FINITE ELEMENT METHOD

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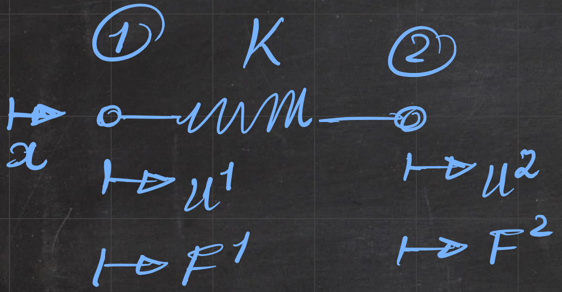
8

Ali Javili

Bilkent University

FINITE ELEMENT METHOD

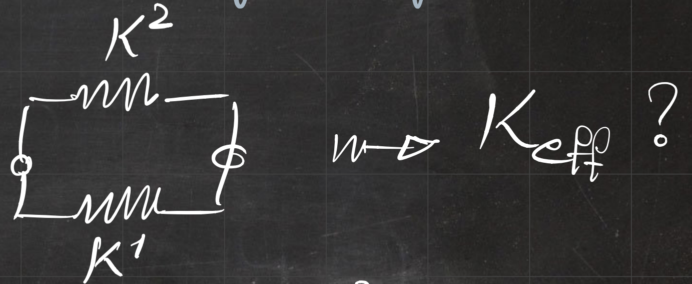
Understanding key ingredients of FEM using springs:



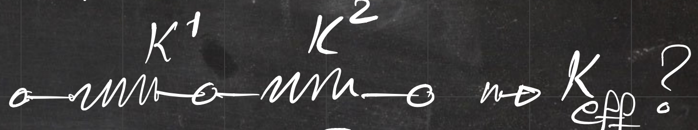
$$\begin{bmatrix} F^1 \\ F^2 \end{bmatrix} = \begin{bmatrix} K & -K \\ -K & K \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix}$$

$$\mathbb{F} = \mathbb{K} \cdot \mathbb{u}$$

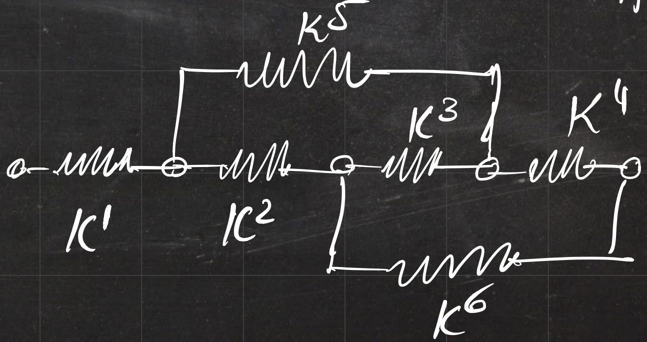
I)



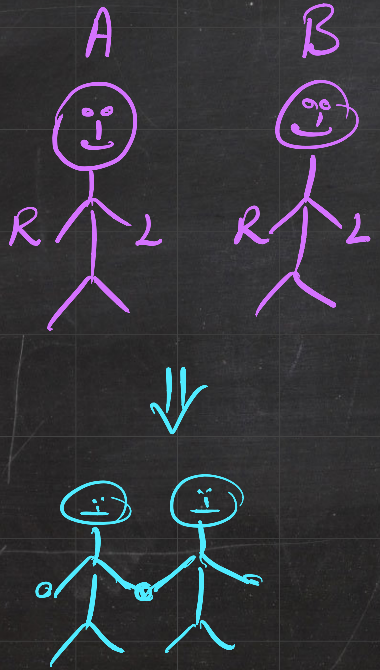
II)

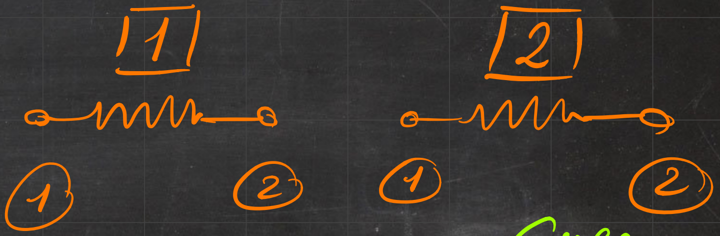


III)



TOWARDS AN ALGORITHMIC APPROACH TO ASSEMBLY:





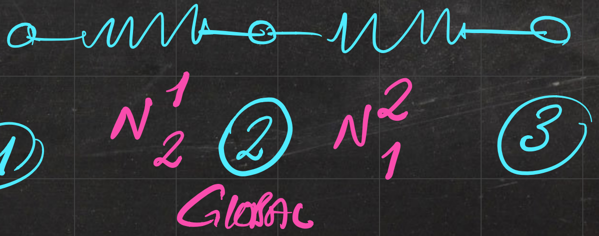
GLOBAL ELEMENT

Superscript: GLOBAL

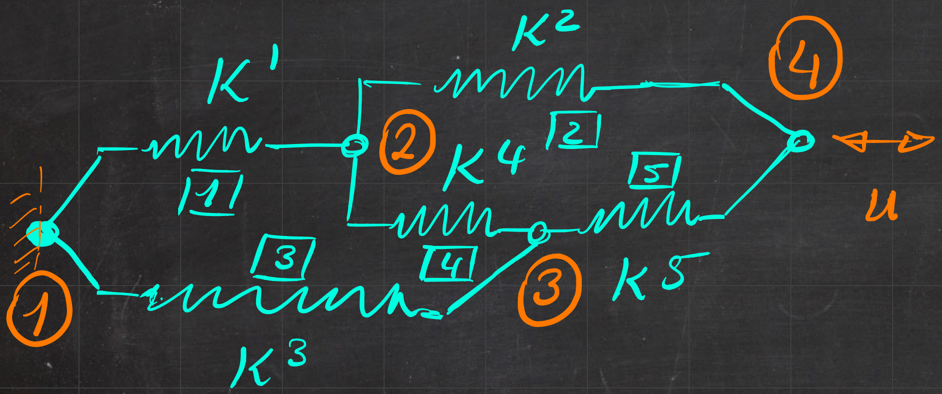
subscript: LOCAL

GLOBAL NODE

$$N^2 = N_2^1 = N_1^2$$



TOWARDS AN ALGORITHMIC APPROACH TO ASSEMBLY:



ELEMENT 5

$$\mathbb{K}^5 = \begin{bmatrix} K^5 & -K^5 \\ -K^5 & K^5 \end{bmatrix}$$

ELEMENT 1

$$\mathbb{K}^1 = \begin{bmatrix} K^1 & -K^1 \\ -K^1 & K^1 \end{bmatrix}$$

BOLD

ELEMENT 2

$$\mathbb{K}^2 = \begin{bmatrix} K^2 & -K^2 \\ -K^2 & K^2 \end{bmatrix}$$

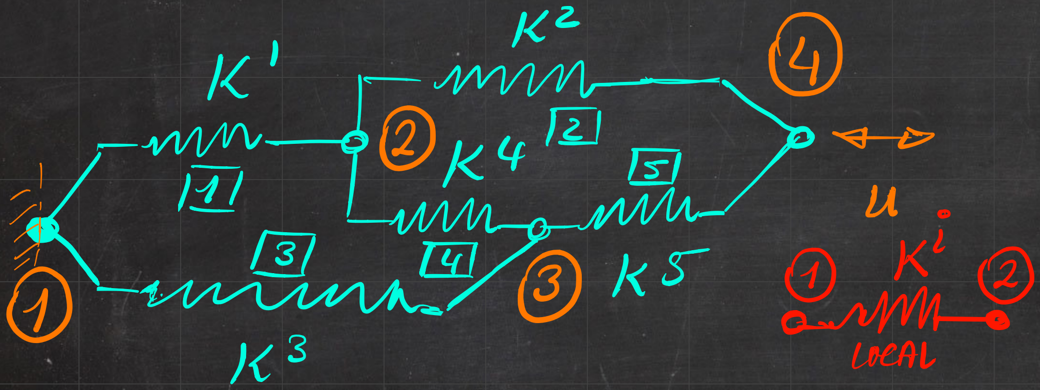
ELEMENT 3

$$\mathbb{K}^3 = \begin{bmatrix} K^3 & -K^3 \\ -K^3 & K^3 \end{bmatrix}$$

ELEMENT 4

$$\mathbb{K}^4 = \begin{bmatrix} K^4 & -K^4 \\ -K^4 & K^4 \end{bmatrix}$$

TOWARDS AN ALGORITHMIC APPROACH TO ASSEMBLY:



$$\mathbb{K}^5 = \begin{array}{cc} \begin{array}{cc} \textcircled{3} & \textcircled{4} \end{array} & \\ \begin{bmatrix} K^5 & -K^5 \\ -K^5 & K^5 \end{bmatrix} & \begin{array}{c} \textcircled{3} \\ \textcircled{4} \end{array} \end{array}$$

$$\mathbb{K}^1 = \begin{array}{cc} \begin{array}{cc} \textcircled{1} & \textcircled{2} \end{array} & \\ \begin{bmatrix} K^1 & -K^1 \\ -K^1 & K^1 \end{bmatrix} & \begin{array}{c} \textcircled{1} \\ \textcircled{2} \end{array} \end{array} \quad \mathbb{K}^2 = \begin{array}{cc} \begin{array}{cc} \textcircled{2} & \textcircled{4} \end{array} & \\ \begin{bmatrix} K^2 & -K^2 \\ -K^2 & K^2 \end{bmatrix} & \begin{array}{c} \textcircled{2} \\ \textcircled{4} \end{array} \end{array}$$

$$\mathbb{K}^3 = \begin{array}{cc} \begin{array}{cc} \textcircled{1} & \textcircled{3} \end{array} & \\ \begin{bmatrix} K^3 & -K^3 \\ -K^3 & K^3 \end{bmatrix} & \begin{array}{c} \textcircled{1} \\ \textcircled{3} \end{array} \end{array} \quad \mathbb{K}^4 = \begin{array}{cc} \begin{array}{cc} \textcircled{2} & \textcircled{3} \end{array} & \\ \begin{bmatrix} K^4 & -K^4 \\ -K^4 & K^4 \end{bmatrix} & \begin{array}{c} \textcircled{2} \\ \textcircled{3} \end{array} \end{array}$$

$$\mathbb{K}^4 = \begin{array}{c|cc|c} & \textcircled{2} & \textcircled{3} & \\ \hline & K^4 & -K^4 & \textcircled{2} \\ & -K^4 & K^4 & \textcircled{3} \end{array}$$

