FINITE ELEMENT METHOD

FINITE ELEMENT METHOD

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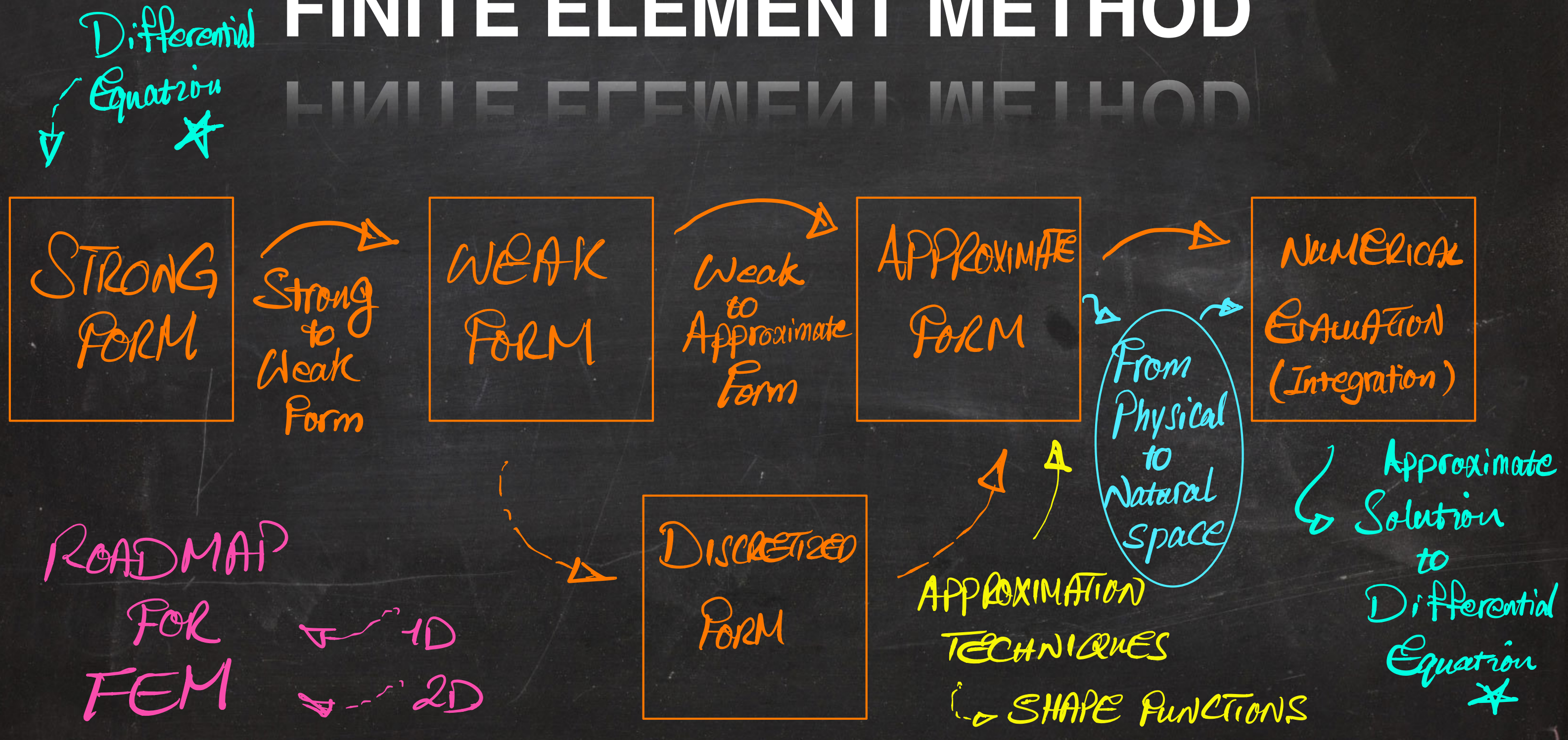
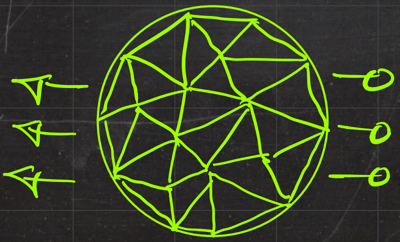


UNDERSTANDING FEM VIA AN ANALOGY (A BRUTAL SIMPLIFICATION)

Approximations in FEM

- Solution Approximation ⇝ inherent to numerical techniques using computers
- Equation Approximation ⇝ diff equation is transformed to weak form + space approximation
- Input Approximation ⇝ space discretization



Discretization (Approximation)
→ DOMAIN (X)
→ SOLUTION (u)
→ TEST (w)

diff. Eq.
→ STRONG FORM
TO
→ WEAK FORM
integral

FINITE ELEMENT METHOD IN A NUTSHELL

2ND. O.D.E.

Diff. Eq. — STRONG FORM

$(EAu')' + b = 0$

test function

(I) MULTIPLY BY w

(II) INTEGRATE

WEAK FORM

INTEGRAL FORM

$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx + w(1)\,u'(1) - w(0)\,u'(0)$$

PIECEWISE

DISCRETIZED FORM

Approximation

Approximate Discretized Weak Form

APPROXIMATE FORM

NUMERICAL INTEGRATION

another source of approx...

ELEMENT-WISE QUANTITIES

PIECEWISE INTEGRALS (SOLUTIONS)

GLOBAL SYSTEM

$[K][u] = [F]$

ASSEMBLY

SOLVE

PostProcess

FROM GLOBAL TO ELEMENTS

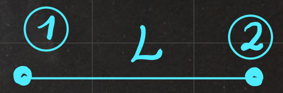
FROM INTEGRAL OVER THE DOMAIN TO SUBINTEGRALS

$$\int_0^1 \dots dx = \int_0^a \dots dx + \int_a^b \dots dx + \dots$$

1D FEM

Overview and Wrap-up

FROM STRONG FORM TO ELEMENT STIFFNESS ← IN PHYSICAL SPACE NOW



$EAu'' = 0$ SUBJECT TO BCs

$\Big\downarrow$ MULTIPLY BY TEST FUNCTION w

$$\begin{cases} \text{DIRICHLET} \rightarrow u \text{ IS PRESCRIBED} \\ \text{NEUMANN} \rightarrow u' \text{ IS PRESCRIBED} \end{cases}$$

$$EAwu'' = 0 \quad \leftarrow \quad wu'' = (wu')' - w'u'$$

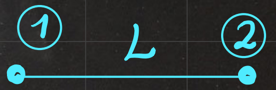
$$EA\left[(wu')' - w'u'\right] = 0 \Rightarrow EA\,w'u' = EA\,(wu')' \quad \leftarrow \text{INTEGRATE}$$

$$\int_L EA\,w'u'\,dx = \int_L EA\,(wu')'\,dx = EA\,wu'\Big]_{1}^{2} = EA\,w^2 u'^2 - EA\,w^1 u'^1$$

FROM STRONG FORM TO ELEMENT STIFFNESS

IN PHYSICAL SPACE NOW

$EA\,u'' = 0$ SUBJECT TO BCs



$$EA \begin{bmatrix} \int_L N^{1\prime} N^{1\prime}\, dx & \int_L N^{1\prime} N^{2\prime}\, dx \\ \int_L N^{2\prime} N^{1\prime}\, dx & \int_L N^{2\prime} N^{2\prime}\, dx \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} = \begin{bmatrix} F^1 \\ F^2 \end{bmatrix}$$

$$\begin{bmatrix} K^{11} & K^{12} \\ K^{21} & K^{22} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} = \begin{bmatrix} F^1 \\ F^2 \end{bmatrix} \leftarrow K^{ij} = EA \int_L N^{i\prime} N^{j\prime}\, dx$$

$$K^{ij} = EA\int_L N^{i\prime} N^{j\prime}\, dx$$ ← PHYSICAL

RECALL:

$$\int_{-1}^{1} g(\xi)\, d\xi = \sum_{gp=1}^{GPE} g(\xi_{gp})\, \alpha_{gp}$$

$$= EA\int_{-1}^{1} \frac{\partial N^i}{\partial \xi} \frac{\partial N^j}{\partial \xi} J^{-1}\, d\xi$$ ← NATURAL

$$= EA \sum_{gp=1}^{GPE} \left\{ \left[\frac{\partial N^i}{\partial \xi} \frac{\partial N^j}{\partial \xi} J^{-1} \right]_{gp} \times \alpha_{gp} \right\}$$

Loop over gp

⋮

END

e.g. in MATLAB

WHAT YOU SEE IN THE CODE ?

