FINITE ELEMENT METHOD

FINITE ELEMENT METHOD

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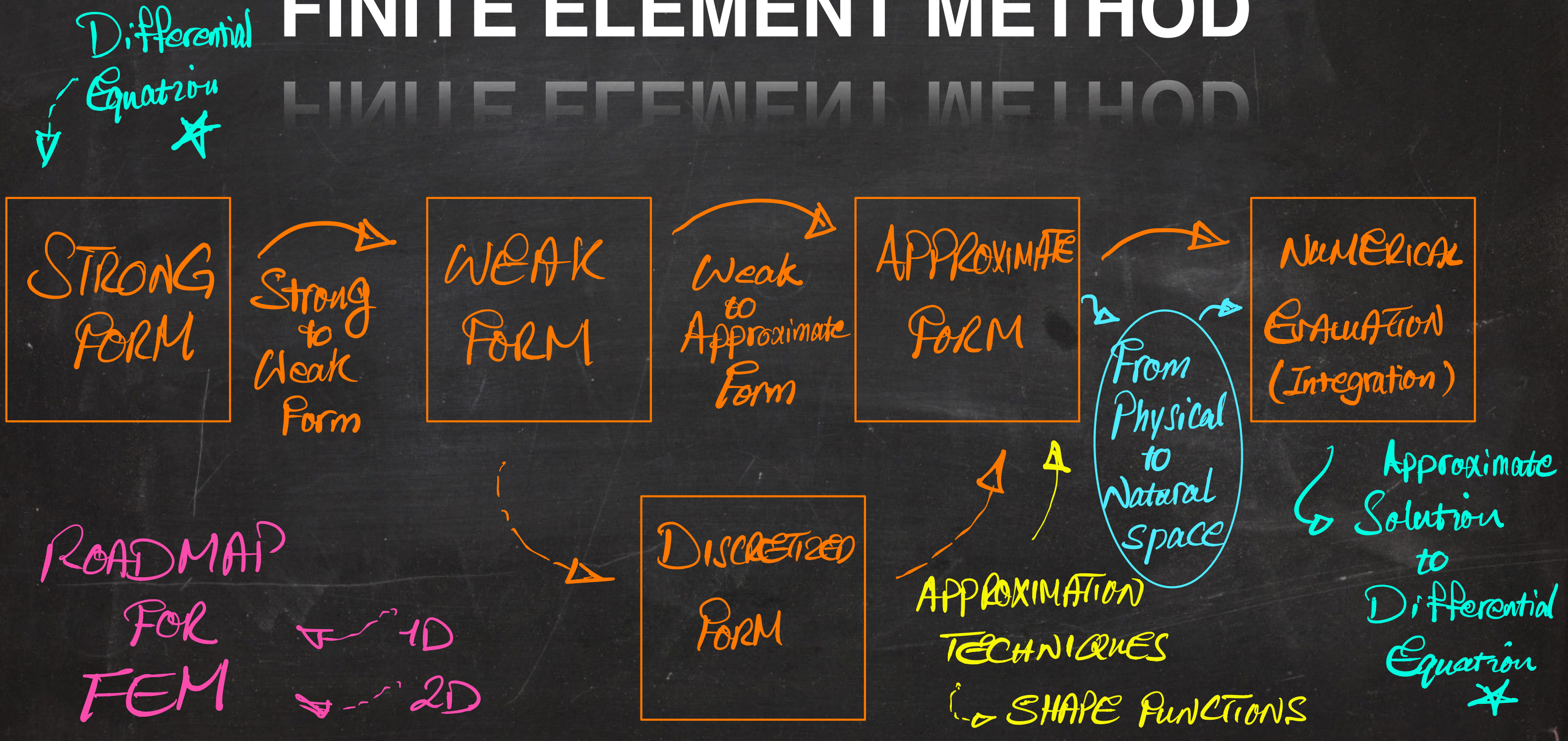
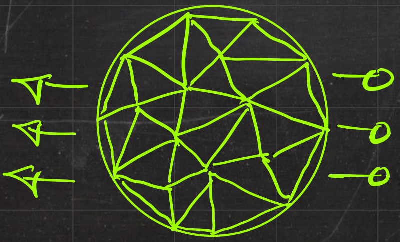


UNDERSTANDING FEM VIA AN ANALOGY (A BRUTAL SIMPLIFICATION)

Approximations in FEM

- → Solution Approximation → inherent to numerical techniques using computers
- → Equation Approximation → diff equation is transformed to weak form + space approximation
- → Input Approximation → space discretization



Discretization (Approximation)

→ DOMAIN (X)

→ SOLUTION (u)

→ TEST (w)

diff. Eq.