GLOBAL

$$\mathbb{K} = \begin{array}{cccc|c} \textcircled{1} & \textcircled{2} & \textcircled{3} & \textcircled{4} & \\ \hline K^1 + K^3 & -K^1 & -K^3 & 0 & \textcircled{1} \\ -K^1 & K^1 + K^2 + K^4 & -K^4 & -K^2 & \textcircled{2} \\ -K^3 & -K^4 & K^3 + K^4 + K^5 & -K^5 & \textcircled{3} \\ 0 & -K^2 & -K^5 & K^2 + K^5 & \textcircled{4} \end{array}$$

$$\mathbb{K}^5 = \begin{array}{cc|c} \textcircled{3} & \textcircled{4} & \\ \hline K^5 & -K^5 & \textcircled{3} \\ -K^5 & K^5 & \textcircled{4} \end{array}$$

$$\mathbb{K}^1 = \begin{array}{cc|c} \textcircled{1} & \textcircled{2} & \\ \hline K^1 & -K^1 & \textcircled{1} \\ -K^1 & K^1 & \textcircled{2} \end{array}$$

$$\mathbb{K}^2 = \begin{array}{cc|c} \textcircled{2} & \textcircled{4} & \\ \hline K^2 & -K^2 & \textcircled{2} \\ -K^2 & K^2 & \textcircled{4} \end{array}$$

$$\mathbb{K}^3 = \begin{array}{cc|c} \textcircled{1} & \textcircled{3} & \\ \hline K^3 & -K^3 & \textcircled{1} \\ -K^3 & K^3 & \textcircled{3} \end{array}$$

$\mathrm{DET}\ \mathbb{K}^{\mathrm{GLOBAL}} = 0$ $\qquad$ $\mathbb{K}^{\mathrm{GLOBAL}}$: SYM

$$\begin{bmatrix} F^1 \\ F^2 \\ F^3 \\ F^4 \end{bmatrix} = \begin{bmatrix} K^{11} & K^{12} & K^{13} & K^{14} \\ K^{21} & K^{22} & K^{23} & K^{24} \\ K^{31} & K^{32} & K^{33} & K^{34} \\ K^{41} & K^{42} & K^{43} & K^{44} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \\ u^4 \end{bmatrix}$$

$$\overset{\text{GLOBAL}}{\underset{\text{NoN x 1}}{\mathbb{F}}} = \overset{\text{GLOBAL}}{\underset{\text{NoN x NoN}}{\mathbb{K}}} \cdot \overset{\text{GLOBAL}}{\underset{\text{NoN x 1}}{\mathbb{u}}}$$

[PD x NoN] x 1

NoN ≡ NoN x PD

1D Problem Dimension

⇒ 4 EQ. & 4 Unknowns ⟵ BCs ?

NEUMANN BCs. ⟸ FORCE BASED ⟵ BCs ⟶ DISPLACEMENT BASED ⇒ DIRICHLET BCs.

$u^1 = 0$

$$\begin{bmatrix} F^1 \\ F^2 \\ F^3 \\ F^4 \end{bmatrix} = \begin{bmatrix} K^{11} & K^{12} & K^{13} & K^{14} \\ K^{21} & K^{22} & K^{23} & K^{24} \\ K^{31} & K^{32} & K^{33} & K^{34} \\ K^{41} & K^{42} & K^{43} & K^{44} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \\ u^4 \end{bmatrix}$$

$u^4 = u$

	Homogeneous ✓	Non Homogeneous
DIRICHLET ↳ Displacement	$=0$	$\neq 0$
NEUMANN ↳ Force	$=0$	$\neq 0$

$$= \begin{bmatrix} K^{11} \\ K^{21} \\ K^{31} \\ K^{41} \end{bmatrix} u^1 + \begin{bmatrix} K^{12} \\ K^{22} \\ K^{32} \\ K^{42} \end{bmatrix} u^2 + \begin{bmatrix} K^{13} \\ K^{23} \\ K^{33} \\ K^{43} \end{bmatrix} u^3 + \begin{bmatrix} K^{14} \\ K^{24} \\ K^{34} \\ K^{44} \end{bmatrix} u^4$$

4 EQN. & 4 Unknowns

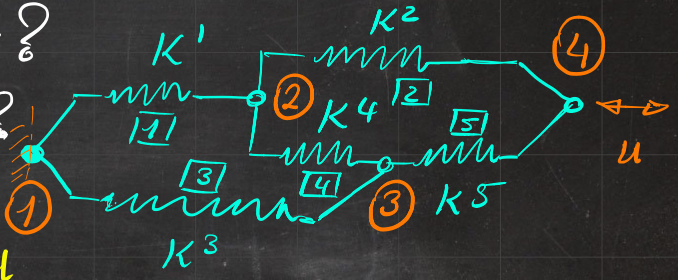
NEUMANN

NEUMANN

$$\begin{bmatrix} \cancelto{?}{F^1} \\ \cancelto{0}{F^2} \\ \cancelto{0}{F^3} \\ \cancelto{?}{F^4} \end{bmatrix} = \begin{bmatrix} K^{11} & K^{12} & K^{13} & K^{14} \\ K^{21} & K^{22} & K^{23} & K^{24} \\ K^{31} & K^{32} & K^{33} & K^{34} \\ K^{41} & K^{42} & K^{43} & K^{44} \end{bmatrix} \begin{bmatrix} \cancelto{0}{u^1} \\ \cancelto{?}{u^2} \\ \cancelto{?}{u^3} \\ \cancelto{u}{u^4} \end{bmatrix}$$

DIRICHLET (Prescribed Displacement)

DIRICHLET (Prescribed Displ.)



$$\begin{bmatrix} F^P \\ \hline F^u \end{bmatrix} = \left[\begin{array}{c|c} K^{Pu} & K^{PP} \\ \hline K^{uu} & K^{uP} \end{array}\right] \begin{bmatrix} u^u \\ \hline u^P \end{bmatrix}$$

NEUMANN

FREE NODES ⇒ DoF ← DEGREES OF FREEDOM

CONSTRAINED NODES ⇒ DoC ← DEGREES OF CONSTRAINT

DIRICHLET

4 EQN. & 4 Unknowns

$$\begin{array}{r} \\ \\ \text{NEUMANN} \\ \text{NEUMANN} \\ \end{array}\begin{bmatrix}\cancelto{?}{F^1}\\ \cancelto{0}{F^2}\\ \cancelto{0}{F^3}\\ \cancelto{?}{F^4}\end{bmatrix} = \begin{bmatrix}K^{11} & K^{12} & K^{13} & K^{14}\\ K^{21} & K^{22} & K^{23} & K^{24}\\ K^{31} & K^{32} & K^{33} & K^{34}\\ K^{41} & K^{42} & K^{43} & K^{44}\end{bmatrix}\begin{bmatrix}\cancelto{0}{u^1}\\ \cancelto{?}{u^2}\\ \cancelto{?}{u^3}\\ \cancelto{u}{u^4}\end{bmatrix}$$

$$[F^p] = [K^{pu}][u^u] + [K^{pp}][u^p]$$

$$\Downarrow$$

$$\underbrace{[\overset{\checkmark}{K^{pu}}]}_{A}\,\underbrace{[u^u]}_{x} = \underbrace{[\overset{\checkmark}{F^p}] - [\overset{\checkmark}{K^{pp}}][\overset{\checkmark}{u^p}]}_{b}$$

$$[x] = [A]^{-1}[b] \Leftarrow \overset{\checkmark}{A}\cdot\overset{?}{x} = \overset{\checkmark}{b}$$

$$\left[\begin{array}{c}F^p\\ \hline F^u\end{array}\right] = \left[\begin{array}{c|c}K^{pu} & K^{pp}\\ \hline K^{uu} & K^{up}\end{array}\right]\left[\begin{array}{c}u^u\\ \hline u^p\end{array}\right]$$

$$\underset{\text{FORCE}}{\left[\begin{array}{c}\text{DoF}\\ \hline \text{DoC}\end{array}\right]} = \underset{\text{STIFFNESS}}{\left[\begin{array}{c|c}\text{DoF}\times\text{DoF} & \text{DoF}\times\text{DoC}\\ \hline \text{DoC}\times\text{DoF} & \text{DoC}\times\text{DoC}\end{array}\right]}\;\underset{\text{DISPLACEMENT}}{\left[\begin{array}{c}\text{DoF}\\ \hline \text{DoC}\end{array}\right]}$$

