FINITE ELEMENT METHOD

4

FINITE ELEMENT METHOD

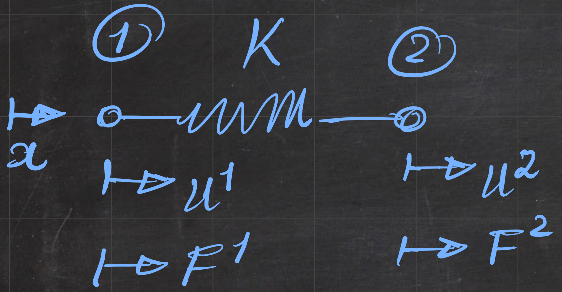
→ UNDERSTAND KEY INGREDIENTS OF FEM USING SPRINGS

→ EXTENDED NODE LIST (ENL) → algorithmic ← MATLAB/Python

- MATRIX
- MEMORY
- NOT OBJECT ORIENTED

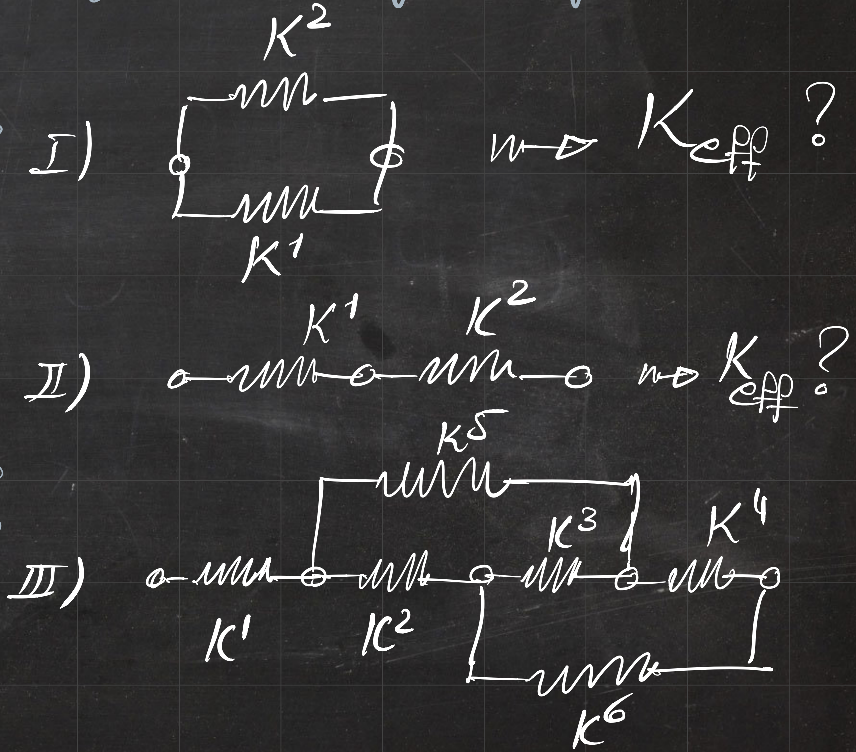
→ STATIC CONDENSATION & REDUCED SYSTEM

Understanding key ingredients of FEM using springs:

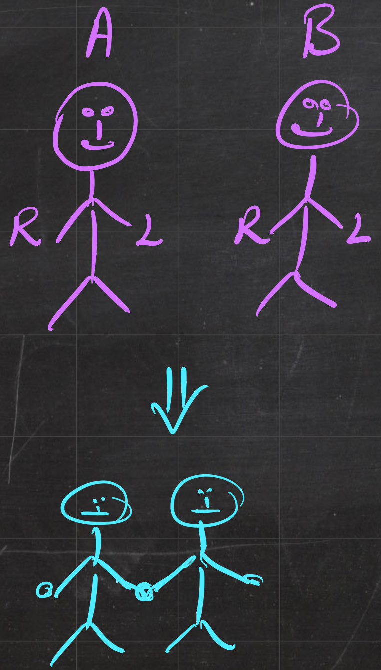


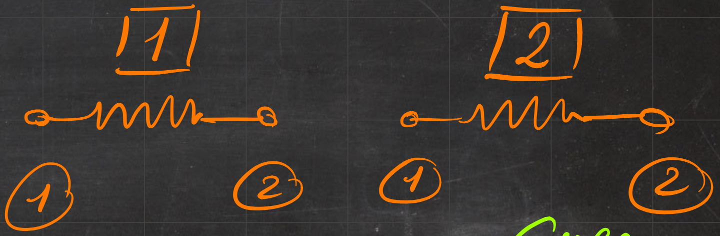
$$\begin{bmatrix} F^1 \\ F^2 \end{bmatrix} = \begin{bmatrix} K & -K \\ -K & K \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix}$$

$$\mathbb{F} = \mathbb{K} \cdot \mathbb{U}$$



TOWARDS AN ALGORITHMIC APPROACH TO ASSEMBLY:





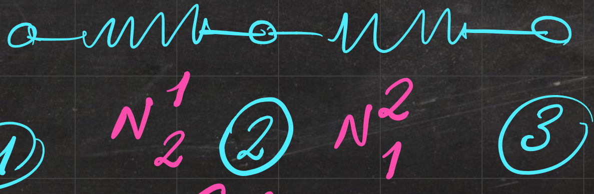
GLOBAL ELEMENT

Superscript: GLOBAL

subscript: LOCAL

GLOBAL NODE

$$N^2 = N_2^1 = N_1^2$$



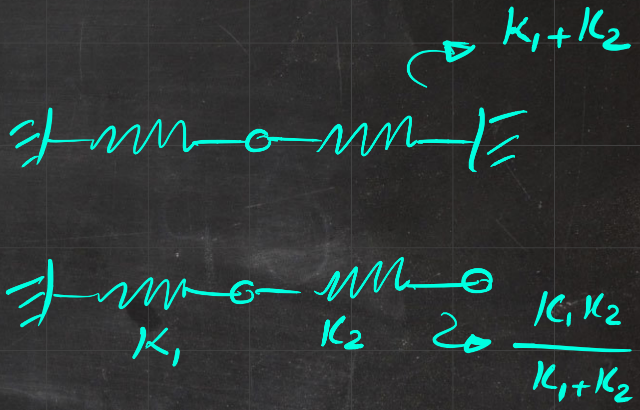
N_2^1 N_1^2

GLOBAL

TOWARDS AN ALGORITHMIC APPROACH TO ASSEMBLY:

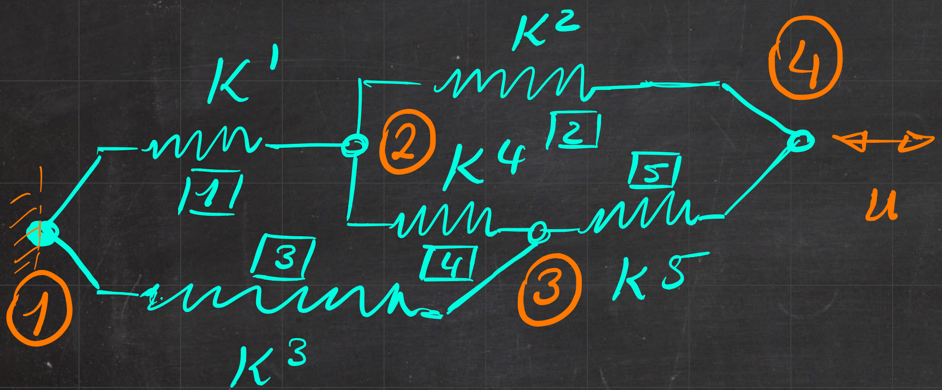
NODE 2 OF ELEMENT 2

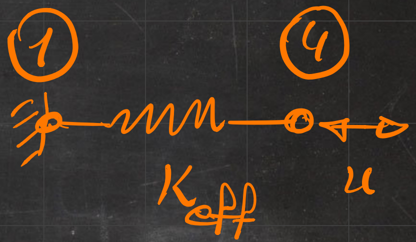
NODE 1 OF ELEMENT 3



$$\Rightarrow K^3 = K^2_2 + K^3_1$$

TOWARDS AN ALGORITHMIC APPROACH TO ASSEMBLY:





ELEMENT 1

$$\mathbb{K}^1 = \begin{bmatrix} K^1 & -K^1 \\ -K^1 & K^1 \end{bmatrix}$$

BOLD

ELEMENT 2

$$\mathbb{K}^2 = \begin{bmatrix} K^2 & -K^2 \\ -K^2 & K^2 \end{bmatrix}$$

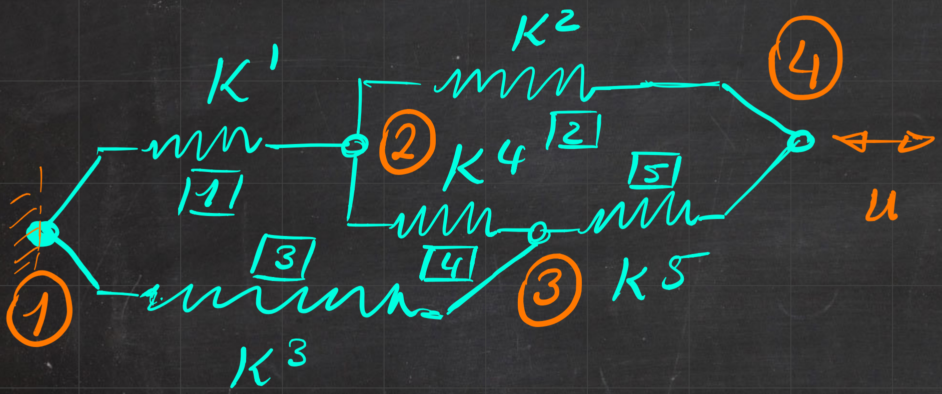
ELEMENT 3

$$\mathbb{K} = \begin{bmatrix} K^3 & -K^3 \\ -K^3 & K^3 \end{bmatrix}$$

ELEMENT

$$\mathbb{K} = \begin{bmatrix} K & -K \\ -K & K \end{bmatrix}$$

TOWARDS AN ALGORITHMIC APPROACH TO ASSEMBLY:



ELEMENT 5

$$\mathbb{K}^5 = \begin{bmatrix} k^5 & -k^5 \\ -k^5 & k^5 \end{bmatrix}$$

ELEMENT 1

$$\mathbb{K}^1 = \begin{bmatrix} k^1 & -k^1 \\ -k^1 & k^1 \end{bmatrix}$$

BOLD

ELEMENT 2

$$\mathbb{K}^2 = \begin{bmatrix} k^2 & -k^2 \\ -k^2 & k^2 \end{bmatrix}$$

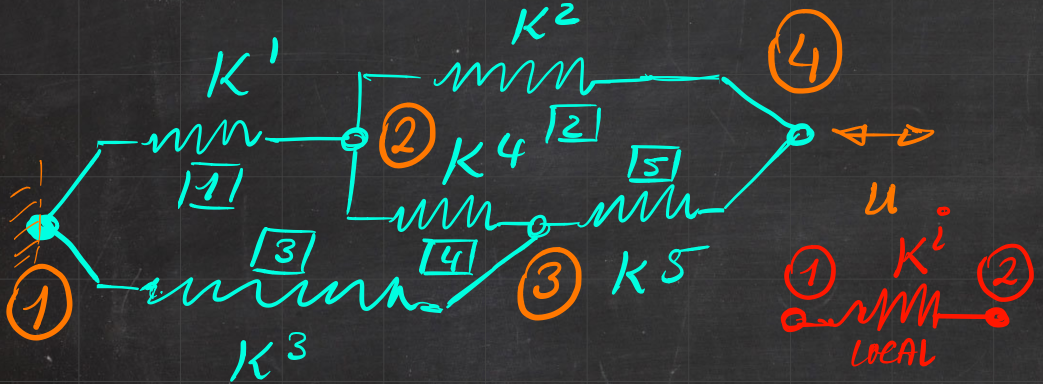
ELEMENT 3

$$\mathbb{K}^3 = \begin{bmatrix} k^3 & -k^3 \\ -k^3 & k^3 \end{bmatrix}$$

ELEMENT 4

$$\mathbb{K}^4 = \begin{bmatrix} k^4 & -k^4 \\ -k^4 & k^4 \end{bmatrix}$$

TOWARDS AN ALGORITHMIC APPROACH TO ASSEMBLY:



$$\mathbb{K}^5 = \begin{array}{c} \\ \\ \end{array}\begin{array}{cc} 3 & 4 \end{array}$$
$$\mathbb{K}^5 = \begin{bmatrix} K^5 & -K^5 \\ -K^5 & K^5 \end{bmatrix}\begin{array}{c} 3 \\ 4 \end{array}$$

$$\mathbb{K}^1 = \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} K^1 & -K^1 \\ -K^1 & K^1 \end{bmatrix} \end{array}\begin{array}{c} 1 \\ 2 \end{array} \quad \mathbb{K}^2 = \begin{array}{c} \begin{array}{cc} 2 & 4 \end{array} \\ \begin{bmatrix} K^2 & -K^2 \\ -K^2 & K^2 \end{bmatrix} \end{array}\begin{array}{c} 2 \\ 4 \end{array} \quad \mathbb{K}^3 = \begin{array}{c} \begin{array}{cc} 1 & 3 \end{array} \\ \begin{bmatrix} K^3 & -K^3 \\ -K^3 & K^3 \end{bmatrix} \end{array}\begin{array}{c} 1 \\ 3 \end{array} \quad \mathbb{K}^4 = \begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} K^4 & -K^4 \\ -K^4 & K^4 \end{bmatrix} \end{array}\begin{array}{c} 2 \\ 3 \end{array}$$

$$\mathbb{K}^4 = \begin{array}{cc} & \begin{array}{cc} ② & ③ \end{array} \\ & \begin{bmatrix} K^4 & -K^4 \\ -K^4 & K^4 \end{bmatrix} \begin{array}{c} ② \\ ③ \end{array} \end{array}$$

$$\mathbb{K}^5 = \begin{array}{cc} & \begin{array}{cc} ③ & ④ \end{array} \\ & \begin{bmatrix} K^5 & -K^5 \\ -K^5 & K^5 \end{bmatrix} \begin{array}{c} ③ \\ ④ \end{array} \end{array}$$