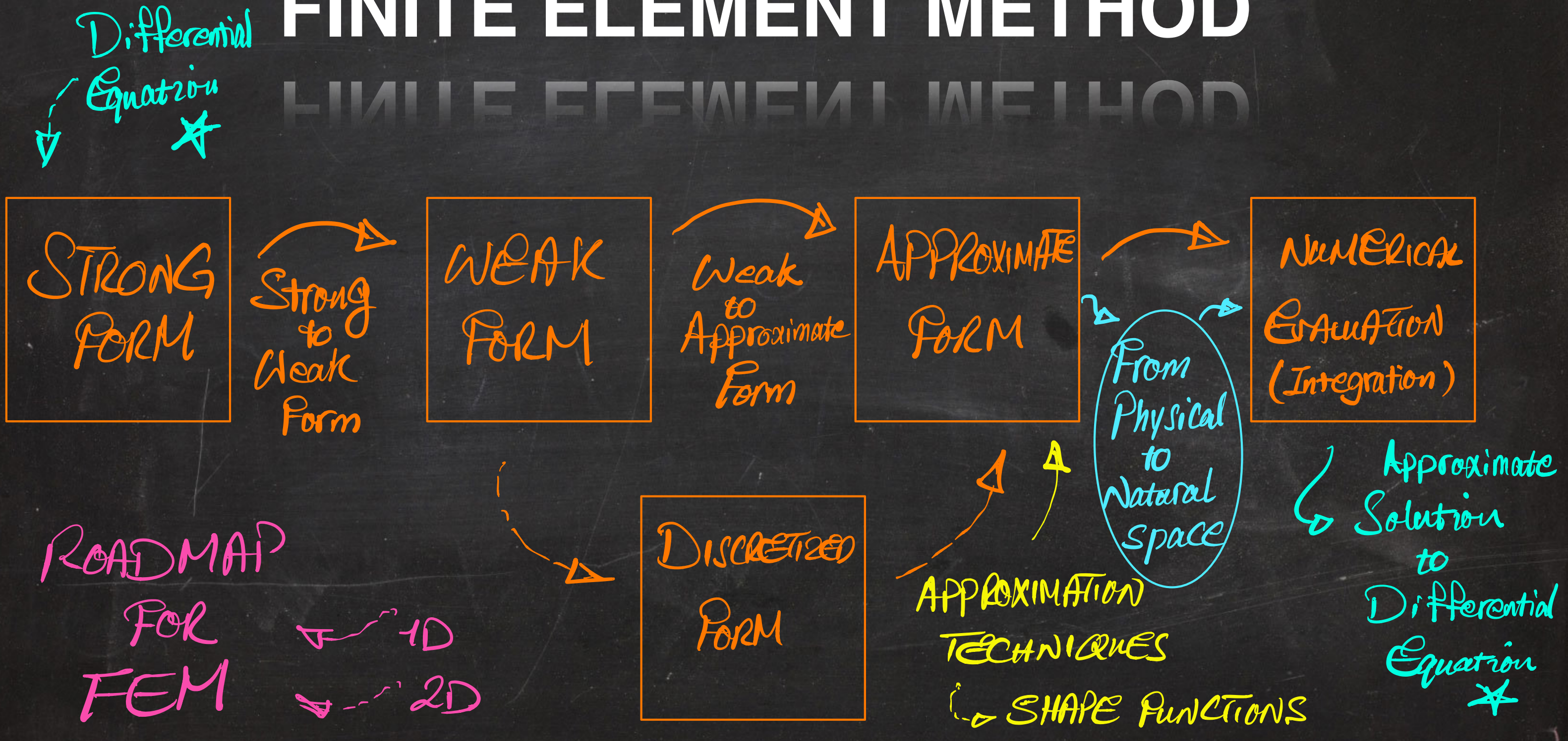
For gp=1:GPE

⋮

End

2D FEM

FINITE ELEMENT METHOD



MATHEMATICAL PRELIMINARIES

EINSTEIN SUMMATION CONVENTION

A little definition for notation convenience

A REPEATED INDEX TWICE MEANS SUMMATION OVER THAT INDEX

also, called "dummy index"

$$\sum_{i=1}^{3} u_i v_i = u_1 v_1 + u_2 v_2 + u_3 v_3 \equiv u_i v_i$$

i is summation index

i: free index

$$\sum_{j=1}^{3} A_{ij} u_j \Rightarrow \begin{cases} i=1 \Rightarrow A_{11} u_1 + A_{12} u_2 + A_{13} u_3 \\ i=2 \Rightarrow A_{21} u_1 + A_{22} u_2 + A_{23} u_3 \\ i=3 \Rightarrow A_{31} u_1 + A_{32} u_2 + A_{33} u_3 \end{cases} \Rightarrow \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \equiv A_{ij} u_j$$

$1 \leq i \leq 3$

j: summation index

$$\mathbb{u}\cdot\mathbb{v} = u_x v_x + u_y v_y + u_z v_z = u_1 v_1 + u_2 v_2 + u_3 v_3 = \sum_{i=1}^{3} u_i v_i = u_i v_i$$

↳ Dot Product $(\mathbb{u}, \mathbb{v}) \rightsquigarrow$ SCALAR $\leftarrow u_i v_i \leftarrow \mathbb{u}\cdot\mathbb{v}$

Double Dot Product $(\mathbb{A}, \mathbb{B}) \rightsquigarrow$ SCALAR $\leftarrow A_{ij} B_{ij} \leftarrow \mathbb{A}:\mathbb{B}$

$\mathbb{u} \otimes \mathbb{v}$ Dyadic Product $(\mathbb{u}, \mathbb{v}) \rightsquigarrow$ MATRIX $\leftarrow u_i v_j \leftarrow [\mathbb{u}\otimes\mathbb{v}]_{ij}$

KRONECKER DELTA $\rightsquigarrow \delta_{ij} = \Phi_i \cdot \Phi_j \Rightarrow \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$

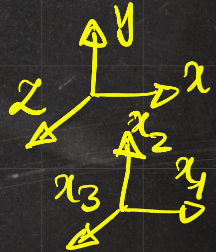
$\boldsymbol{u}\cdot\boldsymbol{v} = u_i v_i$ $\quad [\boldsymbol{A}\cdot\boldsymbol{B}]_{ik} = A_{ij} B_{jk}$ $\quad [\boldsymbol{u}\otimes\boldsymbol{v}]_{ij} = u_i v_j$

$\delta_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{e}_j$ $\quad [\boldsymbol{A}\cdot\boldsymbol{u}]_i = A_{ij} u_j$ $\quad \boldsymbol{A}:\boldsymbol{B} = A_{ij} B_{ij}$

$\mathbb{E}$ $\rightsquigarrow$ FOURTH-ORDER TENSOR (ARRAY) $\rightsquigarrow$ 3x3x3x3 = 81 Components

2nd. 4th. 2nd.