4 EQN. & 4 Unknowns

$$\begin{array}{l} \\ \\ \text{NEUMANN} \\ \text{NEUMANN} \end{array} \begin{bmatrix} \cancel{F^1} \to ? \\ \cancel{F^2} \to 0 \\ \cancel{F^3} \to 0 \\ \cancel{F^4} \to ? \end{bmatrix} = \begin{bmatrix} K^{11} & K^{12} & K^{13} & K^{14} \\ K^{21} & K^{22} & K^{23} & K^{24} \\ K^{31} & K^{32} & K^{33} & K^{34} \\ K^{41} & K^{42} & K^{43} & K^{44} \end{bmatrix} \begin{bmatrix} \cancel{u^1} \to 0 \\ \cancel{u^2} \to ? \\ \cancel{u^3} \to ? \\ \cancel{u^4} \to u \end{bmatrix}$$

$$[F^p] = [K^{pu}][u^u] + [K^{pp}][u^p]$$

$$\Downarrow$$

$$\underbrace{\overset{\checkmark}{[K^{pu}]}}_{\text{REDUCED STIFFNESS}}[u^u] = \overset{\checkmark}{[F^p]} - \overset{\checkmark}{[K^{pp}]}\,\overset{\checkmark}{[u^p]}$$

$$\Rightarrow \underbrace{[u^u] = [K^{pu}]^{-1} \cdot \left\{ [F^p] - [K^{pp}][u^p] \right\}}_{\text{REDUCED SYSTEM}}$$

$$\underset{\text{DOF} \times \text{DOF}}{A} \cdot x = b$$

$$\left[\begin{array}{c} F^p \\ \hline F^u \end{array}\right] = \left[\begin{array}{c|c} K^{pu} & K^{pp} \\ \hline K^{uu} & K^{up} \end{array}\right] \left[\begin{array}{c} u^u \\ \hline u^p \end{array}\right]$$

4 EQN. & 4 Unknowns

$$\begin{array}{c} \\ \\ \text{NEUMANN} \\ \text{NEUMANN} \end{array}\begin{bmatrix} \cancel{F^1}\ ? \\ \cancel{F^2}\ 0 \\ \cancel{F^3}\ 0 \\ \cancel{F^4}\ ? \end{bmatrix} = \begin{bmatrix} K^{11} & K^{12} & K^{13} & K^{14} \\ K^{21} & K^{22} & K^{23} & K^{24} \\ K^{31} & K^{32} & K^{33} & K^{34} \\ K^{41} & K^{42} & K^{43} & K^{44} \end{bmatrix} \begin{bmatrix} \cancel{u^1}\ 0 \\ \cancel{u^2}\ ? \\ \cancel{u^3}\ ? \\ \cancel{u^4}\ u \end{bmatrix}$$

$$[F^P] = [K^{Pu}][u^u] + [K^{PP}][u^P]$$

$$\Downarrow$$

$$\underbrace{\overset{\checkmark}{[K^{Pu}]}[u^u] = \overset{\checkmark}{[F^P]} - \overset{\checkmark}{[K^{PP}]}\overset{\checkmark}{[u^P]}}_{\text{REDUCED SYSTEM}}$$

$$\Rightarrow [u^u] = [K^{Pu}]^{-1} \cdot \left\{ [F^P] - [K^{PP}][u^P] \right\}$$

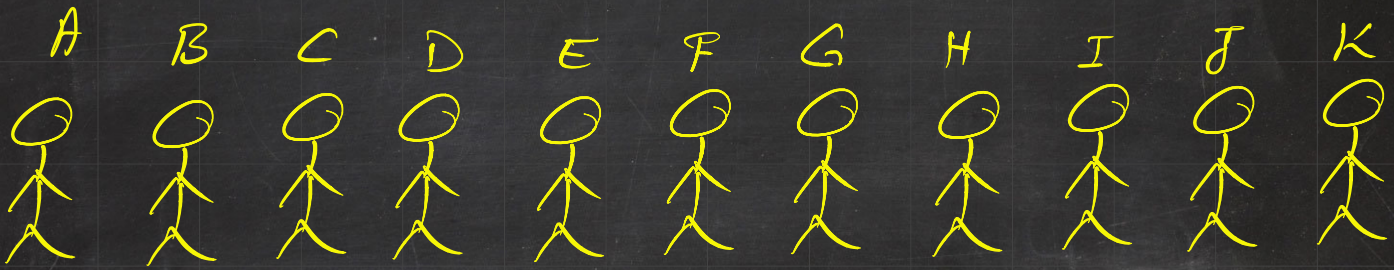
$$\begin{bmatrix} F^P \\ \hline F^u \end{bmatrix} = \left[\begin{array}{c|c} K^{Pu} & K^{PP} \\ \hline K^{uu} & K^{uP} \end{array}\right] \begin{bmatrix} u^u \\ \hline u^P \end{bmatrix}$$

$$\Rightarrow [F^u] = [K^{uu}][u^u] + [K^{uP}][u^P]$$

STATIC CONDENSATION ✓

EXTENDED NODE LIST

→ THE NAMING (NUMBERING) IS ARBITRARY!



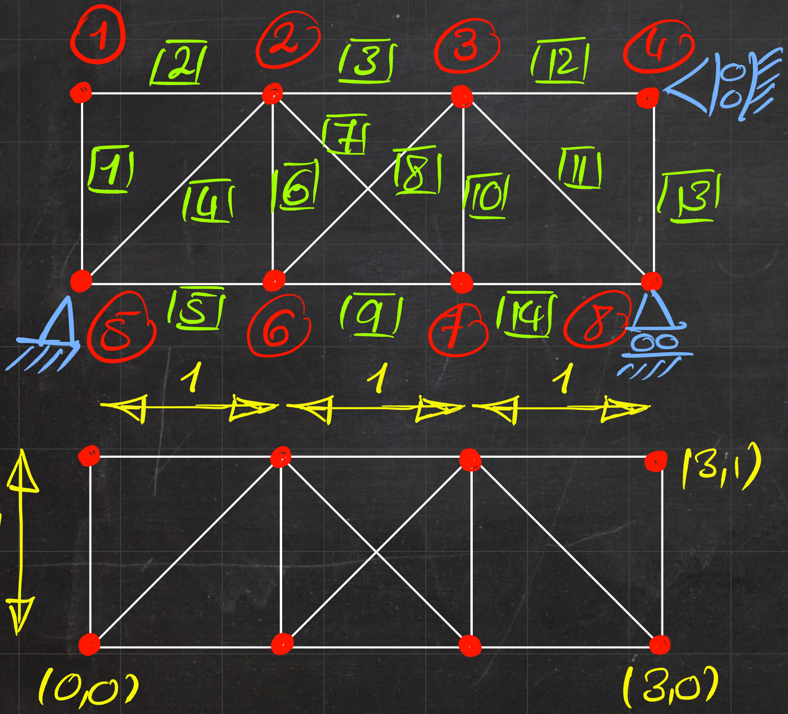
→ EVERY PERSON CAN SAY ONE WORD ← Programming: One Loop!

How many people?

How many right-handed? → Assign Degrees

How many left-handed? → Assign Degrees

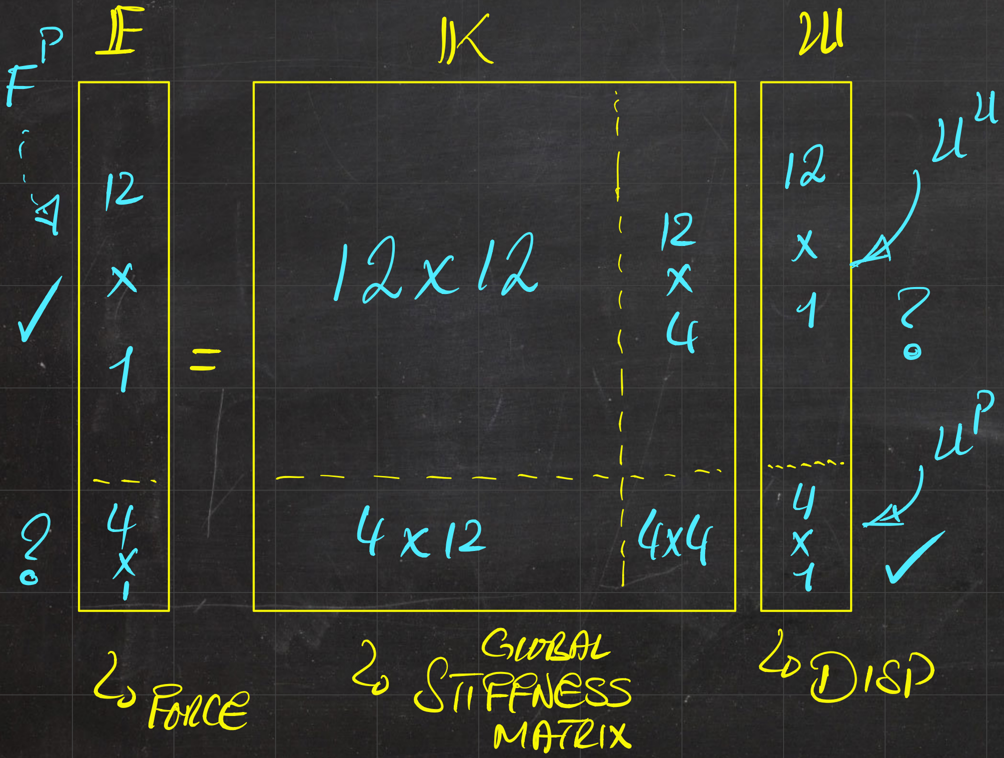
NODE LIST → EXTENDED NODE LIST   CONNECTIVITY