GLOBAL

$$\mathbb{K} = \begin{array}{c} \begin{array}{cccc} ① & ② & ③ & ④ \end{array} \\ \begin{bmatrix} K^1 + K^3 & -K^1 & -K^3 & 0 \\ -K^1 & K^1 + K^2 + K^4 & -K^4 & -K^2 \\ -K^3 & -K^4 & K^3 + K^4 + K^5 & -K^5 \\ 0 & -K^2 & -K^5 & K^2 + K^5 \end{bmatrix} \begin{array}{c} ① \\ ② \\ ③ \\ ④ \end{array} \end{array}$$

$$\mathbb{K}^1 = \begin{array}{cc} & \begin{array}{cc} ① & ② \end{array} \\ & \begin{bmatrix} K^1 & -K^1 \\ -K^1 & K^1 \end{bmatrix} \begin{array}{c} ① \\ ② \end{array} \end{array}$$

$$\mathbb{K}^2 = \begin{array}{cc} & \begin{array}{cc} ② & ④ \end{array} \\ & \begin{bmatrix} K^2 & -K^2 \\ -K^2 & K^2 \end{bmatrix} \begin{array}{c} ② \\ ④ \end{array} \end{array}$$

$$\mathbb{K}^3 = \begin{array}{cc} & \begin{array}{cc} ① & ③ \end{array} \\ & \begin{bmatrix} K^3 & -K^3 \\ -K^3 & K^3 \end{bmatrix} \begin{array}{c} ① \\ ③ \end{array} \end{array}$$

$\mathrm{DET}\ \mathbb{K}^{\mathrm{GLOBAL}} = 0 \qquad \mathbb{K}^{\mathrm{GLOBAL}} : \mathrm{SYM}$

$$\begin{bmatrix} F^1 \\ F^2 \\ F^3 \\ F^4 \end{bmatrix} = \begin{bmatrix} K^{11} & K^{12} & K^{13} & K^{14} \\ K^{21} & K^{22} & K^{23} & K^{24} \\ K^{31} & K^{32} & K^{33} & K^{34} \\ K^{41} & K^{42} & K^{43} & K^{44} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \\ u^4 \end{bmatrix}$$

GLOBAL GLOBAL GLOBAL

$$\mathbb{F} = \mathbb{K} \cdot \mathbb{u}$$

NoN x 1   NoN x NoN   NoN x 1

[PD x NoN] x 1

NoN ≡ NoN x PD

1D   Problem Dimension

⇒ 4 EQ. & 4 Unknowns ⟵ BCs ?

NEUMANN BCs. ⟸ FORCE BASED ⟵ BCs ⟶ DISPLACEMENT BASED ⇒ DIRICHLET BCs.

$$
\begin{bmatrix} F^1 \\ F^2 \\ F^3 \\ F^4 \end{bmatrix} = \begin{bmatrix} K^{11} & K^{12} & K^{13} & K^{14} \\ K^{21} & K^{22} & K^{23} & K^{24} \\ K^{31} & K^{32} & K^{33} & K^{34} \\ K^{41} & K^{42} & K^{43} & K^{44} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \\ u^4 \end{bmatrix}
$$

$\to u^1 = 0$

$\to u^4 = u$

	Homogeneous	Non Homogeneous
Dirichlet $\to$ Displacement	$= 0$	$\neq 0$
Neumann $\to$ Force	$= 0$	$\neq 0$

$$
= \begin{bmatrix} K^{11} \\ K^{21} \\ K^{31} \\ K^{41} \end{bmatrix} u^1 + \begin{bmatrix} K^{12} \\ K^{22} \\ K^{32} \\ K^{42} \end{bmatrix} u^2 + \begin{bmatrix} K^{13} \\ K^{23} \\ K^{33} \\ K^{43} \end{bmatrix} u^3 + \begin{bmatrix} K^{14} \\ K^{24} \\ K^{34} \\ K^{44} \end{bmatrix} u^4
$$

$$\begin{bmatrix} F^1 \\ F^2 \\ F^3 \\ F^4 \end{bmatrix} = \begin{bmatrix} K^{11} & K^{12} & K^{13} & K^{14} \\ K^{21} & K^{22} & K^{23} & K^{24} \\ K^{31} & K^{32} & K^{33} & K^{34} \\ K^{41} & K^{42} & K^{43} & K^{44} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \\ u^4 \end{bmatrix}$$