→ STRONG FORM

TO

integral

→ WEAK FORM

FINITE ELEMENT METHOD IN A NUTSHELL

2ND. O.D.E.

Diff. Eq.

STRONG FORM

$(EAu')' + b = 0$

test function

(I) MULTIPLY BY W

(II) INTEGRATE

WEAK FORM

INTEGRAL FORM

$$\int_0^1 w' u' \, dx = \int_0^1 w \, dx + w(1)\,u'(1) - w(0)\,u'(0)$$

PIECEWISE

DISCRETIZED FORM

Approximate Discretized Weak Form

Approximation

APPROXIMATE FORM

NUMERICAL INTEGRATION

another source of approx...

ELEMENT-WISE QUANTITIES

GLOBAL SYSTEM

$[K][u] = [F]$

SOLVE

PostProcess

FROM GLOBAL TO ELEMENTS

FROM INTEGRAL OVER THE DOMAIN TO SUBINTEGRALS

$$\int_0^1 \dots dx = \int_0^a \dots dx + \int_a^b \dots dx + \dots$$

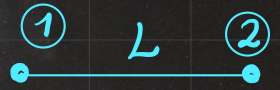
PIECEWISE INTEGRALS (SOLUTIONS)

ASSEMBLY

1D FEM

Overview and Wrap-up

FROM STRONG FORM TO ELEMENT STIFFNESS ← IN PHYSICAL SPACE NOW



$EAu''=0$ SUBJECT TO BCs

MULTIPLY BY TEST FUNCTION w

- DIRICHLET → u IS PRESCRIBED
- NEUMANN → u' IS PRESCRIBED

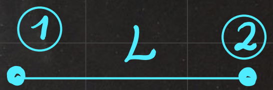
$EAwu''=0 \quad \leftarrow \quad wu'' = (wu')' - w'u'$

$EA\left[(wu')' - w'u'\right] = 0 \Rightarrow EA\,w'u' = EA\,(wu')' \quad \leftarrow$ INTEGRATE

$$\int_L EA\,w'u'\,dx = \int_L EA\,(wu')'\,dx = EA\,wu'\Big]_{1}^{2} = EA\,w^2u'^2 - EA\,w^1u'^1$$

FROM STRONG FORM TO ELEMENT STIFFNESS ← IN PHYSICAL SPACE now

$EA\,u'' = 0$ SUBJECT TO BCs



$$\int_L EA\, w' u'\, dx = \int_L EA\,(w u')'\, dx = EA\, w u' \Big]_{①}^{②} = EA\, w^2 u'^2 - EA\, w^1 u'^1$$

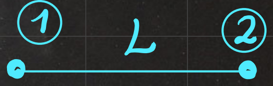
$$u = N^i u^i \Rightarrow u' = N^{i\prime} u^i \qquad w = N^j w^j \Rightarrow w' = N^{j\prime} w^j$$

$$EA\,u' = EA\varepsilon = A\,E\varepsilon = A\sigma = F$$

$$\int_L EA\, N^{j\prime} w^j N^{i\prime} u^i\, dx = \underbrace{EA\, w^2 u'^2}_{F^2} - \underbrace{EA\, w^1 u'^1}_{-F^1} = w^2 F^2 + w^1 F^1$$

FROM STRONG FORM TO ELEMENT STIFFNESS IN PHYSICAL SPACE NOW

$EA\,u'' = 0$ SUBJECT TO BCs



$$\int_L EA\, N^{j\prime} w^j N^{i\prime} u^i \, dx = w^2 F^2 + w^1 F^1$$

$$w^j \left[\int_L EA\, N^{j\prime} N^{i\prime} \, dx \right] u^i = w^2 F^2 + w^1 F^1$$

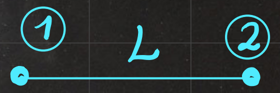
w: ARBITRARY

$$\left[\int_L EA\, N^{1\prime} N^{i\prime} \, dx \right] u^i = F^1 \quad \Leftarrow \quad \text{e.g. } w^1 = 1,\ w^2 = 0$$

$$\left[\int_L EA\, N^{2\prime} N^{i\prime} \, dx \right] u^i = F^2 \quad \Leftarrow \quad \text{e.g. } w^1 = 0,\ w^2 = 1$$