$\boldsymbol{\sigma} = \mathbb{E} : \boldsymbol{\varepsilon} \rightsquigarrow [\boldsymbol{\sigma}]_{ij} = [\mathbb{E}]_{ijkl}[\boldsymbol{\varepsilon}]_{kl}$



INSTEAD OF $x, y, z \rightsquigarrow 1, 2, 3 \rightsquigarrow x_1, x_2, x_3$ $\quad \boldsymbol{e}_x \rightsquigarrow \boldsymbol{e}_1$

DERIVATIVES ← e.g. STRONG FORM ⟶ u''

$$y = f(x) \longrightarrow y' = f'(x) \leftarrow \{\circ\}' = \frac{\partial \{\circ\}}{\partial x}$$

$$\boldsymbol{u} = \boldsymbol{u}(x, y, z) \longrightarrow \text{GRAD } \boldsymbol{u} \quad \text{OR} \quad \text{DIV } \boldsymbol{u} \quad \text{OR} \quad \text{CURL } \boldsymbol{u}$$

$$\frac{\partial \boldsymbol{u}}{\partial x_i} \otimes \boldsymbol{e}_i \qquad \frac{\partial \boldsymbol{u}}{\partial x_i} \cdot \boldsymbol{e}_i \qquad \frac{\partial \boldsymbol{u}}{\partial x_i} \times \boldsymbol{e}_i$$

DERIVATIVES $\rightarrow$ e.g. STRONG FORM $\rightarrow u''$

$y = f(x) \rightarrow y' = f'(x) \quad \{\circ\}' = \dfrac{\partial\{\circ\}}{\partial x}$

$\rightarrow \text{GRAD}\, \boldsymbol{u} = \dfrac{\partial \boldsymbol{u}}{\partial x_i} \otimes \boldsymbol{e}_i$

$\boldsymbol{u} = \boldsymbol{u}(x, y, z) \rightarrow \text{GRAD}\, \boldsymbol{u}$ OR $\text{DIV}\, \boldsymbol{u}$

$\rightarrow \text{DIV}\, \boldsymbol{u} = \dfrac{\partial \boldsymbol{u}}{\partial x_i} \cdot \boldsymbol{e}_i$

$\omega = f(x, y, z) \rightarrow \text{GRAD}\, f \rightarrow \dfrac{\partial f}{\partial x_i} \boldsymbol{e}_i$

$\mathbb{A} = \mathbb{A}(x, y, z) \rightarrow \text{GRAD}\, \mathbb{A}$ OR $\text{DIV}\, \mathbb{A}$

$\rightarrow \dfrac{\partial \mathbb{A}}{\partial x_i} \otimes \boldsymbol{e}_i \qquad \rightarrow \dfrac{\partial \mathbb{A}}{\partial x_i} \cdot \boldsymbol{e}_i$

SCALAR VALUED FUNCTION $\phi(x) \approx \phi(x_1, x_2, x_3)$

$\phi(x, y, z)$

$$\text{GRAD}\, \phi = \frac{\partial \phi}{\partial x_i} e_i$$

$$= \frac{\partial \phi}{\partial x_1} e_1 + \frac{\partial \phi}{\partial x_2} e_2 + \frac{\partial \phi}{\partial x_3} e_3$$

$$= \frac{\partial \phi}{\partial x} e_x + \frac{\partial \phi}{\partial y} e_y + \frac{\partial \phi}{\partial z} e_z$$

$$\text{GRAD}\, \phi = \begin{bmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \\ \frac{\partial \phi}{\partial z} \end{bmatrix}$$

VECTOR-VALUED FUNCTION $\mathbf{u}(\mathbf{x})$ $\checkmark$ $\begin{bmatrix} u_1(x_1, x_2, x_3) \\ u_2(x_1, x_2, x_3) \\ u_3(x_1, x_2, x_3) \end{bmatrix}$

$$\text{GRAD } \mathbf{u} = \frac{\partial \mathbf{u}}{\partial x_i} \otimes \mathbf{e}_i$$

$$= \frac{\partial (u_j \mathbf{e}_j)}{\partial x_i} \otimes \mathbf{e}_i$$

$$= \frac{\partial u_j}{\partial x_i} \mathbf{e}_j \otimes \mathbf{e}_i$$

$$\text{GRAD } \mathbf{u} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

VECTOR-VALUED FUNCTION $\mathbf{u}(\mathbf{x})$

$$\text{DIV}\,\mathbf{u} = \frac{\partial \mathbf{u}}{\partial x_i} \cdot \mathbf{e}_i$$

$$= \frac{\partial (u_j \mathbf{e}_j)}{\partial x_i} \cdot \mathbf{e}_i$$

$$= \frac{\partial u_j}{\partial x_i} \underbrace{\mathbf{e}_j \cdot \mathbf{e}_i}_{\delta_{ij}} = \frac{\partial u_i}{\partial x_i}$$

$$\text{GRAD}\,\mathbf{u} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

$$\text{DIV}\,\mathbf{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

MATRIX-VALUED FUNCTION $\mathbb{A}(x)$

$$\operatorname{Div}\mathbb{A} = \frac{\partial \mathbb{A}}{\partial x_i}\cdot \mathbf{e}_i$$

$$= \frac{\partial (A_{jk}\,\mathbf{e}_j\otimes\mathbf{e}_k)}{\partial x_i}\cdot \mathbf{e}_i$$

$$= \frac{\partial A_{jk}}{\partial x_i}\,\mathbf{e}_j\,\underbrace{\mathbf{e}_k\cdot\mathbf{e}_i}_{\delta_{ki}} = \frac{\partial A_{ji}}{\partial x_i}\,\mathbf{e}_j$$

$$\rightsquigarrow [\operatorname{Div}\mathbb{A}]_j = \frac{\partial A_{ji}}{\partial x_i}$$

$$\operatorname{Div}\mathbb{A} = \begin{bmatrix} \dfrac{\partial A_{11}}{\partial x_1} + \dfrac{\partial A_{12}}{\partial x_2} + \dfrac{\partial A_{13}}{\partial x_3} \\ \dfrac{\partial A_{21}}{\partial x_1} + \dfrac{\partial A_{22}}{\partial x_2} + \dfrac{\partial A_{23}}{\partial x_3} \\ \dfrac{\partial A_{31}}{\partial x_1} + \dfrac{\partial A_{32}}{\partial x_2} + \dfrac{\partial A_{33}}{\partial x_3} \end{bmatrix}$$

SCALAR → 0, VECTOR → 1, MATRIX → 2

GRADIENT INCREASES THE ORDER BY 1

DIVERGENCE REDUCES THE ORDER BY 1

$$\mathrm{GRAD}\,\phi = \begin{bmatrix} \partial\phi/\partial x_1 \\ \partial\phi/\partial x_2 \\ \partial\phi/\partial x_3 \end{bmatrix} \qquad \mathrm{GRAD}\,u = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

$$\mathrm{DIV}\,u = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

$$\mathrm{DIV}\,A = \begin{bmatrix} \frac{\partial A_{11}}{\partial x_1} + \frac{\partial A_{12}}{\partial x_2} + \frac{\partial A_{13}}{\partial x_3} \\ \frac{\partial A_{21}}{\partial x_1} + \frac{\partial A_{22}}{\partial x_2} + \frac{\partial A_{23}}{\partial x_3} \\ \frac{\partial A_{31}}{\partial x_1} + \frac{\partial A_{32}}{\partial x_2} + \frac{\partial A_{33}}{\partial x_3} \end{bmatrix}$$