$u^1 = 0$

$u^4 = u$

$$F^i \leftarrow K^{ij} \, u^j$$

K^{ij} How much force F^i is related to displacement u^j

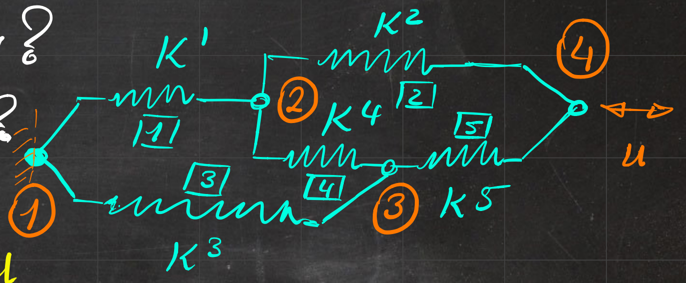
$$= \begin{bmatrix} K^{11} \\ K^{21} \\ K^{31} \\ K^{41} \end{bmatrix} u^1 + \begin{bmatrix} K^{12} \\ K^{22} \\ K^{32} \\ K^{42} \end{bmatrix} u^2 + \begin{bmatrix} K^{13} \\ K^{23} \\ K^{33} \\ K^{43} \end{bmatrix} u^3 + \begin{bmatrix} K^{14} \\ K^{24} \\ K^{34} \\ K^{44} \end{bmatrix} u^4$$

4 EQN. & 4 Unknowns

$$\begin{array}{l} \\ \\ \text{NEUMANN} \\ \text{NEUMANN} \end{array} \begin{bmatrix} \cancel{F^1} \\ \cancel{F^2} \\ \cancel{F^3} \\ \cancel{F^4} \end{bmatrix} \begin{array}{l} ? \\ 0 \\ 0 \\ ? \end{array} \begin{bmatrix} K^{11} & K^{12} & K^{13} & K^{14} \\ K^{21} & K^{22} & K^{23} & K^{24} \\ K^{31} & K^{32} & K^{33} & K^{34} \\ K^{41} & K^{42} & K^{43} & K^{44} \end{bmatrix} \begin{bmatrix} \cancel{u^1} \\ \cancel{u^2} \\ \cancel{u^3} \\ \cancel{u^4} \end{bmatrix} \begin{array}{l} 0 \\ ? \\ ? \\ u \end{array}$$

DIRICHLET (Prescribed Displacement)

DIRICHLET (Prescribed Displ.)



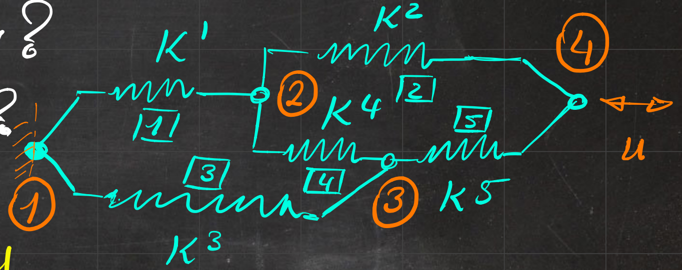
$$\overset{\checkmark}{\mathbb{b}} = \mathbb{A} \, \overset{?}{\mathbb{X}}$$

$$\mathbb{A} \cdot \mathbb{X} = \mathbb{b} \Rightarrow \mathbb{X} = \mathbb{A}^{-1} \cdot \mathbb{b}$$

4 EQN. & 4 Unknowns

$$\begin{array}{rc} & ? \\ & 0 \\ \text{NEUMANN} & 0 \\ \text{NEUMANN} & ? \end{array}\begin{bmatrix} \cancel{F^1} \\ \cancel{F^2} \\ \cancel{F^3} \\ \cancel{F^4} \end{bmatrix} = \begin{bmatrix} K^{11} & K^{12} & K^{13} & K^{14} \\ K^{21} & K^{22} & K^{23} & K^{24} \\ K^{31} & K^{32} & K^{33} & K^{34} \\ K^{41} & K^{42} & K^{43} & K^{44} \end{bmatrix} \begin{bmatrix} \cancel{u^1} \\ \cancel{u^2} \\ \cancel{u^3} \\ \cancel{u^4} \end{bmatrix}\begin{array}{l} 0 \\ ? \\ ? \\ u \end{array}$$

DIRICHLET (Prescribed Displacement)



DIRICHLET (Prescribed Displ.)

$$\begin{bmatrix} F^P \\ \hline F^u \end{bmatrix} = \begin{bmatrix} K^{Pu} & K^{PP} \\ \hline K^{uu} & K^{uP} \end{bmatrix} \begin{bmatrix} u^u \\ \hline u^P \end{bmatrix}$$