FROM STRONG FORM TO ELEMENT STIFFNESS ← IN PHYSICAL SPACE NOW

$EAu'' = 0$ SUBJECT TO BCs



$$\left[\int_L EA\, N^{1\prime} N^{i\prime}\, dx\right] u^i = F^1$$

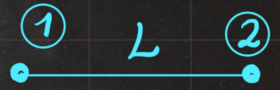
$$\left[\int_L EA\, N^{2\prime} N^{i\prime}\, dx\right] u^i = F^2$$

$$\int_L EA\, N^{1\prime} N^{1\prime}\, dx\; u^1 + \int_L EA\, N^{1\prime} N^{2\prime}\, dx\; u^2 = F^1$$

$$\int_L EA\, N^{2\prime} N^{1\prime}\, dx\; u^1 + \int_L EA\, N^{2\prime} N^{2\prime}\, dx\; u^2 = F^2$$

FROM STRONG FORM TO ELEMENT STIFFNESS ← IN PHYSICAL SPACE NOW

$EA\,u'' = 0$ SUBJECT TO BCs



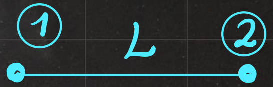
$$\int_L EA\, N^{1\prime} N^{1\prime}\, dx\; u^1 + \int_L EA\, N^{1\prime} N^{2\prime}\, dx\; u^2 = F^1$$

$$\int_L EA\, N^{2\prime} N^{1\prime}\, dx\; u^1 + \int_L EA\, N^{2\prime} N^{2\prime}\, dx\; u^2 = F^2$$

$$EA \begin{bmatrix} \int_L N^{1\prime} N^{1\prime}\, dx & \int_L N^{1\prime} N^{2\prime}\, dx \\ \int_L N^{2\prime} N^{1\prime}\, dx & \int_L N^{2\prime} N^{2\prime}\, dx \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} = \begin{bmatrix} F^1 \\ F^2 \end{bmatrix}$$

FROM STRONG FORM TO ELEMENT STIFFNESS ← IN PHYSICAL SPACE NOW

$EA\,u'' = 0$ SUBJECT TO BCs



$$EA \begin{bmatrix} \int_L N^{1\prime} N^{1\prime}\, dx & \int_L N^{1\prime} N^{2\prime}\, dx \\ \int_L N^{2\prime} N^{1\prime}\, dx & \int_L N^{2\prime} N^{2\prime}\, dx \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} = \begin{bmatrix} F^1 \\ F^2 \end{bmatrix}$$

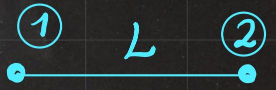
↳ NEXT, WE WRITE $x = x(\xi)$ USING SHAPE FUNCTIONS.

(ISOPARAMETRIC CONCEPT)

↳ ALTERNATIVELY, WE COULD HAVE WRITTEN THE SHAPE FUNCTIONS IN NATURAL SPACE ξ FROM SCRATCH!

FROM STRONG FORM TO ELEMENT STIFFNESS ← IN PHYSICAL SPACE NOW

$EA\,u'' = 0$ SUBJECT TO BCs



$$EA \begin{bmatrix} \int_L N^{1\prime} N^{1\prime}\, dx & \int_L N^{1\prime} N^{2\prime}\, dx \\ \int_L N^{2\prime} N^{1\prime}\, dx & \int_L N^{2\prime} N^{2\prime}\, dx \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} = \begin{bmatrix} F^1 \\ F^2 \end{bmatrix}$$

$$\begin{bmatrix} K^{11} & K^{12} \\ K^{21} & K^{22} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} = \begin{bmatrix} F^1 \\ F^2 \end{bmatrix} \leftarrow K^{ij} = EA \int_L N^{i\prime} N^{j\prime}\, dx$$

$$K^{ij} = EA\int_L N^{i\prime} N^{j\prime}\,dx = EA\int_L \frac{\partial N^i}{\partial x}\frac{\partial N^j}{\partial x}\,dx \quad J\,d\xi$$

$$x = x(\xi) \Rightarrow dx = \frac{\partial x}{\partial \xi}\,d\xi \Rightarrow dx = J\,d\xi \quad \text{with } J = \frac{\partial x}{\partial \xi}$$

$$= EA\int_{-1}^{1} \frac{\partial N^i}{\partial x}\frac{\partial N^j}{\partial \xi}\,J\,d\xi \qquad \frac{\partial N^i}{\partial x} = \frac{\partial N^i}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{\partial N^i}{\partial \xi}J^{-1}$$