	x,y
NL	COOR
1	0,1
2	1,1
3	2,1
4	3,1
5	0,0
6	1,0
7	2,0
8	3,0

EL	
1	5,1
2	1,2
3	2,3
4	5,2
5	5,6
6	6,2
7	2,7
8	∘
9	∘
10	∘
11	∘
⋮	

EXTENDED NODE LIST

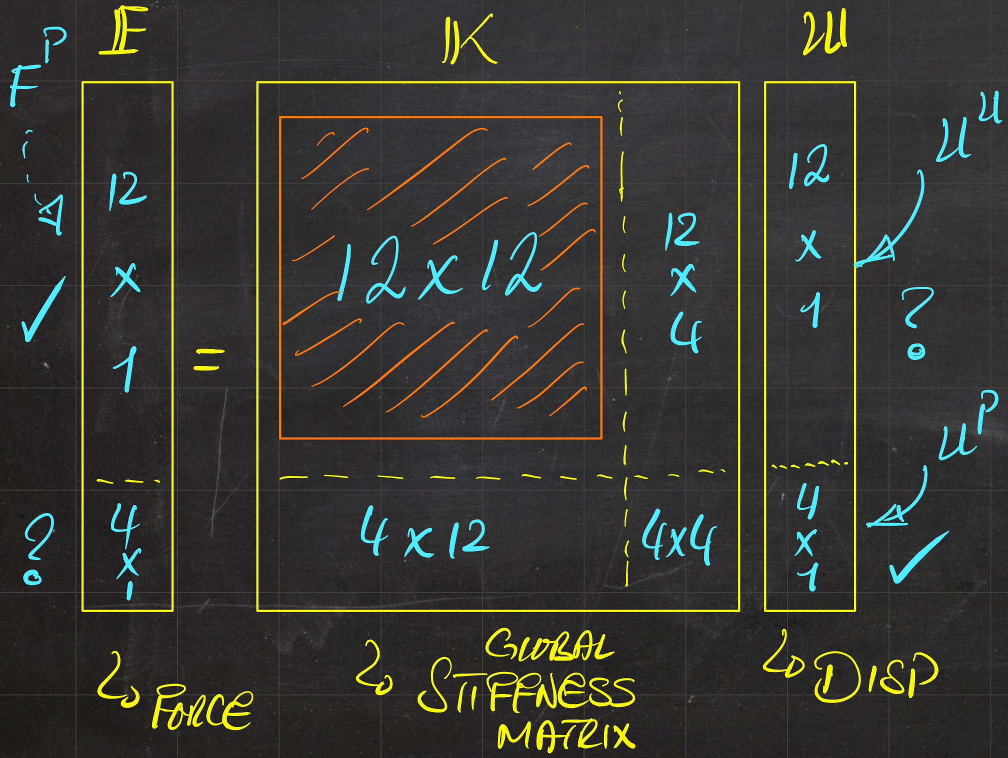


(GLOBAL) DEGREE

X	Y
1	2
3	4
5	6
13	7
14	15
8	9
10	11
12	16

12 DoF

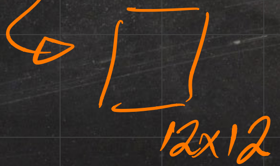
EXTENDED NODE LIST



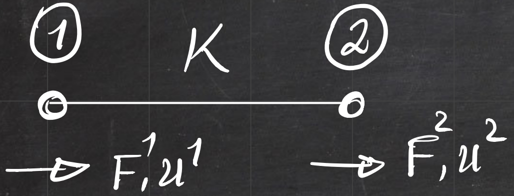
STATIC CONDENSATION

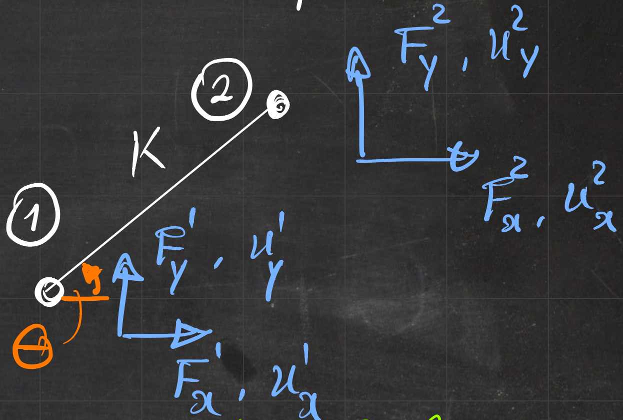
REDUCED SYSTEM

$$\mathbb{A} \cdot x = \mathbb{b}$$

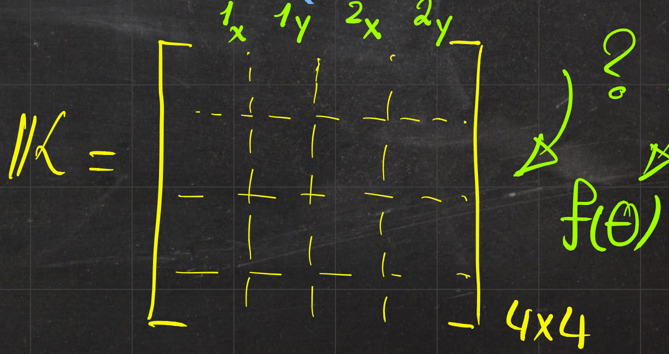


To compute stiffness of 1D element in 2D space

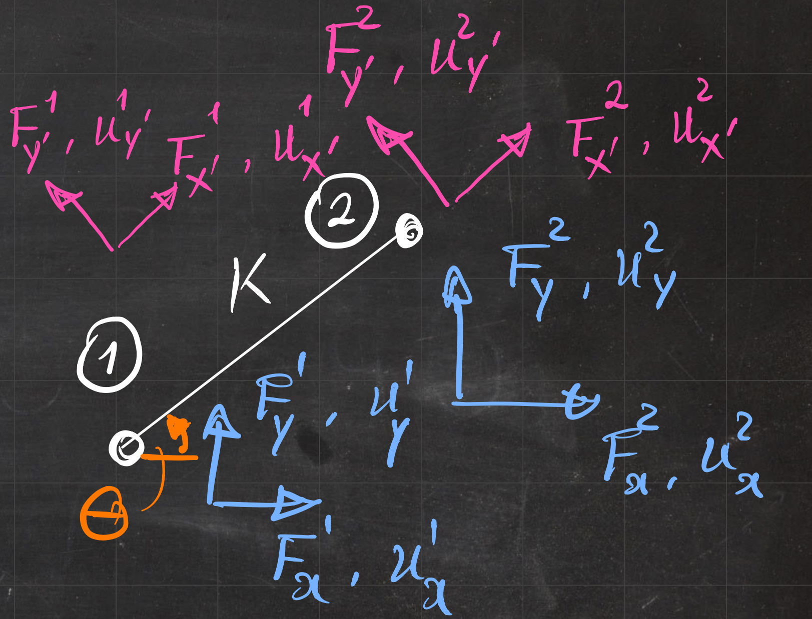




$$\begin{bmatrix}F^1\\F^2\end{bmatrix} = \begin{bmatrix}K & -K\\ -K & K\end{bmatrix}\begin{bmatrix}u^1\\u^2\end{bmatrix}$$



$$\begin{cases} \boldsymbol{e}_{x'} = \cos\theta\, \boldsymbol{e}_x + \sin\theta\, \boldsymbol{e}_y \\ \boldsymbol{e}_{y'} = -\sin\theta\, \boldsymbol{e}_x + \cos\theta\, \boldsymbol{e}_y \end{cases}$$