Big Picture of Mechanics (Mechanical Problems & Thermal Problems)

$$\begin{array}{ccccc}
\text{Def. } \mathbf{u} & \overset{?}{\longleftrightarrow} & \text{Load. } \mathbf{t} \\
\Big\updownarrow \; \mathbf{E} = \text{GRAD}^{\text{SYM}}\, \mathbf{u} & & \Big\updownarrow \; \overset{\text{CAUCHY}}{\mathbf{S} \cdot \mathbf{n} = \mathbf{t}} \\
\text{STRAIN } \mathbf{E} & \overset{\text{CONST.}}{\longleftrightarrow} & \text{STRESS } \mathbf{S}
\end{array}$$

$$\mathbf{S} = \mathbb{C} : \mathbf{E}$$

↳ HOOKE'S LAW

$$\begin{array}{ccccc}
\text{Temp. } \theta & \overset{?}{\longleftrightarrow} & \text{Heat Flux } q_n \\
\Big\updownarrow \; \mathbf{G} = \text{GRAD}\, \theta & & \Big\updownarrow \; \overset{\text{CAUCHY}}{\mathbf{q} \cdot \mathbf{n} = q_n} \\
\text{Temp. GRADIENT } \mathbf{G} & \overset{\text{CONST.}}{\longleftrightarrow} & \text{Heat Flux VECTOR } \mathbf{q}
\end{array}$$

$$\mathbf{q} = -\mathbb{K} \cdot \mathbf{G}$$

↳ FOURIER'S LAW

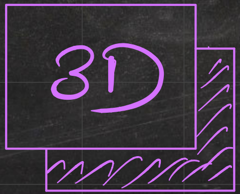
STRONG FORM (GENERIC FORM) $\rightsquigarrow$ $\mathrm{Div}\,\boldsymbol{\sigma} + \mathbf{b} = \mathbf{0}$, $\mathrm{Div}\,\mathbf{q} + c = 0$

$$\frac{\partial \sigma_{jk}}{\partial x_k} + b_j = 0$$

$$\begin{cases} \dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{xy}}{\partial y} + \dfrac{\partial \sigma_{xz}}{\partial z} + b_x = 0 \\[2ex] \dfrac{\partial \sigma_{yx}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} + \dfrac{\partial \sigma_{yz}}{\partial z} + b_y = 0 \\[2ex] \dfrac{\partial \sigma_{zx}}{\partial x} + \dfrac{\partial \sigma_{zy}}{\partial y} + \dfrac{\partial \sigma_{zz}}{\partial z} + b_z = 0 \end{cases}$$

$$\frac{\partial q_i}{\partial x_i} + c = 0$$

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} + c = 0$$



STRONG FORM (GENERIC FORM) $\rightsquigarrow$ $\text{Div}\,\boldsymbol{\sigma} + \mathbf{b} = \mathbf{0}$, $\text{Div}\,\mathbf{q} + c = 0$

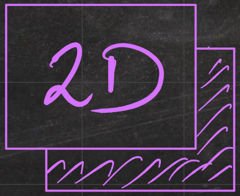
$$\frac{\partial \sigma_{jk}}{\partial x_k} + b_j = 0$$

$$\frac{\partial q_i}{\partial x_i} + c = 0$$

$$\begin{cases} \dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{xy}}{\partial y} + b_x = 0 \\[2ex] \dfrac{\partial \sigma_{yx}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} + b_y = 0 \end{cases}$$

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + c = 0$$

2D $\rightsquigarrow$ Plane Strain
Plane Stress $\Big\}\ \boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\varepsilon}$

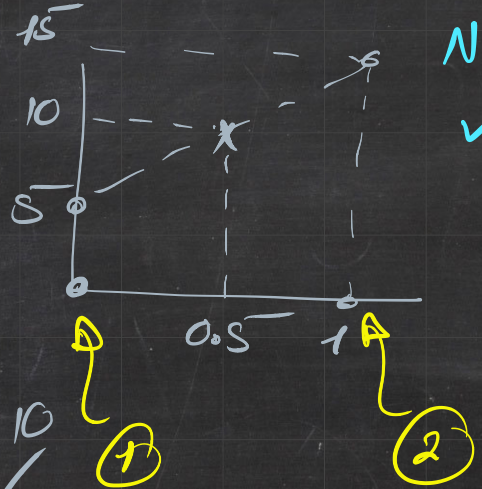


APPROXIMATION USING 2D FINITE ELEMENTS ← Shape Functions

$f(0) = 5$

$f(1) = 15$

¿? $f(0.5) \approx 10$ ✓



$N^1 = [x-1]/_{-1}$ ✓

$$f(x) = N^1 \overset{5}{f^1} + N^2 \overset{15}{f^2}$$

$= \ldots$

$= 10x + 5$

1D EXAMPLE

(I) intuitive

(II) FE approach

APPROXIMATION USING 2D FINITE ELEMENTS ← Shape Functions

EXAMPLE I:

$f(-1,-1) = 1$

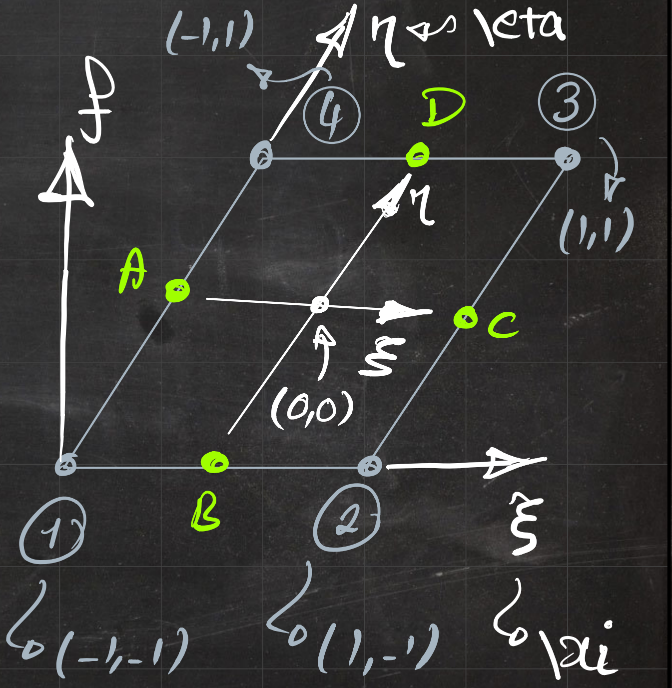
$f(1,-1) = 2$

$f(1,1) = -1$

$f(-1,1) = 4$

$$f(\xi,\eta) = N^1 f^1 + N^2 f^2 + N^3 f^3 + N^4 f^4$$

$\to f(0,0) = ?\ ,\ f(\xi,\eta) = ?$



APPROXIMATION USING 2D FINITE ELEMENTS ← Shape Functions

EXAMPLE II: ← EXERCISE

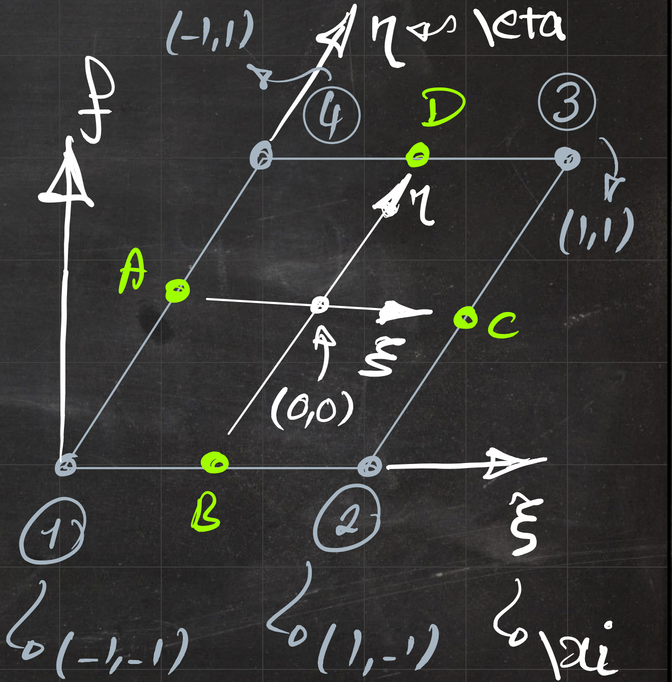
$f(A) = f(-1,0) = 2.5$

$f(B) = f(0,-1) = 1.5$

$f(C) = f(1,0) = 0.5$

$f(D) = f(0,1) = 1.5$

$N^i f^i$



→ $f(0,0) = ?$, $f(\xi,\eta) = ?$

APPROXIMATION USING 2D FINITE ELEMENTS ← ✓ Shape Functions

$N^i \rightsquigarrow 1$ @ NODE i , $\emptyset$ @ OTHERS

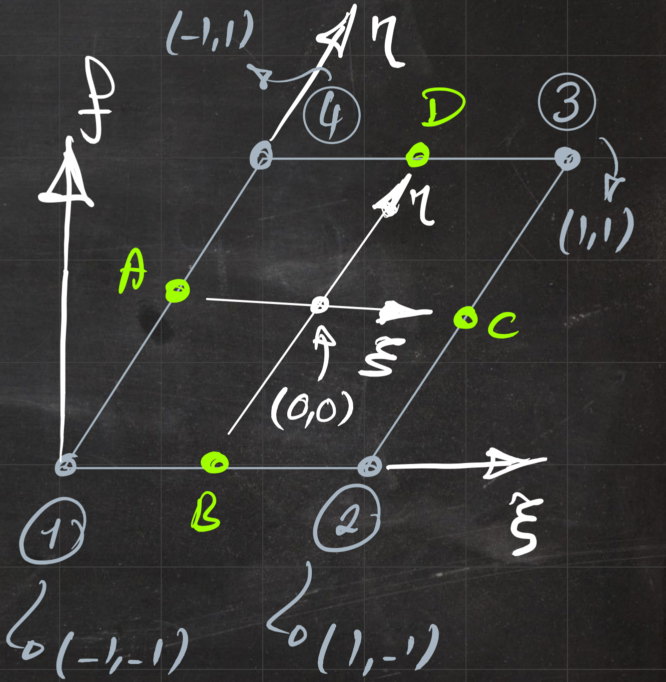
$$N^1 = \frac{1}{4}[\xi - 1][\eta - 1]$$