$$= EA\int_{-1}^{1} \frac{\partial N^i}{\partial \xi}\frac{\partial N^j}{\partial \xi}\,J^{-1}\underbrace{J^{-1}J}_{1}\,d\xi \qquad \frac{\partial N^j}{\partial x} = \frac{\partial N^j}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{\partial N^j}{\partial \xi}J^{-1}$$

$$K^{ij} = EA \int_L N^{i\prime} N^{j\prime} \, dx$$ PHYSICAL

RECALL:

$$\int_{-1}^{1} g(\xi)\, d\xi = \sum_{gp=1}^{GPE} g(\xi_{gp})\, \alpha_{gp}$$

NATURAL

$$= EA \int_{-1}^{1} \frac{\partial N^i}{\partial \xi} \frac{\partial N^j}{\partial \xi} J^{-1} \, d\xi$$

Loop over gp

⋮

END

$$= EA \sum_{gp=1}^{GPE} \left\{ \left[\frac{\partial N^i}{\partial \xi} \frac{\partial N^j}{\partial \xi} J^{-1} \right]_{gp} \times \alpha_{gp} \right\}$$

e.g. in MATLAB

WHAT YOU SEE IN THE CODE!

For gp=1:GPE

⋮

End

2D FEM

FINITE ELEMENT METHOD

Differential Equation ✱

STRONG FORM

Strong to Weak Form

WEAK FORM

Weak to Approximate Form

APPROXIMATE FORM

From Physical to Natural space

NUMERICAL EVALUATION (Integration)

Approximate Solution to Differential Equation ✱

DISCRETIZED FORM

APPROXIMATION TECHNIQUES

SHAPE FUNCTIONS

ROADMAP FOR FEM

1D

2D

MATHEMATICAL PRELIMINARIES

EINSTEIN SUMMATION CONVENTION

A little definition for notation convenience

A REPEATED INDEX TWICE MEANS SUMMATION OVER THAT INDEX

also, called "dummy index"

$$\sum_{i=1}^{3} u_i v_i = u_1 v_1 + u_2 v_2 + u_3 v_3 \equiv u_i v_i$$

i is summation index

i: free index

$$\sum_{j=1}^{3} A_{ij} u_j \Rightarrow \begin{cases} i=1 \Rightarrow A_{11} u_1 + A_{12} u_2 + A_{13} u_3 \\ i=2 \Rightarrow A_{21} u_1 + A_{22} u_2 + A_{23} u_3 \\ i=3 \Rightarrow A_{31} u_1 + A_{32} u_2 + A_{33} u_3 \end{cases} \Rightarrow \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \equiv A_{ij} u_j$$

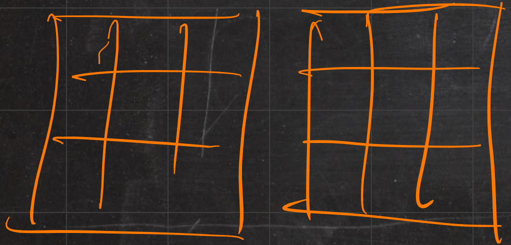
$1 \leq i \leq 3$

j: summation index

$$\mathbb{u} \circ \mathbb{v} = u_x v_x + u_y v_y + u_z v_z = u_1 v_1 + u_2 v_2 + u_3 v_3 = \sum_{i=1}^{3} u_i v_i = u_i v_i$$

$$\underset{3\times3}{[\mathbb{A}]}\,\underset{3\times3}{[\mathbb{B}]} = \underset{3\times3}{[\mathbb{C}]} \Rightarrow A_{ij} B_{jk} = C_{ik} \leftarrow i=1,\ k=2$$

$$\Rightarrow A_{1j} B_{j2} = C_{12}$$



$$\hookrightarrow A_{11} B_{12} + A_{12} B_{22} + A_{13} B_{32}$$

$$\mathbb{u} \cdot \mathbb{v} = u_x v_x + u_y v_y + u_z v_z = u_1 v_1 + u_2 v_2 + u_3 v_3 = \sum_{i=1}^{3} u_i v_i = u_i v_i$$

↳ Dot Product $(\mathbb{u}, \mathbb{v}) \mapsto$ SCALAR $\leftarrow u_i v_i \leftarrow \mathbb{u} \cdot \mathbb{v}$

Double Dot Product $(\mathbb{A}, \mathbb{B}) \mapsto$ SCALAR $\leftarrow A_{ij} B_{ij} \leftarrow \mathbb{A} : \mathbb{B}$