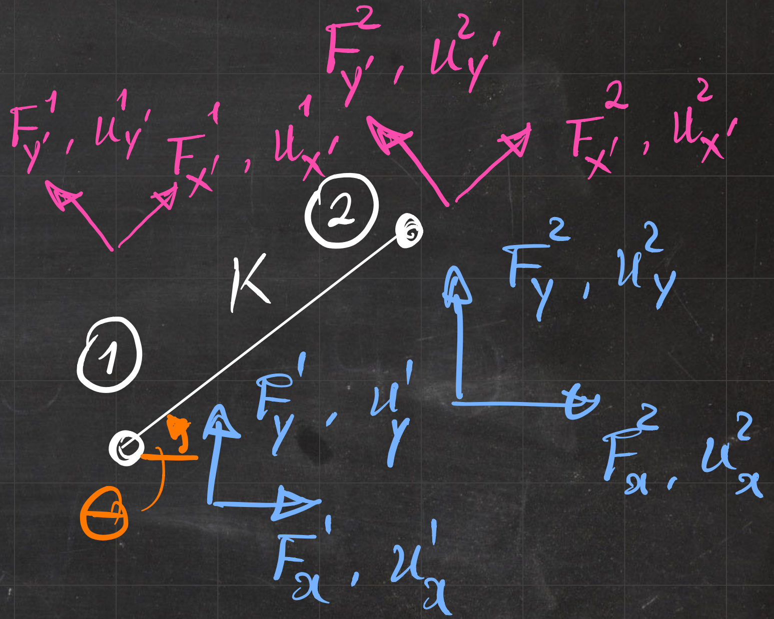


$$\boldsymbol{F} = F_{x'}\, \boldsymbol{e}_{x'} + F_{y'}\, \boldsymbol{e}_{y'}$$

$$= F_x\, \boldsymbol{e}_x + F_y\, \boldsymbol{e}_y \quad \ldots$$

$$\begin{bmatrix} F_x \\ F_y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ +\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} F_{x'} \\ F_{y'} \end{bmatrix} \Longleftrightarrow \begin{bmatrix} F_{x'} \\ F_{y'} \end{bmatrix} = \begin{bmatrix} \cos\theta & +\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix}$$

$$\begin{bmatrix} F^1_{x'} \\ F^1_{y'} \\ F^2_{x'} \\ F^2_{y'} \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix}}_{\mathbb{R}} \begin{bmatrix} F^1_x \\ F^1_y \\ F^2_x \\ F^2_y \end{bmatrix}$$



$$\begin{bmatrix} F_x \\ F_y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ +\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} F_{x'} \\ F_{y'} \end{bmatrix} \Longleftrightarrow \begin{bmatrix} F_{x'} \\ F_{y'} \end{bmatrix} = \begin{bmatrix} \cos\theta & +\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix}$$

$$\begin{bmatrix} F^1_x \\ F^1_y \\ F^2_x \\ F^2_y \end{bmatrix} = \begin{array}{cccc} 1_x & 1_y & 2_x & 2_y \\ \hline & & & \\ & & & \\ & & & \\ & & & \end{array} \begin{bmatrix} u^1_x \\ u^1_y \\ u^2_x \\ u^2_y \end{bmatrix}$$

$$\mathbb{F} = \mathbb{K}\,\mathbb{u}$$

$$\mathbb{K} = \mathbb{R}^T_\theta\,\mathbb{K}_0\,\mathbb{R}_\theta$$

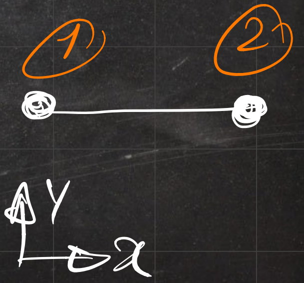
$$\boxed{\begin{matrix} F^1_x \\ F^1_y \\ F^2_x \\ F^2_y \end{matrix}} = K \; \frac{EA}{L} \boxed{\begin{matrix} \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ \sin\theta\cos\theta & \sin^2\theta & -\sin\theta\cos\theta & -\sin^2\theta \\ -\cos^2\theta & -\cos\theta\sin\theta & \cos^2\theta & \cos\theta\sin\theta \\ -\sin\theta\cos\theta & -\sin^2\theta & \sin\theta\cos\theta & \sin^2\theta \end{matrix}} \boxed{\begin{matrix} u^1_x \\ u^1_y \\ u^2_x \\ u^2_y \end{matrix}}$$

$$\underbrace{\begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix}}_{\mathbb{R}(\theta)}$$

$$\begin{bmatrix} F_x^1 \\ F_y^1 \\ F_x^2 \\ F_y^2 \end{bmatrix} = \underset{\frac{EA}{L}}{K} \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ \sin\theta\cos\theta & \sin^2\theta & -\sin\theta\cos\theta & -\sin^2\theta \\ -\cos^2\theta & -\cos\theta\sin\theta & \cos^2\theta & \cos\theta\sin\theta \\ -\sin\theta\cos\theta & -\sin^2\theta & \sin\theta\cos\theta & \sin^2\theta \end{bmatrix} \begin{bmatrix} u_x^1 \\ u_y^1 \\ u_x^2 \\ u_y^2 \end{bmatrix}$$

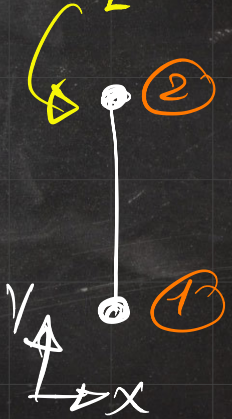
$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \downarrow$$

$$\theta = 0 \Rightarrow \mathbb{K} = \frac{EA}{L} \begin{array}{c} \begin{array}{cccc} 1x & 1y & 2x & 2y \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$



$$\begin{bmatrix} F^1_x \\ F^1_y \\ F^2_x \\ F^2_y \end{bmatrix} = \underset{\frac{EA}{L}}{K} \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ \sin\theta\cos\theta & \sin^2\theta & -\sin\theta\cos\theta & -\sin^2\theta \\ -\cos^2\theta & -\cos\theta\sin\theta & \cos^2\theta & \cos\theta\sin\theta \\ -\sin\theta\cos\theta & -\sin^2\theta & \sin\theta\cos\theta & \sin^2\theta \end{bmatrix} \begin{bmatrix} u^1_x \\ u^1_y \\ u^2_x \\ u^2_y \end{bmatrix}$$

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

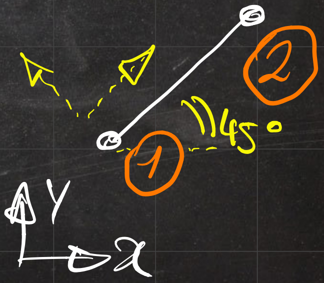


$$\theta = 90^\circ \Rightarrow \mathbb{K} = \frac{EA}{L} \begin{array}{cccc} 1x & 1y & 2x & 2y \\ \hline \end{array}$$

$$\begin{array}{c} \\ \mathbb{K} = \frac{EA}{L} \end{array} \left[\begin{array}{cccc} 1x & 1y & 2x & 2y \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right]$$

$$\begin{bmatrix} F^1_x \\ F^1_y \\ F^2_x \\ F^2_y \end{bmatrix} = \underset{\frac{EA}{L}}{K} \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ \sin\theta\cos\theta & \sin^2\theta & -\sin\theta\cos\theta & -\sin^2\theta \\ -\cos^2\theta & -\cos\theta\sin\theta & \cos^2\theta & \cos\theta\sin\theta \\ -\sin\theta\cos\theta & -\sin^2\theta & \sin\theta\cos\theta & \sin^2\theta \end{bmatrix} \begin{bmatrix} u^1_x \\ u^1_y \\ u^2_x \\ u^2_y \end{bmatrix}$$

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

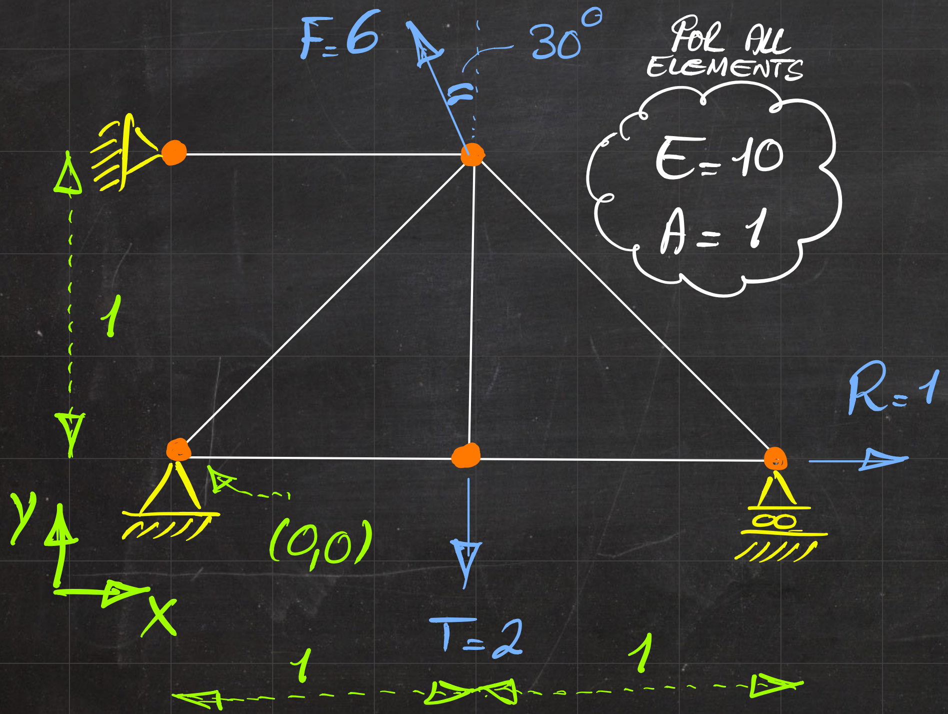


$$\theta = 45^\circ \Rightarrow \mathbb{K} = \frac{1}{2}\frac{EA}{L}\begin{array}{c} \begin{array}{cccc} 1x & 1y & 2x & 2y \end{array} \\ \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix} \end{array}$$