$N^2 =$ -1 -1

$N^3 =$

$N^4 =$ -2 -2

4



APPROXIMATION USING 2D FINITE ELEMENTS ← Shape Functions

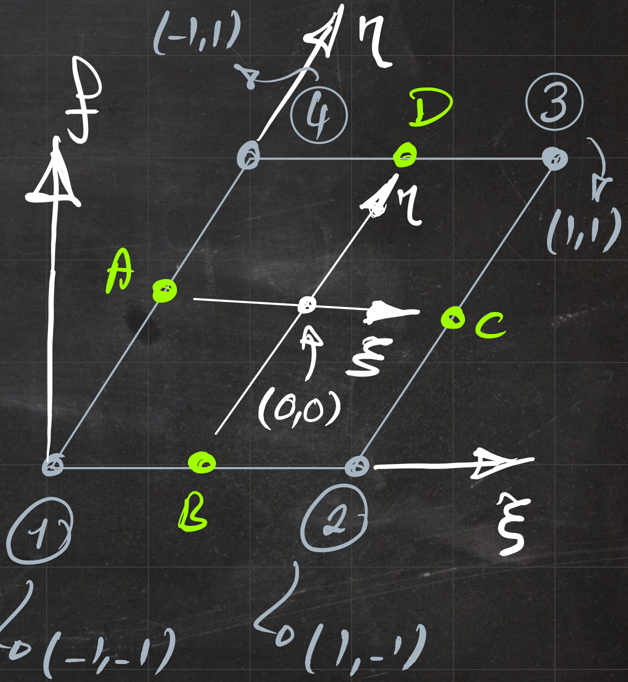
$N^i \rightsquigarrow 1$ @ NODE i, 0 @ OTHERS

$$N^1(\xi,\eta) = \frac{1}{4}[\xi - 1][\eta - 1]$$

$$N^2(\xi,\eta) = -\frac{1}{4}[\xi + 1][\eta - 1]$$

$$N^3(\xi,\eta) = \frac{1}{4}[\xi + 1][\eta + 1]$$

$$N^4(\xi,\eta) = -\frac{1}{4}[\xi - 1][\eta + 1]$$



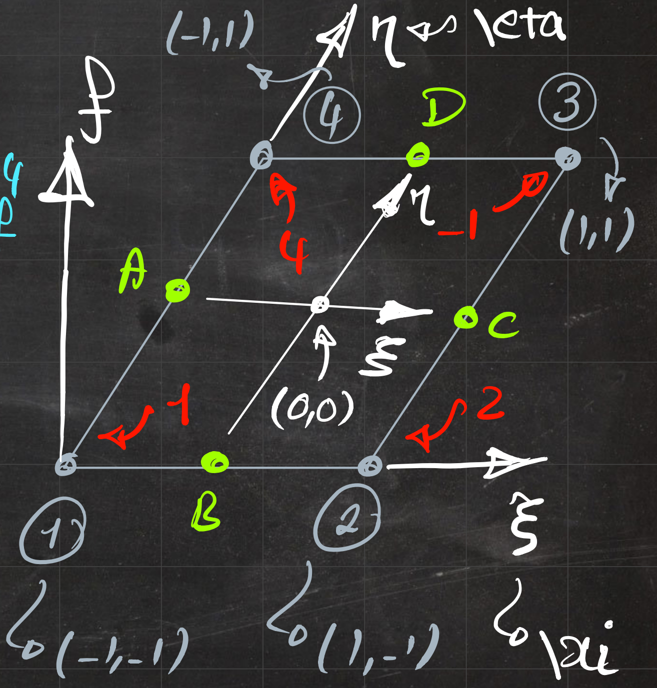
APPROXIMATION USING 2D FINITE ELEMENTS ← Shape Functions

EXAMPLE I:

$$f(\xi,\eta) = N^1(\xi,\eta)\, f^1 + N^2(\xi,\eta)\, f^2 + N^3(\xi,\eta)\, f^3 + N^4(\xi,\eta)\, f^4$$

$$= \sum_{i=1}^{NPE} N^i(\xi,\eta)\, f^i$$

$$= N^i f^i$$



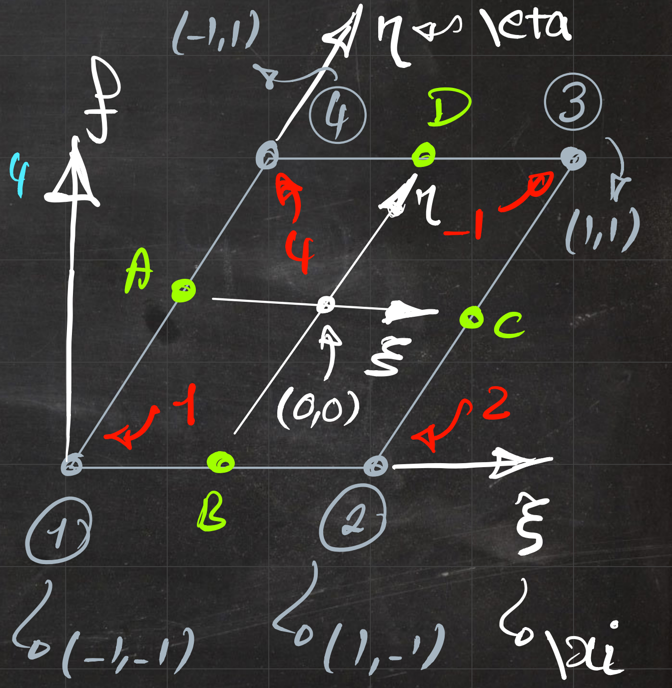
APPROXIMATION USING 2D FINITE ELEMENTS ← Shape Functions

EXAMPLE I:

$$N^1(\xi,\eta) = \frac{1}{4}[\xi-1][\eta-1]$$
$$N^2(\xi,\eta) = -\frac{1}{4}[\xi+1][\eta-1]$$
$$N^3(\xi,\eta) = \frac{1}{4}[\xi+1][\eta+1]$$
$$N^4(\xi,\eta) = -\frac{1}{4}[\xi-1][\eta+1]$$

$$f(\xi,\eta) = N^i f^i$$

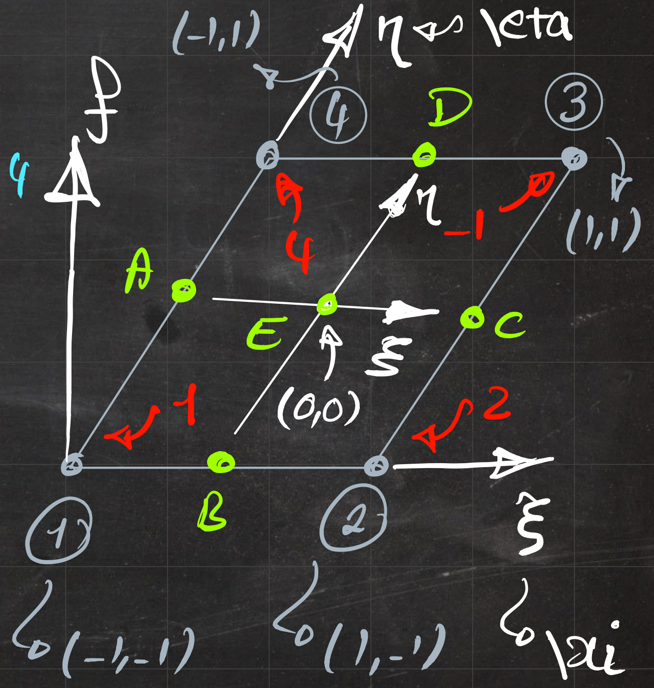
$$= \frac{1}{4}[\xi-1][\eta-1] \times 1$$
$$- \frac{1}{4}[\xi+1][\eta-1] \times 2$$
$$+ \frac{1}{4}[\xi+1][\eta+1] \times -1$$
$$- \frac{1}{4}[\xi-1][\eta+1] \times 4$$
$$= \dots = f(\xi,\eta) \checkmark$$