↳ $A_{11} B_{11} + A_{12} B_{12} + \dots + A_{32} B_{32} + A_{33} B_{33}$

$$\mathbb{u} \cdot \mathbb{v} = u_x v_x + u_y v_y + u_z v_z = u_1 v_1 + u_2 v_2 + u_3 v_3 = \sum_{i=1}^{3} u_i v_i = u_i v_i$$

$\hookrightarrow$ Dot Product $(\mathbb{u}, \mathbb{v}) \rightsquigarrow$ SCALAR $\leftarrow u_i v_i \leftarrow \mathbb{u} \cdot \mathbb{v}$

Double Dot Product $(\mathbb{A}, \mathbb{B}) \rightsquigarrow$ SCALAR $\leftarrow A_{ij} B_{ij} \leftarrow \mathbb{A} : \mathbb{B}$

$$\mathbb{u} \otimes \mathbb{v} \dashrightarrow \begin{bmatrix} u_1 v_1 & u_1 v_2 & u_1 v_3 \\ u_2 v_1 & u_2 v_2 & u_2 v_3 \\ u_3 v_1 & u_3 v_2 & u_3 v_3 \end{bmatrix} \qquad e_1 \otimes e_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\hookrightarrow$ Dyadic Product $(\mathbb{u}, \mathbb{v}) \rightsquigarrow$ MATRIX $\leftarrow u_i v_j \leftarrow [\mathbb{u} \otimes \mathbb{v}]_{ij}$

$$\mathbb{u} \cdot \mathbb{v} = u_x v_x + u_y v_y + u_z v_z = u_1 v_1 + u_2 v_2 + u_3 v_3 = \sum_{i=1}^{3} u_i v_i = u_i v_i$$

$\hookrightarrow$ Dot Product $(\mathbb{u}, \mathbb{v}) \rightsquigarrow$ SCALAR $\leftarrow u_i v_i \leftarrow \mathbb{u} \cdot \mathbb{v}$

Double Dot Product $(\mathbb{A}, \mathbb{B}) \rightsquigarrow$ SCALAR $\leftarrow A_{ij} B_{ij} \leftarrow \mathbb{A} : \mathbb{B}$

$\mathbb{u} \otimes \mathbb{v}$ Dyadic Product $(\mathbb{u}, \mathbb{v}) \rightsquigarrow$ MATRIX $\leftarrow u_i v_j \leftarrow [\mathbb{u} \otimes \mathbb{v}]_{ij}$

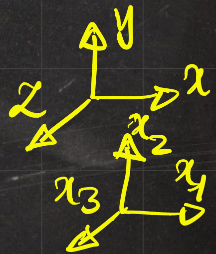
KRONECKER DELTA $\rightsquigarrow \delta_{ij} = \Phi_i \cdot \Phi_j \Rightarrow \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

$\mathbb{u}\cdot\mathbb{v} = u_i v_i$ $\qquad [\mathbb{A}\cdot\mathbb{B}]_{ik} = A_{ij} B_{jk}$ $\qquad [\mathbb{u}\otimes\mathbb{v}]_{ij} = u_i v_j$

$\delta_{ij} = \mathbb{e}_i\cdot\mathbb{e}_j$ $\qquad [\mathbb{A}\cdot\mathbb{u}]_i = A_{ij} u_j$ $\qquad \mathbb{A}:\mathbb{B} = A_{ij} B_{ij}$

$\mathbb{E}$ ⟶ FOURTH-ORDER TENSOR (ARRAY) ⟶ $3\times3\times3\times3 = 81$ Components

$$\underbrace{\boldsymbol{\sigma}}_{\text{2nd.}} = \underbrace{\mathbb{E}}_{\text{4th.}} : \underbrace{\boldsymbol{\varepsilon}}_{\text{2nd.}} \quad \rightsquigarrow \quad [\boldsymbol{\sigma}]_{ij} = [\mathbb{E}]_{ijkl}\,[\boldsymbol{\varepsilon}]_{kl}$$



INSTEAD OF x, y, z ⟶ $1, 2, 3$ ⟶ x_1, x_2, x_3 $\qquad \mathbb{e}_x \rightsquigarrow \mathbb{e}_1$

DERIVATIVES $\leftarrow$ e.g. STRONG FORM $\rightsquigarrow u''$

$$y = f(x) \rightsquigarrow y' = f'(x) \leftarrow \{\circ\}' = \frac{\partial \{\circ\}}{\partial x}$$

$$u = u(x, y, z) \rightsquigarrow \text{GRAD } u \quad \text{OR} \quad \text{DIV } u \quad \text{OR} \quad \text{CURL } u$$

$$\frac{\partial u}{\partial x_i} \otimes e_i \qquad \frac{\partial u}{\partial x_i} \cdot e_i \qquad \frac{\partial u}{\partial x_i} \times e_i$$