$$\cos\theta = \sin\theta = \frac{1}{\sqrt{2}}$$

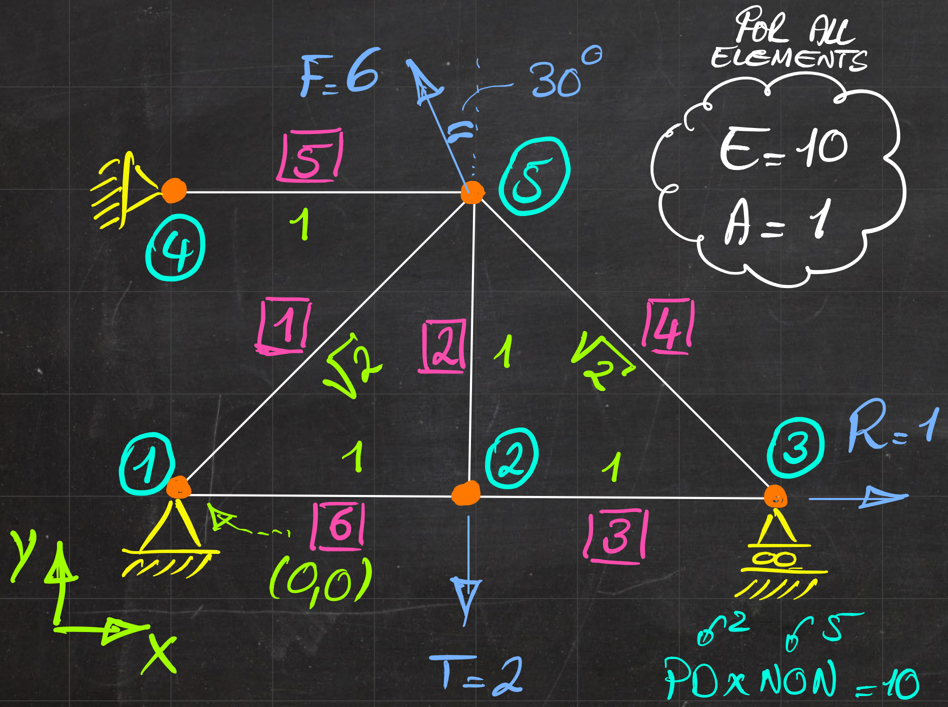
EXAMPLE:

CALCULATE THE DISPLACEMENTS OF ALL THE NODES.



- NUMBER NODES & ELEMENTS
  - ↳ Create NL & EL
- CREATE ENL
- COMPUTE ELEMENTS STIFFNESSES
- ASSEMBLE STIFFNESS
- SOLVE

EXAMPLE:

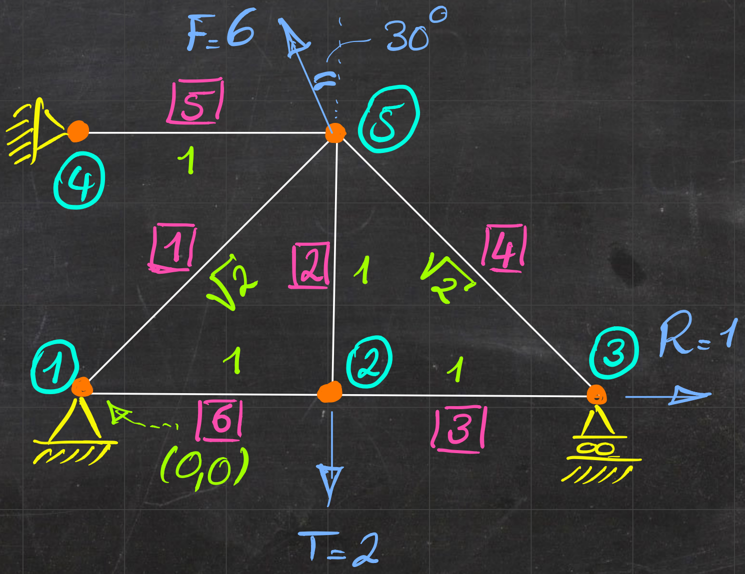


CALCULATE THE DISPLACEMENTS OF ALL THE NODES.

* NUMBER NODES & ELEMENTS
  ↳ Create NL & EL
* CREATE ENL
* COMPUTE ELEMENTS STIFFNESSES
* ASSEMBLE STIFFNESS
* SOLVE

EXAMPLE:

	COOR	
NL	X	Y
1	0	0
2	1	0
3	2	0
4	0	1
5	1	1

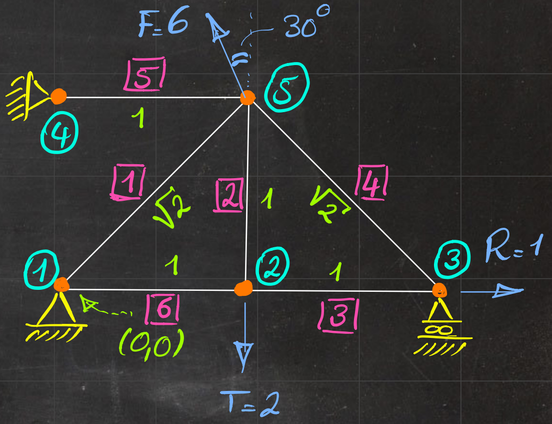


CONNECTIVITY

	CORNERS	
EL	N1	N2
1	1	5
2	2	5
3	2	3
4	3	5
5	4	5
6	1	2

EXAMPLE:

NL	COOR		BC INFO		TMP DEG.	
	X	Y	X	Y	X	Y
1	0	0	D	D	-1	-2
2	1	0	N	N	1	2
3	2	0	N	D	3	-3
4	0	1	D	D	-4	-5
5	1	1	N	N	4	5



$\text{DoF} = 5$

$\text{DoC} = 5 \leftarrow |S|$

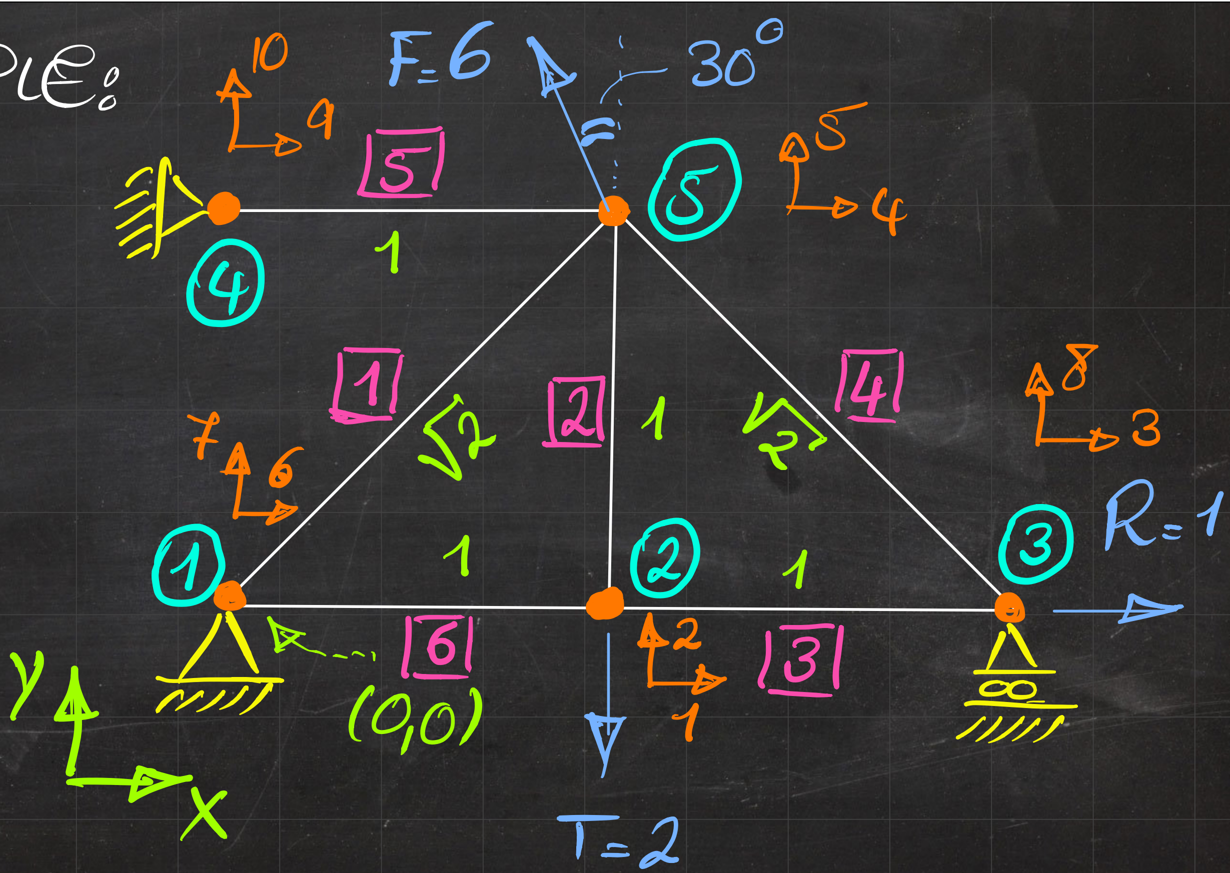
$+ \rightarrow 10 = 2 \times 5 \leftarrow \text{NoN}$