APPROXIMATION USING 2D FINITE ELEMENTS ← Shape Functions

EXAMPLE I:

$$f(\xi,\eta) = \frac{1}{4}[\xi-1][\eta-1] \times 1 - \frac{1}{4}[\xi+1][\eta-1] \times 2 + \frac{1}{4}[\xi+1][\eta+1] \times -1 - \frac{1}{4}[\xi-1][\eta+1] \times 4$$



$$f(0,0) = \frac{1}{4} + \frac{2}{4} - \frac{1}{4} + 1 = 1.5$$

$$\Rightarrow f_E = 1.5 \checkmark$$

APPROXIMATION USING 2D FINITE ELEMENTS ← Shape Functions

EXAMPLE I:

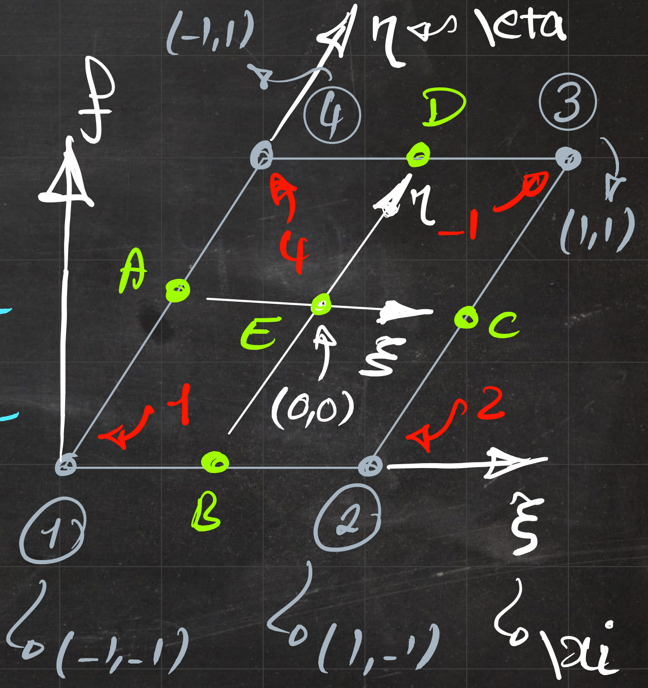
2. $f(\xi,\eta) = \dots$

@A ⟶ $\xi=-1, \eta=0$ ⟶ $f_A=2.5$

@B ⟶ $\xi=0, \eta=-1$ ⟶ $f_B=1.5$

@C ⟶ $\xi=1, \eta=0$ ⟶ $f_C=0.5$

@D ⟶ $\xi=0, \eta=1$ ⟶ $f_D=1.5$



TWO-DIMENSIONAL FINITE ELEMENTS:

D2QU4N

D2QU9N

D2QU8N

D2TR3N

D2TR6N

2D Finite Element Library

two-dimensional finite elements library

Numerical Analysis and High-level Applications Lab NAHAL

- two-dimensional 4-noded quadrilateral element (D2QU4N)

  a.k.a. bilinear quadrilateral element

- two-dimensional 9-noded quadrilateral element (D2QU9N)

  a.k.a. Lagrange biquadratic quadrilateral element

- two-dimensional 8-noded quadrilateral element (D2QU8N)

  a.k.a. serendipity biquadratic quadrilateral element

- two-dimensional 3-noded triangular element (D2TR3N)

  a.k.a. constant strain triangle

- two-dimensional 6-noded triangular element (D2TR6N)

  a.k.a. quadratic triangle

- two-dimensional quadrature rule

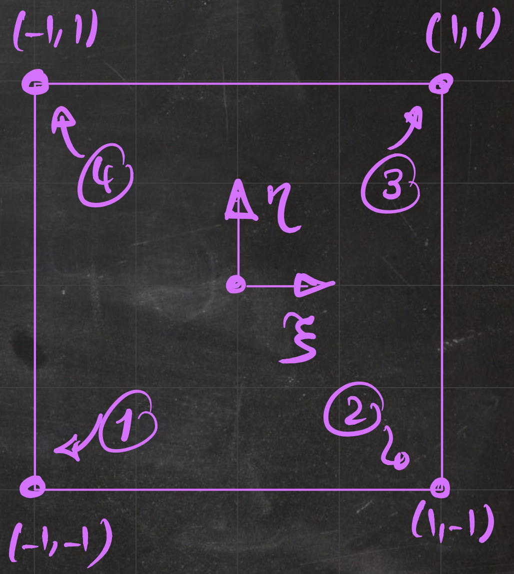
D2QU4N → LINEAR QUADRILATERAL ELEMENT

$$N^1 = \frac{1}{4}[1-\xi][1-\eta]$$

$$N^2 = \frac{1}{4}[1+\xi][1-\eta]$$

$$N^3 = \frac{1}{4}[1+\xi][1+\eta]$$

$$N^4 = \frac{1}{4}[1-\xi][1+\eta]$$



2D Finite Element Library

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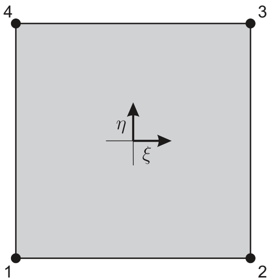
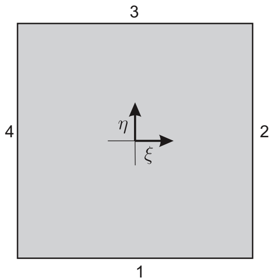
D2QU4N — bilinear quadrilateral element

Nodes



Edges



Node Number	Coordinates	
	ξ	η
1	-1	-1
2	1	-1
3	1	1
4	-1	1

$$N^1 = \frac{1}{4}\left(1-\xi\right)\left(1-\eta\right)$$

$$N^2 = \frac{1}{4}\left(1+\xi\right)\left(1-\eta\right)$$

$$N^3 = \frac{1}{4}\left(1+\xi\right)\left(1+\eta\right)$$

$$N^4 = \frac{1}{4}\left(1-\xi\right)\left(1+\eta\right)$$

$$N^1_{,\xi} = -\frac{1}{4}\left(1-\eta\right) \qquad N^1_{,\eta} = -\frac{1}{4}\left(1-\xi\right)$$

$$N^2_{,\xi} = +\frac{1}{4}\left(1-\eta\right) \qquad N^2_{,\eta} = -\frac{1}{4}\left(1+\xi\right)$$

$$N^3_{,\xi} = +\frac{1}{4}\left(1+\eta\right) \qquad N^3_{,\eta} = +\frac{1}{4}\left(1+\xi\right)$$

$$N^4_{,\xi} = -\frac{1}{4}\left(1+\eta\right) \qquad N^4_{,\eta} = +\frac{1}{4}\left(1-\xi\right)$$

D2QU9N ← Quadratic Quadrilateral Element (Lagrange)