unknown

prescribed

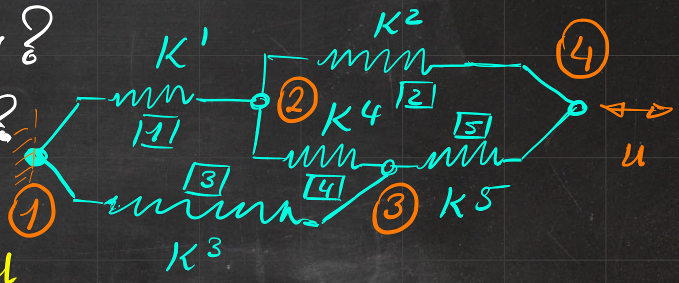
4 EQN. & 4 Unknowns

$$\begin{array}{c} ? \\ 0 \\ 0 \\ ? \end{array} \begin{bmatrix} \cancel{F^1} \\ \cancel{F^2} \\ \cancel{F^3} \\ \cancel{F^4} \end{bmatrix} \begin{array}{c} \\ \\ = \\ \\ \end{array} \begin{bmatrix} K^{11} & K^{12} & K^{13} & K^{14} \\ K^{21} & K^{22} & K^{23} & K^{24} \\ K^{31} & K^{32} & K^{33} & K^{34} \\ K^{41} & K^{42} & K^{43} & K^{44} \end{bmatrix} \begin{bmatrix} \cancel{u^1} \\ \cancel{u^2} \\ \cancel{u^3} \\ \cancel{u^4} \end{bmatrix} \begin{array}{c} 0 \\ ? \\ ? \\ u \end{array}$$

NEUMANN

NEUMANN

DIRICHLET (Prescribed Displacement)



DIRICHLET (Prescribed Displ.)

$$\boxed{\begin{array}{c} F^p \\ \hline F^u \end{array}} = \boxed{\begin{array}{c|c} K^{pu} & K^{pp} \\ \hline K^{uu} & K^{up} \end{array}} \boxed{\begin{array}{c} u^u \\ \hline u^p \end{array}}$$

NEUMANN

FREE NODES $\Rightarrow$ DoF $\leftarrow$ DEGREES OF FREEDOM

CONSTRAINED NODES $\Rightarrow$ DoC $\leftarrow$ DEGREES OF CONSTRAINT

DIRICHLET

4 EQN. & 4 Unknowns

$$\begin{bmatrix} \cancel{F^1} \\ \cancel{F^2} \\ \cancel{F^3} \\ \cancel{F^4} \end{bmatrix} = \begin{bmatrix} K^{11} & K^{12} & K^{13} & K^{14} \\ K^{21} & K^{22} & K^{23} & K^{24} \\ K^{31} & K^{32} & K^{33} & K^{34} \\ K^{41} & K^{42} & K^{43} & K^{44} \end{bmatrix} \begin{bmatrix} \cancel{u^1} \\ \cancel{u^2} \\ \cancel{u^3} \\ \cancel{u^4} \end{bmatrix}$$

F^1: ?, F^2: 0, F^3: 0 (NEUMANN), F^4: ? (NEUMANN)

u^1: 0, u^2: ?, u^3: ?, u^4: u

$$[F^p] = [K^{pu}][u^u] + [K^{pp}][u^p]$$

$$\Downarrow$$

$$\underbrace{[K^{pu}]}_{A}\underbrace{[u^u]}_{x} = \underbrace{[F^p] - [K^{pp}][u^p]}_{b}$$

$$[x] = [A]^{-1}[b] \Leftarrow A \cdot x = b$$

$$\begin{bmatrix} F^p \\ \hline F^u \end{bmatrix} = \left[\begin{array}{c|c} K^{pu} & K^{pp} \\ \hline K^{uu} & K^{up} \end{array}\right] \begin{bmatrix} u^u \\ \hline u^p \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} \text{DoF} \\ \hline \text{DoC} \end{bmatrix}}_{\text{FORCE}} = \underbrace{\left[\begin{array}{c|c} \text{DoF} \times \text{DoF} & \text{DoF} \times \text{DoC} \\ \hline \text{DoC} \times \text{DoF} & \text{DoC} \times \text{DoC} \end{array}\right]}_{\text{STIFFNESS}} \underbrace{\begin{bmatrix} \text{DoF} \\ \hline \text{DoC} \end{bmatrix}}_{\text{DISPLACEMENT}}$$

4 EQN. & 4 Unknowns

$$
\begin{array}{l}
\\ \\ \text{NEUMANN} \\ \text{NEUMANN}
\end{array}
\begin{bmatrix} \cancel{F^1} \\ \cancel{F^2} \\ \cancel{F^3} \\ \cancel{F^4} \end{bmatrix}
\begin{array}{l} ? \\ 0 \\ 0 \\ ? \end{array}
=
\begin{bmatrix}
K^{11} & K^{12} & K^{13} & K^{14} \\
K^{21} & K^{22} & K^{23} & K^{24} \\
K^{31} & K^{32} & K^{33} & K^{34} \\
K^{41} & K^{42} & K^{43} & K^{44}
\end{bmatrix}
\begin{bmatrix} \cancel{u^1} \\ \cancel{u^2} \\ \cancel{u^3} \\ \cancel{u^4} \end{bmatrix}
\begin{array}{l} 0 \\ ? \\ ? \\ u \end{array}
$$