EXAMPLE:

EXTENDED NODE LIST

NL	COOR		BC INFO		TMP DEG.		GLOBAL DEG		DISP		EXTERNALLY PRESCRIBED & FORCE	
	X	Y	X	Y	X	Y	X	Y	X	Y	X	Y
1	0	0	D	D	-1	-2	6	7	0	0	?	?
2	1	0	N	N	1	2	1	2	?	?	0	-2
3	2	0	N	D	3	-3	3	8	?	0	1	?
4	0	1	D	D	-4	-5	9	10	0	0	?	?
5	1	1	N	N	4	5	4	5	?	?	$-F\sin 30^\circ$	$F\cos 30^\circ$

EXAMPLE:



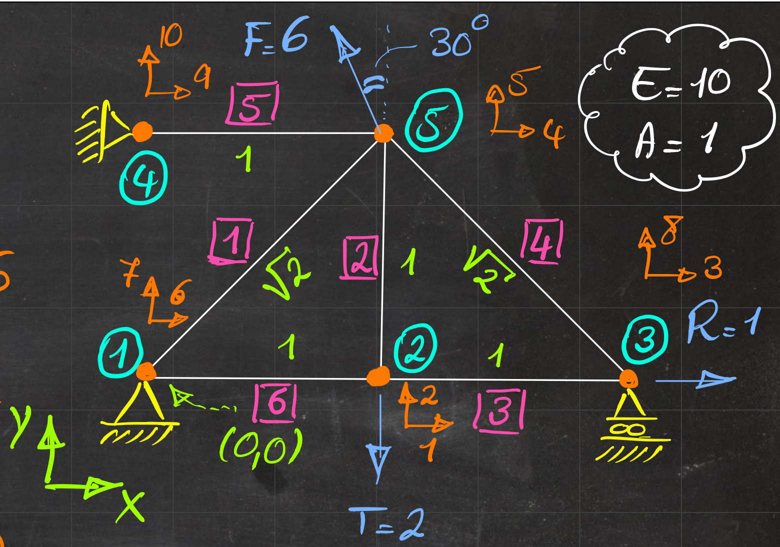
EXAMPLE:

BETWEEN ① - ⑤

$$\mathbb{K}^1 = \frac{EA}{2\sqrt{2}}$$

$$\left\{ \begin{array}{l} E = 10 \\ A = 1 \\ L = \sqrt{2} \end{array} \right.$$

$$\begin{array}{c|cccc} & \overset{6}{1_x} & \overset{7}{1_y} & \overset{4}{5_x} & \overset{5}{5_y} \\ \hline 1_x \; 6 & 1 & 1 & -1 & -1 \\ 1_y \; 7 & 1 & 1 & -1 & -1 \\ 5_x \; 4 & -1 & -1 & 1 & 1 \\ 5_y \; 5 & -1 & -1 & 1 & 1 \end{array}$$



$$\mathbb{K}_\theta = \frac{EA}{L} \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ \sin\theta\cos\theta & \sin^2\theta & -\sin\theta\cos\theta & -\sin^2\theta \\ -\cos^2\theta & -\cos\theta\sin\theta & \cos^2\theta & \cos\theta\sin\theta \\ -\sin\theta\cos\theta & -\sin^2\theta & \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

EXAMPLE:

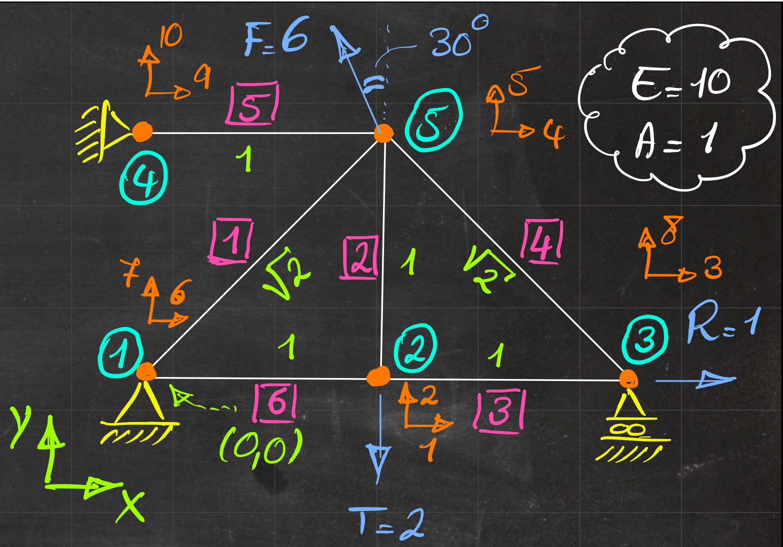
BETWEEN ① - ⑤

$$\overset{1}{\mathbb{K}} = \frac{EA}{2\sqrt{2}} \quad \begin{array}{cccc|c} 6 & 7 & 4 & 5 & \\ \hline 1 & 1 & -1 & -1 & 6 \\ 1 & 1 & -1 & -1 & 7 \\ -1 & -1 & 1 & 1 & 4 \\ -1 & -1 & 1 & 1 & 5 \end{array}$$

$E = 10$

$A = 1$

$L = \sqrt{2}$



$$\underset{\theta}{\mathbb{K}} = \frac{EA}{L} \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ \sin\theta\cos\theta & \sin^2\theta & -\sin\theta\cos\theta & -\sin^2\theta \\ -\cos^2\theta & -\cos\theta\sin\theta & \cos^2\theta & \cos\theta\sin\theta \\ -\sin\theta\cos\theta & -\sin^2\theta & \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

EXAMPLE:

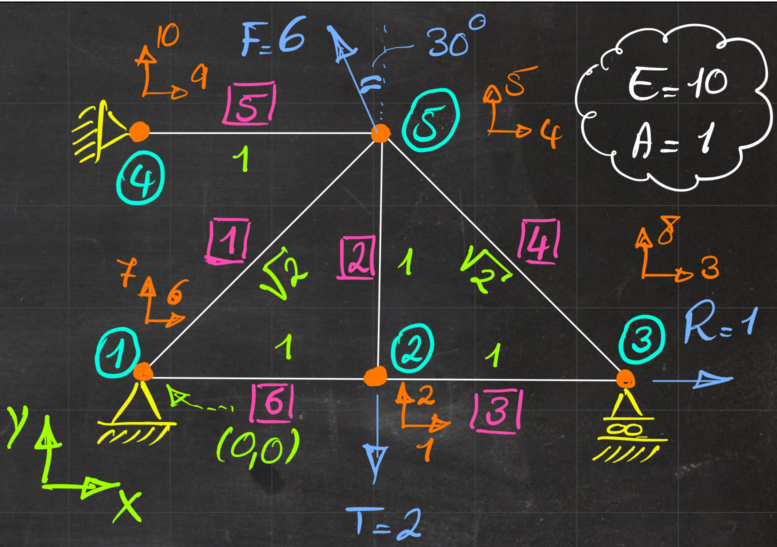
BETWEEN ②-⑤

$$\mathbb{K}^{2} = \frac{EA}{1}\begin{array}{c|cccc} & 1 & 2 & 4 & 5 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & -1 \\ 4 & 0 & 0 & 0 & 0 \\ 5 & 0 & -1 & 0 & 1 \end{array}$$

$E = 10$

$A = 1$

$L = 1$

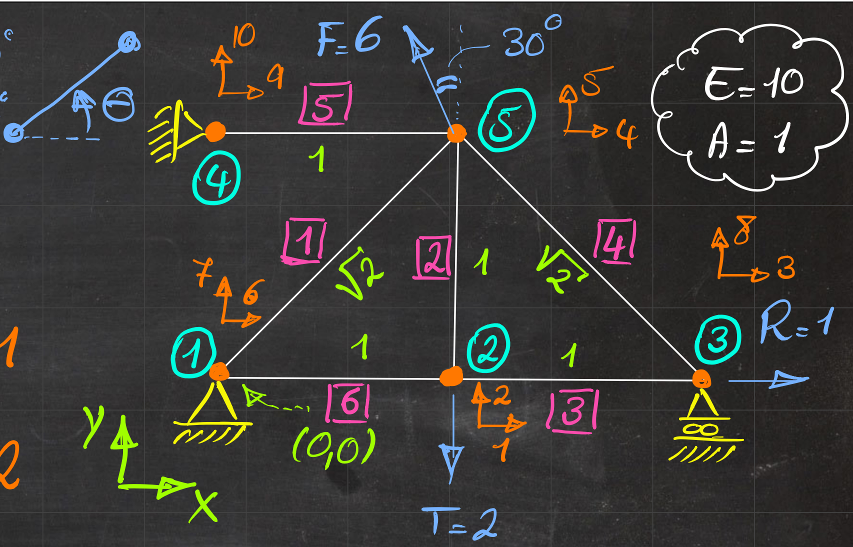


$$\mathbb{K}_{\theta} = \frac{EA}{L}\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ \sin\theta\cos\theta & \sin^2\theta & -\sin\theta\cos\theta & -\sin^2\theta \\ -\cos^2\theta & -\cos\theta\sin\theta & \cos^2\theta & \cos\theta\sin\theta \\ -\sin\theta\cos\theta & -\sin^2\theta & \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