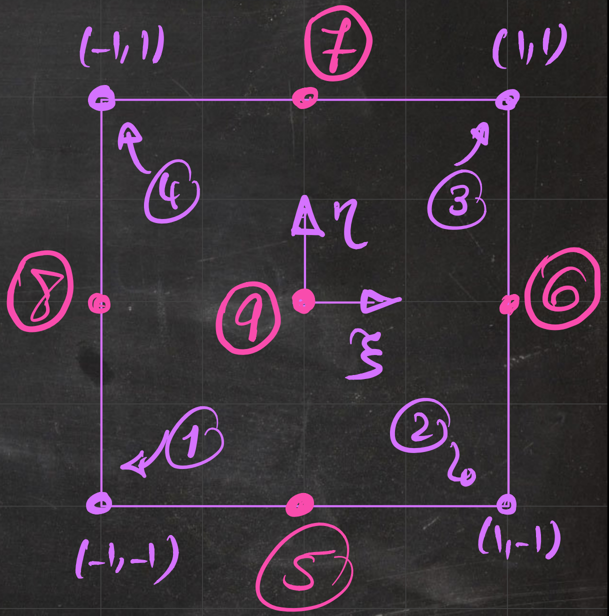
$$N^1 = \frac{1}{4}[1-\xi]\,\xi\,[1-\eta]\,\eta$$

$\circ\ \circ\ \circ$

$$N^5 = -\frac{1}{2}[1-\xi][1+\xi][1-\eta]\,\eta$$

$\circ\ \circ\ \circ$

$$N^9 = [1-\xi][1+\xi][1-\eta][1+\eta]$$



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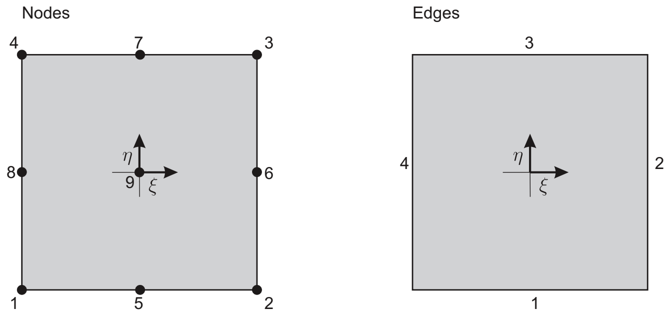
D2QU9N — Lagrange biquadratic quadrilateral element



Node Number	Coordinates	
	ξ	η
1	-1	-1
2	1	-1
3	1	1
4	-1	1
5	0	-1
6	1	0
7	0	1
8	-1	0
9	0	0

$$N^1 = +\frac{1}{4}(1-\xi)\,\xi\,(1-\eta)\,\eta$$
$$N^2 = -\frac{1}{4}(1+\xi)\,\xi\,(1-\eta)\,\eta$$
$$N^3 = +\frac{1}{4}(1+\xi)\,\xi\,(1+\eta)\,\eta$$
$$N^4 = -\frac{1}{4}(1-\xi)\,\xi\,(1+\eta)\,\eta$$
$$N^5 = -\frac{1}{2}(1-\xi)(1+\xi)(1-\eta)\,\eta$$
$$N^6 = +\frac{1}{2}(1+\xi)\,\xi\,(1-\eta)(1+\eta)$$
$$N^7 = +\frac{1}{2}(1-\xi)(1+\xi)(1+\eta)\,\eta$$
$$N^8 = -\frac{1}{2}(1-\xi)\,\xi\,(1-\eta)(1+\eta)$$
$$N^9 = (1-\xi)(1+\xi)(1-\eta)(1+\eta)$$

$$N^1_{,\xi} = +\frac{1}{4}(1-2\xi)(1-\eta)\,\eta$$
$$N^2_{,\xi} = -\frac{1}{4}(1+2\xi)(1-\eta)\,\eta$$
$$N^3_{,\xi} = +\frac{1}{4}(1+2\xi)(1+\eta)\,\eta$$
$$N^4_{,\xi} = -\frac{1}{4}(1-2\xi)(1+\eta)\,\eta$$
$$N^5_{,\xi} = \xi\eta\,(1-\eta)$$
$$N^6_{,\xi} = \frac{1}{2}(1+2\xi)(1-\eta)(1+\eta)$$
$$N^7_{,\xi} = -\xi\eta\,(1+\eta)$$
$$N^8_{,\xi} = -\frac{1}{2}(1-2\xi)(1-\eta)(1+\eta)$$
$$N^9_{,\xi} = -2\xi\,(1-\eta)(1+\eta)$$

$$N^1_{,\eta} = +\frac{1}{4}(1-\xi)\,\xi\,(1-2\eta)$$
$$N^2_{,\eta} = -\frac{1}{4}(1+\xi)\,\xi\,(1-2\eta)$$
$$N^3_{,\eta} = +\frac{1}{4}(1+\xi)\,\xi\,(1+2\eta)$$
$$N^4_{,\eta} = -\frac{1}{4}(1-\xi)\,\xi\,(1+2\eta)$$
$$N^5_{,\eta} = \frac{1}{2}(1-\xi)(1+\xi)(2\eta-1)$$
$$N^6_{,\eta} = -(1+\xi)\,\xi\eta$$
$$N^7_{,\eta} = \frac{1}{2}(1-\xi)(1+\xi)(1+2\eta)$$
$$N^8_{,\eta} = (1-\xi)\,\xi\eta$$
$$N^9_{,\eta} = -2\,(1-\xi)(1+\xi)\,\eta$$

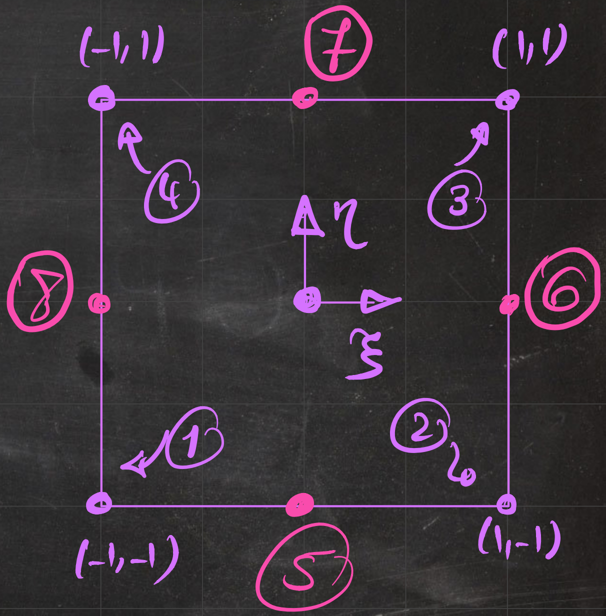
D2QU8N → Quadratic Quadrilateral Element (Serendipity)

$$N^1 = -\frac{1}{4}[1-\xi][1-\eta][1+\xi+\eta]$$

$\dots$

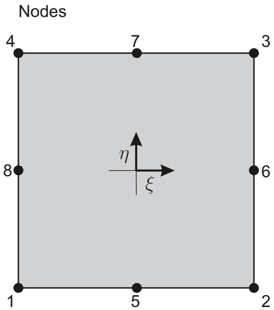
$$N^5 = \frac{1}{2}[1-\xi][1+\xi][1-\eta]$$