DERIVATIVES → e.g. STRONG FORM ⟶ u''

$$y = f(x) \longrightarrow y' = f'(x) \quad \rightarrow \quad \{\circ\}' = \frac{\partial\{\circ\}}{\partial x}$$

$$u = u(x,y,z) \longrightarrow \text{GRAD } u \quad \text{OR} \quad \text{DIV } u$$

$$\omega = f(x,y,z) \longrightarrow \text{GRAD } f \rightarrow \frac{\partial f}{\partial x_i} e_i$$

DERIVATIVES $\rightarrow$ e.g. STRONG FORM $\rightarrow$ u''

$$y = f(x) \rightarrow y' = f'(x) \quad \{\circ\}' = \frac{\partial \{\circ\}}{\partial x}$$

$$\boldsymbol{u} = \boldsymbol{u}(x,y,z) \rightarrow \text{GRAD}\,\boldsymbol{u} \text{ OR } \text{DIV}\,\boldsymbol{u}$$

$$\rightarrow \text{GRAD}\,\boldsymbol{u} = \frac{\partial \boldsymbol{u}}{\partial x_i} \otimes \boldsymbol{e}_i$$

$$\rightarrow \text{DIV}\,\boldsymbol{u} = \frac{\partial \boldsymbol{u}}{\partial x_i} \cdot \boldsymbol{e}_i$$

$$\omega = f(x,y,z) \rightarrow \text{GRAD}\,f \quad \frac{\partial f}{\partial x_i}\boldsymbol{e}_i$$

$$\mathbb{A} = \mathbb{A}(x,y,z) \rightarrow \text{GRAD}\,\mathbb{A} \text{ OR } \text{DIV}\,\mathbb{A}$$

$$\hookrightarrow \frac{\partial \mathbb{A}}{\partial x_i} \otimes \boldsymbol{e}_i \qquad \hookrightarrow \frac{\partial \mathbb{A}}{\partial x_i} \cdot \boldsymbol{e}_i$$

SCALAR VALUED FUNCTION $\phi(\boldsymbol{x}) \approx \phi(x_1, x_2, x_3)$

$\uparrow \phi(x, y, z)$

$$\mathrm{GRAD}\,\phi = \frac{\partial \phi}{\partial x_i} \boldsymbol{e}_i$$

$$= \frac{\partial \phi}{\partial x_1} \boldsymbol{e}_1 + \frac{\partial \phi}{\partial x_2} \boldsymbol{e}_2 + \frac{\partial \phi}{\partial x_3} \boldsymbol{e}_3$$

$$= \frac{\partial \phi}{\partial x} \boldsymbol{e}_x + \frac{\partial \phi}{\partial y} \boldsymbol{e}_y + \frac{\partial \phi}{\partial z} \boldsymbol{e}_z$$

$$\mathrm{GRAD}\,\phi = \begin{bmatrix} \dfrac{\partial \phi}{\partial x} \\ \dfrac{\partial \phi}{\partial y} \\ \dfrac{\partial \phi}{\partial z} \end{bmatrix}$$

SCALAR VALUED FUNCTION $\phi(\mathbf{x}) \rightsquigarrow \phi(x_1, x_2, x_3)$

$\phi(x, y, z)$

$$\mathrm{GRAD}\,\phi = \frac{\partial \phi}{\partial x_i}\,\mathbf{e}_i$$

$y = f(x) \rightsquigarrow f' = \frac{dy}{dx}$

$\Rightarrow dy = f'\,dx$

$$= \frac{\partial \phi}{\partial x_1}\,\mathbf{e}_1 + \frac{\partial \phi}{\partial x_2}\,\mathbf{e}_2 + \frac{\partial \phi}{\partial x_3}\,\mathbf{e}_3$$

$$\mathrm{GRAD}\,\phi = \begin{bmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \\ \frac{\partial \phi}{\partial z} \end{bmatrix}$$

$$= \frac{\partial \phi}{\partial x}\,\mathbf{e}_x + \frac{\partial \phi}{\partial y}\,\mathbf{e}_y + \frac{\partial \phi}{\partial z}\,\mathbf{e}_z$$