$$[F^P] = [K^{Pu}][u^u] + [K^{PP}][u^P]$$

$$\Downarrow$$

$$\underbrace{[K^{Pu}]}_{\text{REDUCED STIFFNESS}}[u^u] = [F^P] - [K^{PP}][u^P]$$

$$\Rightarrow [u^u] = [K^{Pu}]^{-1} \cdot \underbrace{\{[F^P] - [K^{PP}][u^P]\}}$$

$$
\begin{bmatrix} F^P \\ \hline F^u \end{bmatrix}
=
\left[\begin{array}{c|c} K^{Pu} & K^{PP} \\ \hline K^{uu} & K^{uP} \end{array}\right]
\begin{bmatrix} u^u \\ \hline u^P \end{bmatrix}
$$

REDUCED SYSTEM

$$\underset{\text{DOF} \times \text{DOF}}{A} \cdot x = b$$

4 EQN. & 4 Unknowns

$$\begin{array}{c} ? \\ 0 \\ 0 \\ ? \end{array}\quad \begin{bmatrix} \cancel{F^1} \\ \cancel{F^2} \\ \cancel{F^3} \\ \cancel{F^4} \end{bmatrix} = \begin{bmatrix} K^{11} & K^{12} & K^{13} & K^{14} \\ K^{21} & K^{22} & K^{23} & K^{24} \\ K^{31} & K^{32} & K^{33} & K^{34} \\ K^{41} & K^{42} & K^{43} & K^{44} \end{bmatrix} \begin{bmatrix} \cancel{u^1} \\ \cancel{u^2} \\ \cancel{u^3} \\ \cancel{u^4} \end{bmatrix} \quad \begin{array}{c} 0 \\ ? \\ ? \\ u \end{array}$$

NEUMANN

NEUMANN

$$[F^P] = [K^{Pu}][u^u] + [K^{PP}][u^P]$$

$$\Downarrow$$

$$\underbrace{[K^{Pu}][u^u] = [F^P] - [K^{PP}][u^P]}_{\text{REDUCED SYSTEM}}$$

$$\Rightarrow [u^u] = [K^{Pu}]^{-1} \cdot \left\{ [F^P] - [K^{PP}][u^P] \right\}$$

$$\begin{bmatrix} F^P \\ \hline F^u \end{bmatrix} = \left[\begin{array}{c|c} K^{Pu} & K^{PP} \\ \hline K^{uu} & K^{uP} \end{array}\right] \begin{bmatrix} u^u \\ \hline u^P \end{bmatrix}$$

$$\Rightarrow [F^u] = [K^{uu}][u^u] + [K^{uP}][u^P]$$

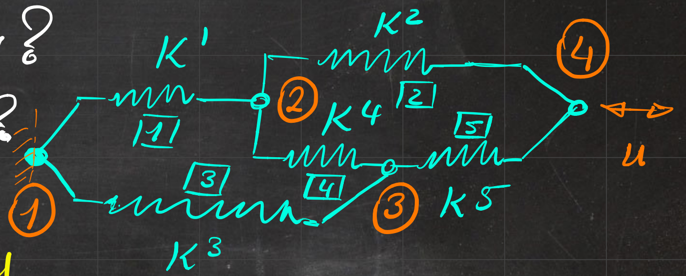
STATIC CONDENSATION ✓

4 EQN. & 4 Unknowns

NEUMANN

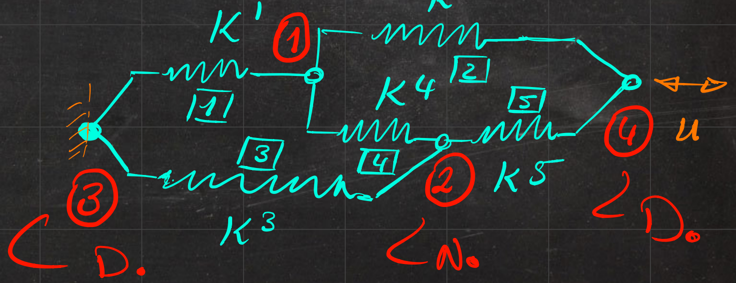
NEUMANN

$$\begin{bmatrix} \cancel{F^1} \to ? \\ \cancel{F^2} \to 0 \\ \cancel{F^3} \to 0 \\ \cancel{F^4} \to ? \end{bmatrix} = \begin{bmatrix} K^{11} & K^{12} & K^{13} & K^{14} \\ K^{21} & K^{22} & K^{23} & K^{24} \\ K^{31} & K^{32} & K^{33} & K^{34} \\ K^{41} & K^{42} & K^{43} & K^{44} \end{bmatrix} \begin{bmatrix} \cancel{u^1} \to 0 \\ \cancel{u^2} \to ? \\ \cancel{u^3} \to ? \\ \cancel{u^4} \to u \end{bmatrix}$$



NUMBERING CAREFULLY FROM THE ONSET

N.