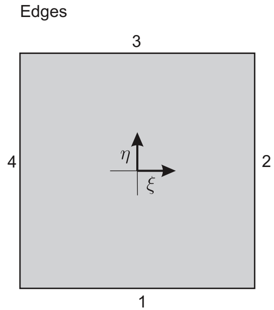
$\dots$



2D Finite Element Library

D2QU8N — serendipity biquadratic quadrilateral element





Node Number	Coordinates	
	ξ	η
1	-1	-1
2	1	-1
3	1	1
4	-1	1
5	0	-1
6	1	0
7	0	1
8	-1	0

$$N^1 = -\frac{1}{4}(1-\xi)(1-\eta)(1+\xi+\eta) \qquad N^1_{,\xi} = +\frac{1}{4}(1-\eta)(2\xi+\eta) \qquad N^1_{,\eta} = +\frac{1}{4}(1-\xi)(\xi+2\eta)$$

$$N^2 = -\frac{1}{4}(1+\xi)(1-\eta)(1-\xi+\eta) \qquad N^2_{,\xi} = +\frac{1}{4}(1-\eta)(2\xi-\eta) \qquad N^2_{,\eta} = +\frac{1}{4}(1+\xi)(-\xi+2\eta)$$

$$N^3 = -\frac{1}{4}(1+\xi)(1+\eta)(1-\xi-\eta) \qquad N^3_{,\xi} = +\frac{1}{4}(1+\eta)(2\xi+\eta) \qquad N^3_{,\eta} = +\frac{1}{4}(1+\xi)(\xi+2\eta)$$

$$N^4 = -\frac{1}{4}(1-\xi)(1+\eta)(1+\xi-\eta) \qquad N^4_{,\xi} = +\frac{1}{4}(1+\eta)(2\xi-\eta) \qquad N^4_{,\eta} = +\frac{1}{4}(1-\xi)(-\xi+2\eta)$$

$$N^5 = \frac{1}{2}(1-\xi)(1+\xi)(1-\eta) \qquad N^5_{,\xi} = -\xi(1-\eta) \qquad N^5_{,\eta} = -\frac{1}{2}(1-\xi)(1+\xi)$$

$$N^6 = \frac{1}{2}(1+\xi)(1+\eta)(1-\eta) \qquad N^6_{,\xi} = +\frac{1}{2}(1-\eta)(1+\eta) \qquad N^6_{,\eta} = -(1+\xi)\eta$$

$$N^7 = \frac{1}{2}(1-\xi)(1+\xi)(1+\eta) \qquad N^7_{,\xi} = -\xi(1+\eta) \qquad N^7_{,\eta} = +\frac{1}{2}(1-\xi)(1+\xi)$$

$$N^8 = \frac{1}{2}(1-\xi)(1+\eta)(1-\eta) \qquad N^8_{,\xi} = -\frac{1}{2}(1-\eta)(1+\eta) \qquad N^8_{,\eta} = -(1-\xi)\eta$$

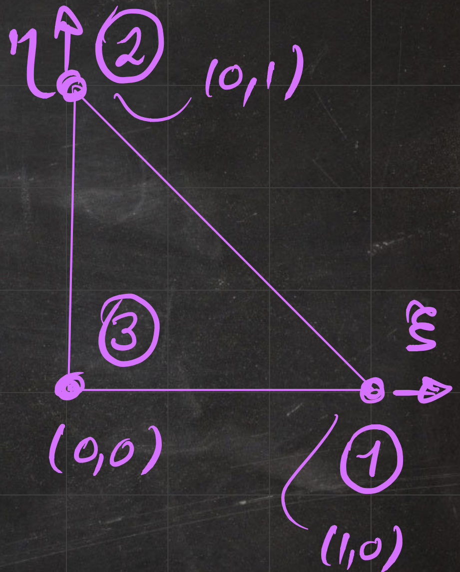
D2TR3N ⤶ LINEAR TRIANGULAR ELEMENT (CST)

(CONSTANT STRAIN TRIANGLE)

$N^1 = \xi \quad \leadsto \quad N^1_{,\xi} = 1 \qquad N^1_{,\eta} = 0$

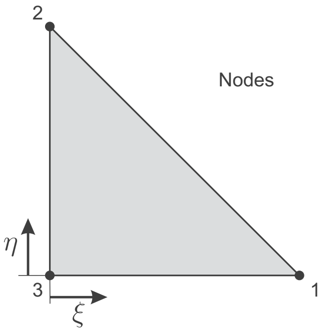
$N^2 = \eta \quad \leadsto \quad N^2_{,\xi} = 0 \qquad N^2_{,\eta} = 1$

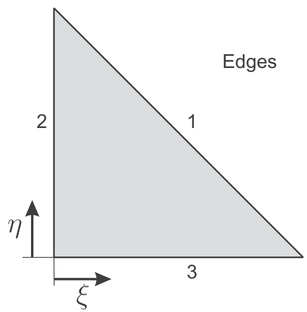
$N^3 = 1 - \xi - \eta \quad \leadsto \quad N^3_{,\xi} = -1 \qquad N^3_{,\eta} = -1$



2D Finite Element Library

D2TR3N — constant strain triangle (CST)





Node Number	Coordinates	
	ξ	η
1	1	0
2	0	1
3	0	0

$$N^1 = \xi \qquad N^1_{,\xi} = 1 \qquad N^1_{,\eta} = 0$$

$$N^2 = \eta \qquad N^2_{,\xi} = 0 \qquad N^2_{,\eta} = 1$$

$$N^3 = (1 - \xi - \eta) \qquad N^3_{,\xi} = -1 \qquad N^3_{,\eta}(\xi, \eta) = -1$$

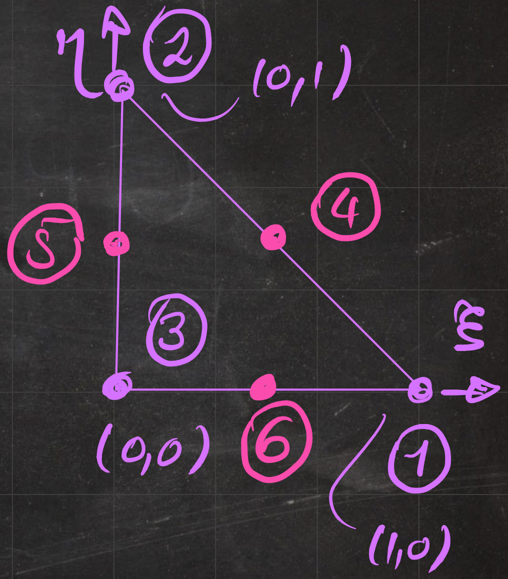
D2TR6N $\rightarrow$ QUADRATIC TRIANGULAR ELEMENT

$$N^1 = \xi\,[2\xi - 1]$$

$\dots$

$$N^4 = 4\,\xi\,\eta$$

$\dots$



2D Finite Element Library

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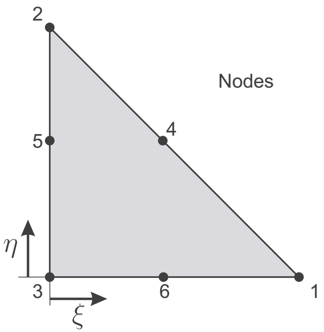
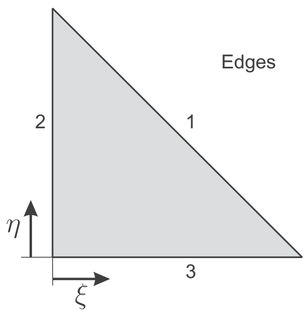
D2TR6N

quadratic triangle





Node Number	Coordinates	
	ξ	η
1	1	0
2	0	1
3	0	0
4	1/2	1/2
5	0	1/2
6	1/2	0

$$N^1 = \xi(2\xi - 1) \qquad N^1_{,\xi} = -1 + 4\xi \qquad N^1_{,\eta} = 0$$

$$N^2 = \eta(2\eta - 1) \qquad N^2_{,\xi} = 0 \qquad N^2_{,\eta} = -1 + 4\eta$$

$$N^3 = (1 - \xi - \eta)(1 - 2\xi - 2\eta) \qquad N^3_{,\xi} = -3 + 4\xi + 4\eta \qquad N^3_{,\eta} = -3 + 4\xi + 4\eta$$

$$N^4 = 4\xi\eta \qquad N^4_{,\xi} = 4\eta \qquad N^4_{,\eta} = 4\xi$$

$$N^5 = 4\eta(1 - \xi - \eta) \qquad N^5_{,\xi} = -4\eta \qquad N^5_{,\eta} = -4(-1 + 2\eta + \xi)$$

$$N^6 = 4\xi(1 - \xi - \eta) \qquad N^6_{,\xi} = -4(-1 + \eta + 2\xi) \qquad N^6_{,\eta} = -4\xi$$