$\Rightarrow d\phi = \mathrm{GRAD}\,\phi \cdot d\mathbf{x}$

VECTOR-VALUED FUNCTION $\boldsymbol{u}(\boldsymbol{x})$

$$\begin{bmatrix} u_1(x_1, x_2, x_3) \\ u_2(x_1, x_2, x_3) \\ u_3(x_1, x_2, x_3) \end{bmatrix}$$

$$\boldsymbol{u} = u_j \boldsymbol{e}_j = u_1 \boldsymbol{e}_1 + \cdots$$

$$\text{GRAD } \boldsymbol{u} = \frac{\partial \boldsymbol{u}}{\partial x_i} \otimes \boldsymbol{e}_i$$

$$= \frac{\partial (u_j \boldsymbol{e}_j)}{\partial x_i} \otimes \boldsymbol{e}_i$$

$$= \frac{\partial u_j}{\partial x_i} \boldsymbol{e}_j \otimes \boldsymbol{e}_i$$

$\boldsymbol{e}_2 \otimes \boldsymbol{e}_1$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\partial u_2 / \partial x_1$

VECTOR-VALUED FUNCTION $\boldsymbol{u}(\boldsymbol{x}) \rightarrow \begin{bmatrix} u_1(x_1, x_2, x_3) \\ u_2(x_1, x_2, x_3) \\ u_3(x_1, x_2, x_3) \end{bmatrix}$

$$\text{GRAD}\, \boldsymbol{u} = \frac{\partial \boldsymbol{u}}{\partial x_i} \otimes \boldsymbol{e}_i$$

$$= \frac{\partial (u_j \boldsymbol{e}_j)}{\partial x_i} \otimes \boldsymbol{e}_i$$

$$= \frac{\partial u_j}{\partial x_i} \boldsymbol{e}_j \otimes \boldsymbol{e}_i$$

$$\text{GRAD}\, \boldsymbol{u} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

VECTOR-VALUED FUNCTION $\mathbf{u}(\mathbf{x})$

$$\text{DIV}\,\mathbf{u} = \frac{\partial \mathbf{u}}{\partial x_i} \cdot \mathbf{e}_i$$

$$= \frac{\partial (u_j \mathbf{e}_j)}{\partial x_i} \cdot \mathbf{e}_i$$

$$= \frac{\partial u_j}{\partial x_i} \underbrace{\mathbf{e}_j \cdot \mathbf{e}_i}_{\delta_{ij}} = \frac{\partial u_i}{\partial x_i}$$

$$\text{GRAD}\,\mathbf{u} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

$$\text{DIV}\,\mathbf{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

MATRIX-VALUED FUNCTION $\mathbb{A}(\boldsymbol{x})$

$$\operatorname{Div} \mathbb{A} = \frac{\partial \mathbb{A}}{\partial x_i} \cdot \boldsymbol{e}_i$$

$$= \frac{\partial (A_{jk}\, \boldsymbol{e}_j \otimes \boldsymbol{e}_k)}{\partial x_i} \cdot \boldsymbol{e}_i$$

$$= \frac{\partial A_{jk}}{\partial x_i}\, \boldsymbol{e}_j \underbrace{\boldsymbol{e}_k \cdot \boldsymbol{e}_i}_{\delta_{ki}} = \frac{\partial A_{ji}}{\partial x_i}\, \boldsymbol{e}_j$$

$$\rightsquigarrow [\operatorname{Div} \mathbb{A}]_j = \frac{\partial A_{ji}}{\partial x_i}$$

$$\operatorname{Div} \mathbb{A} = \begin{bmatrix} \frac{\partial A_{11}}{\partial x_1} + \frac{\partial A_{12}}{\partial x_2} + \frac{\partial A_{13}}{\partial x_3} \\ \frac{\partial A_{21}}{\partial x_1} + \frac{\partial A_{22}}{\partial x_2} + \frac{\partial A_{23}}{\partial x_3} \\ \frac{\partial A_{31}}{\partial x_1} + \frac{\partial A_{32}}{\partial x_2} + \frac{\partial A_{33}}{\partial x_3} \end{bmatrix}$$

SCALAR → 0, VECTOR → 1, MATRIX → 2

GRADIENT INCREASES THE ORDER BY 1

DIVERGENCE REDUCES THE ORDER BY 1

$$\text{GRAD } \phi = \begin{bmatrix} \partial\phi/\partial x_1 \\ \partial\phi/\partial x_2 \\ \partial\phi/\partial x_3 \end{bmatrix}$$

$$\text{GRAD } \boldsymbol{u} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

$$\text{DIV } \boldsymbol{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

$$\text{DIV } \boldsymbol{A} = \begin{bmatrix} \frac{\partial A_{11}}{\partial x_1} + \frac{\partial A_{12}}{\partial x_2} + \frac{\partial A_{13}}{\partial x_3} \\ \frac{\partial A_{21}}{\partial x_1} + \frac{\partial A_{22}}{\partial x_2} + \frac{\partial A_{23}}{\partial x_3} \\ \frac{\partial A_{31}}{\partial x_1} + \frac{\partial A_{32}}{\partial x_2} + \frac{\partial A_{33}}{\partial x_3} \end{bmatrix}$$