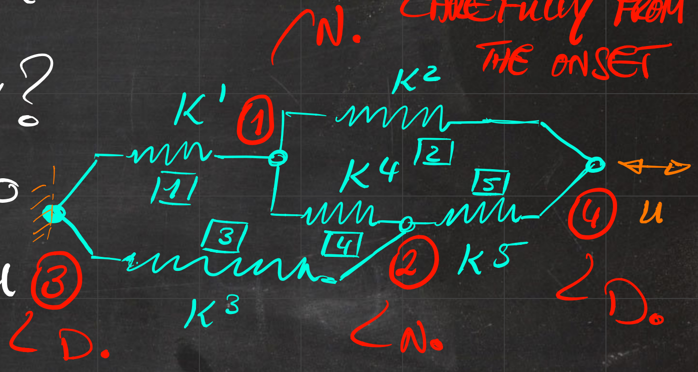


$$\begin{bmatrix} F^p \\ \hline F^u \end{bmatrix} = \begin{bmatrix} K^{pu} & K^{pp} \\ \hline K^{uu} & K^{up} \end{bmatrix} \begin{bmatrix} u^u \\ \hline u^p \end{bmatrix} \begin{matrix} \} \text{N.} \\ \} \text{D.} \end{matrix}$$

4 EQN. & 4 Unknowns

$$\begin{bmatrix} \cancelto{0}{F^1} \\ \cancelto{0}{F^2} \\ \cancelto{?}{F^3} \\ \cancelto{?}{F^4} \end{bmatrix} = \left[\begin{array}{cc|cc} K^{11} & K^{12} & K^{13} & K^{14} \\ K^{21} & K^{22} & K^{23} & K^{24} \\ \hline K^{31} & K^{32} & K^{33} & K^{34} \\ K^{41} & K^{42} & K^{43} & K^{44} \end{array}\right] \begin{bmatrix} \cancelto{?}{u^1} \\ \cancelto{?}{u^2} \\ \cancelto{0}{u^3} \\ \cancel{u^4} \end{bmatrix}$$

NUMBERING CAREFULLY FROM THE ONSET



$$\Downarrow$$

$$\left[\begin{array}{c} F^P \\ \hline F^u \end{array}\right] = \left[\begin{array}{c|c} K^{Pu} & K^{PP} \\ \hline K^{uu} & K^{uP} \end{array}\right] \left[\begin{array}{c} u^u \\ \hline u^P \end{array}\right] \begin{array}{l} \}\,\text{N.} \\ \}\,\text{D.} \end{array}$$

THE PROBLEM IS WE DO NOT KNOW A PRIORI HOW MANY DOFs WE HAVE! ✓ Efficiency Matters

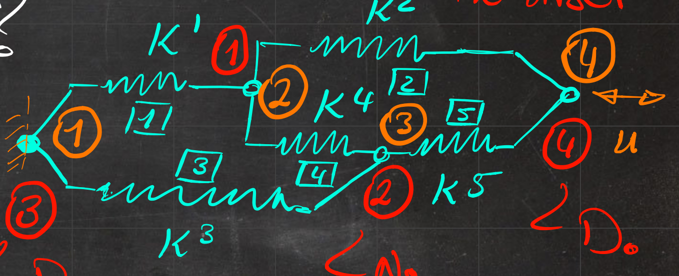
4 EQN. & 4 Unknowns

$$\begin{bmatrix} \cancel{F^1} \to 0 \\ \cancel{F^2} \to 0 \\ \cancel{F^3} \to ? \\ \cancel{F^4} \to ? \end{bmatrix} = \left[\begin{array}{cc|cc} K^{11} & K^{12} & K^{13} & K^{14} \\ K^{21} & K^{22} & K^{23} & K^{24} \\ \hline K^{31} & K^{32} & K^{33} & K^{34} \\ K^{41} & K^{42} & K^{43} & K^{44} \end{array}\right] \begin{bmatrix} \cancel{u^1} \to ? \\ \cancel{u^2} \to ? \\ \cancel{u^3} \to 0 \\ \cancel{u^4} \to ? \end{bmatrix}$$

$$\Downarrow$$

$$\left[\begin{array}{c} F^P \\ \hline F^u \end{array}\right] = \left[\begin{array}{c|c} K^{Pu} & K^{PP} \\ \hline K^{uu} & K^{uP} \end{array}\right] \left[\begin{array}{c} u^u \\ \hline u^P \end{array}\right]$$

NUMBERING CAREFULLY FROM THE ONSET



	NL.		DEGREE	ENL
N.	(x,y)	1	→	3
	(x,y)	2	→	1
D.	(x,y)	3	→	2
	(x,y)	4	→	4

Loop over nodes

ASSIGN DEGREES TO NODES

end