_1 + [1-a] D_1 F_1$$

$$= a C_1 \left[1 - \tfrac{1}{2} a\right] + D_1 [1-a]\left[\tfrac{1}{2} - \tfrac{1}{2} a\right] \qquad \forall C_1, D_1$$

$$\Rightarrow E_1 = 1 - \tfrac{1}{2} a \;, \quad F_1 = \tfrac{1}{2} - \tfrac{1}{2} a$$

$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx \qquad x\in[0,1]$$

$$u'' + 1 = 0 \qquad 0\leqslant x\leqslant 1$$

D: $u(0) = 0$ ← prescribed

N: $u'(1) = 0$

→ 2-PIECE LINEAR (GENERIC) APPROXIMATION

$$x\in[0,a] \Rightarrow u = \left[1-\tfrac{1}{2}a\right]x$$

$$x\in[a,1] \Rightarrow u = \left[\tfrac{1}{2}-\tfrac{1}{2}a\right]x + \tfrac{1}{2}a$$

$$w: \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \end{cases}$$

$a = 0.5$

$$\begin{cases} u = 0.75x \\ u = 0.25x + 0.25 \end{cases}$$

$a = 0.6$

$$\begin{cases} u = 0.7x \\ u = 0.2x + 0.3 \end{cases}$$

$a = 0.4$

$$\begin{cases} u = 0.8x \\ u = 0.3x + 2 \end{cases}$$