STRONG FORM $\rightsquigarrow$ $EAu'' + b = 0$ force/[m] $\checkmark$ 1D force density

1D

$Eu'' + \dfrac{b}{A} = 0$ force/$[m]^3$ $\checkmark$ 3D force density

EAu''

$= (EAu')'$

$u'' = (u')' = \varepsilon'$

$\underbrace{(E\varepsilon)'}_{\sigma'} + \dfrac{b}{A} = 0 \rightsquigarrow \sigma' + b_{3D} = 0$

$\leadsto b_{3D}$

STRONG FORM $\rightsquigarrow$ $\boldsymbol{\sigma}' + \mathbf{b} = \boldsymbol{0}$ $\leftarrow$ 3D & 2D & 1D STRONG FORM

$$\sigma' + b_{3D} = 0$$

$$\Rightarrow \operatorname{Div} \boldsymbol{\sigma} + \mathbf{b} = \boldsymbol{0}$$

STRONG FORM (GENERIC FORM) $\rightsquigarrow$ $\mathrm{Div}\,\mathbb{G} + \mathbb{b} = 0$

Big Picture of Mechanics (Mechanical Problems & Thermal Problems)

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\text{Def. } \mathbf{u} \arrow[r, leftrightarrow, "?"] \arrow[d, leftrightarrow, "\boldsymbol{\varepsilon} = \mathrm{Grad}^{\mathrm{sym}}\, \mathbf{u}"] & \text{Load. } \mathbf{t} \arrow[d, leftrightarrow, "\text{Cauchy: } \boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{t}"] \\
\text{Strain } \boldsymbol{\varepsilon} \arrow[r, leftrightarrow, "\text{Const.}"] & \text{Stress } \boldsymbol{\sigma}
\end{tikzcd}
$$

$$\boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\varepsilon}$$

Hooke's law

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\text{Temp. } \theta \arrow[r, leftrightarrow, "?"] \arrow[d, leftrightarrow, "\mathbf{G} = \mathrm{Grad}\, \theta"] & \text{Heat Flux } q_n \arrow[d, leftrightarrow, "\text{Cauchy: } \mathbf{q} \cdot \mathbf{n} = q_n"] \\
\text{Temp. Gradient } \mathbf{G} \arrow[r, leftrightarrow, "\text{Const.}"] & \text{Heat Flux Vector } \mathbf{q}
\end{tikzcd}
$$

$$\mathbf{q} = -\mathbb{K} \cdot \mathbf{G}$$

Fourier's law

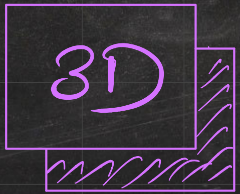
Strong Form (Generic Form) $\rightarrow$ $\mathrm{Div}\,\boldsymbol{\sigma} + \mathbf{b} = \mathbf{0}$, $\mathrm{Div}\,\mathbf{q} + c = 0$

$$\frac{\partial \sigma_{jk}}{\partial x_k} + b_j = 0$$

$$\frac{\partial q_i}{\partial x_i} + c = 0$$

$$\left\{\begin{aligned} &\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + b_x = 0 \\ &\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + b_y = 0 \\ &\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + b_z = 0 \end{aligned}\right.$$

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} + c = 0$$



Strong Form (Generic Form) $\rightsquigarrow$ $\mathrm{Div}\,\boldsymbol{\sigma} + \mathbf{b} = \mathbf{0}$, $\mathrm{Div}\,\mathbf{q} + c = 0$

$$\frac{\partial \sigma_{jk}}{\partial x_k} + b_j = 0$$

$$\frac{\partial q_i}{\partial x_i} + c = 0$$

$$\begin{cases} \dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{xy}}{\partial y} + b_x = 0 \\[2ex] \dfrac{\partial \sigma_{yx}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} + b_y = 0 \end{cases}$$

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + c = 0$$

2D $\rightsquigarrow$ Plane Strain, Plane Stress $\Big\}$ $\boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\varepsilon}$