EXAMPLE:

$② \rightarrow ③ \Rightarrow \theta = 0°$

$③ \rightarrow ② \Rightarrow \theta = 180°$



BETWEEN ②-③

$$\mathbb{K}^{3} = \frac{EA}{1} \begin{array}{c} \\ \\ \\ \\ \end{array} \begin{array}{cccc|c} 1 & 2 & 3 & 8 & \\ \hline 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 \\ -1 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 8 \end{array}$$

E=10

A=1

L=1

$$\mathbb{K}_{\theta} = \frac{EA}{L} \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ \sin\theta\cos\theta & \sin^2\theta & -\sin\theta\cos\theta & -\sin^2\theta \\ -\cos^2\theta & -\cos\theta\sin\theta & \cos^2\theta & \cos\theta\sin\theta \\ -\sin\theta\cos\theta & -\sin^2\theta & \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

EXAMPLE:

$\theta = 135^\circ$

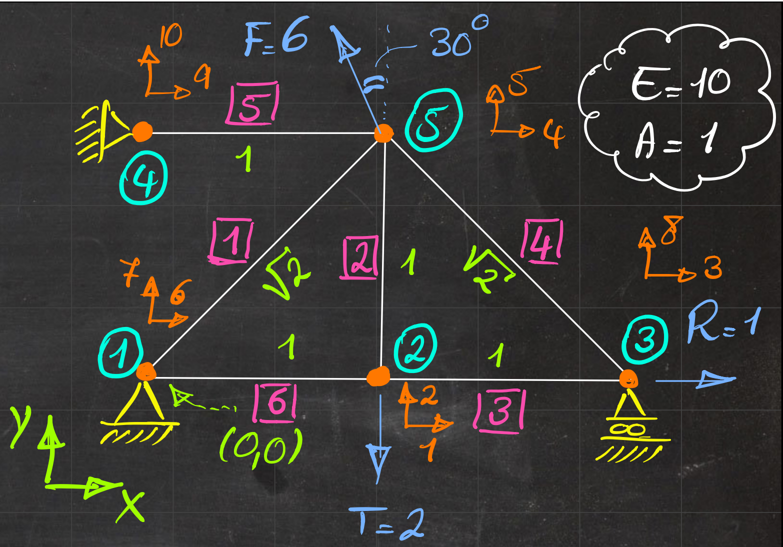
BETWEEN ③ - ⑤

$$\mathbb{K}^4 = \frac{EA}{2\sqrt{2}} \begin{array}{cc} \begin{array}{cccc} 3 & 8 & 4 & 5 \end{array} & \\ \left[\begin{array}{cccc} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{array}\right] & \begin{array}{c} 3 \\ 8 \\ 4 \\ 5 \end{array} \end{array}$$

$E = 10$

$A = 1$

$L = \sqrt{2}$



$$\mathbb{K}_\theta = \frac{EA}{L} \boxed{\begin{matrix} \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ \sin\theta\cos\theta & \sin^2\theta & -\sin\theta\cos\theta & -\sin^2\theta \\ -\cos^2\theta & -\cos\theta\sin\theta & \cos^2\theta & \cos\theta\sin\theta \\ -\sin\theta\cos\theta & -\sin^2\theta & \sin\theta\cos\theta & \sin^2\theta \end{matrix}}$$

EXAMPLE:

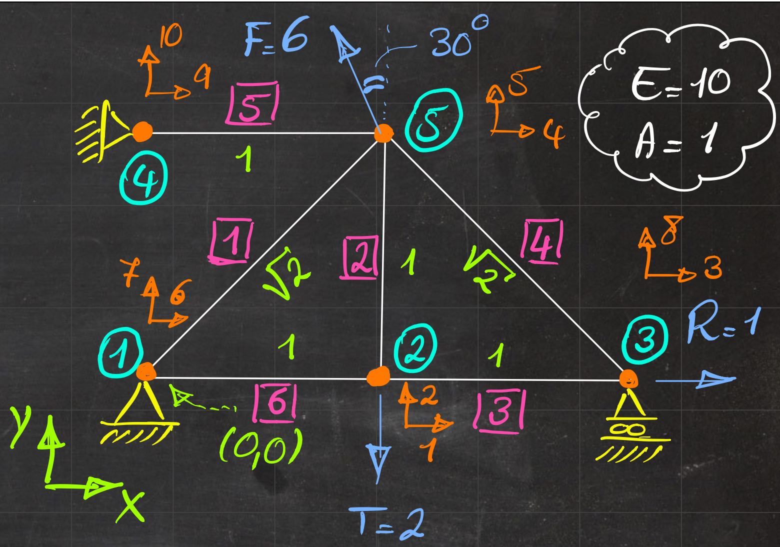
BETWEEN ④ - ⑤

$$\mathbb{K}^{5} = \frac{EA}{1}\left[\begin{array}{cccc|c} 9 & 10 & 4 & 5 & \\ 1 & 0 & -1 & 0 & 9 \\ 0 & 0 & 0 & 0 & 10 \\ -1 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 5 \end{array}\right]$$

$E = 10$

$A = 1$

$L = 1$



$$\underset{\theta}{\mathbb{K}} = \frac{EA}{L}\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ \sin\theta\cos\theta & \sin^2\theta & -\sin\theta\cos\theta & -\sin^2\theta \\ -\cos^2\theta & -\cos\theta\sin\theta & \cos^2\theta & \cos\theta\sin\theta \\ -\sin\theta\cos\theta & -\sin^2\theta & \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

EXAMPLE:

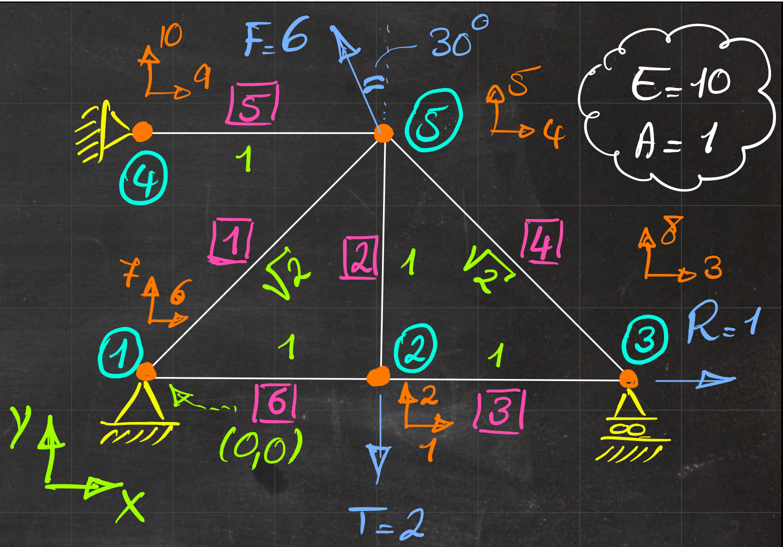
BETWEEN ①-②

$$\mathbb{K}^{6} = \frac{EA}{1}\left[\begin{array}{cccc|c} 6 & 7 & 1 & 2 & \\ 1 & 0 & -1 & 0 & 6 \\ 0 & 0 & 0 & 0 & 7 \\ -1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right]$$

$E=10$

$A=1$

$L=1$



$$\mathbb{K}_{\theta} = \frac{EA}{L}\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ \sin\theta\cos\theta & \sin^2\theta & -\sin\theta\cos\theta & -\sin^2\theta \\ -\cos^2\theta & -\cos\theta\sin\theta & \cos^2\theta & \cos\theta\sin\theta \\ -\sin\theta\cos\theta & -\sin^2\theta & \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

GLOBAL STIFFNESS

↓

10×10

1	2	3	4	5	6	7	8	9	10	
										1
										2
										3
			K^1_{33}	K^1_{34}	K^1_{31}	K^1_{32}				4
			K^1_{43}	K^1_{44}	K^1_{41}	K^1_{42}				5
			K^1_{13}	K^1_{14}	K^1_{11}	K^1_{12}				6
			K^1_{23}	K^1_{24}	K^1_{21}	K^1_{22}				7
										8
										9
										10

$$\mathbb{K}^1 = \frac{EA}{2\sqrt{2}} \begin{array}{cccc|c} 6 & 7 & 4 & 5 & \\ \hline 1 & 1 & -1 & -1 & 6 \\ 1 & 1 & -1 & -1 & 7 \\ -1 & -1 & 1 & 1 & 4 \\ -1 & -1 & 1 & 1 & 5 \end{array}$$

$$\Downarrow$$

El. No.

$$\mathbb{K}^1 = \begin{array}{cccc|c} 6 & 7 & 4 & 5 & \\ \hline K^1_{11} & K^1_{12} & K^1_{13} & K^1_{14} & 6 \\ K^1_{21} & K^1_{22} & K^1_{23} & K^1_{24} & 7 \\ K^1_{31} & K^1_{32} & K^1_{33} & K^1_{34} & 4 \\ K^1_{41} & K^1_{42} & K^1_{43} & K^1_{44} & 5 \end{array}$$

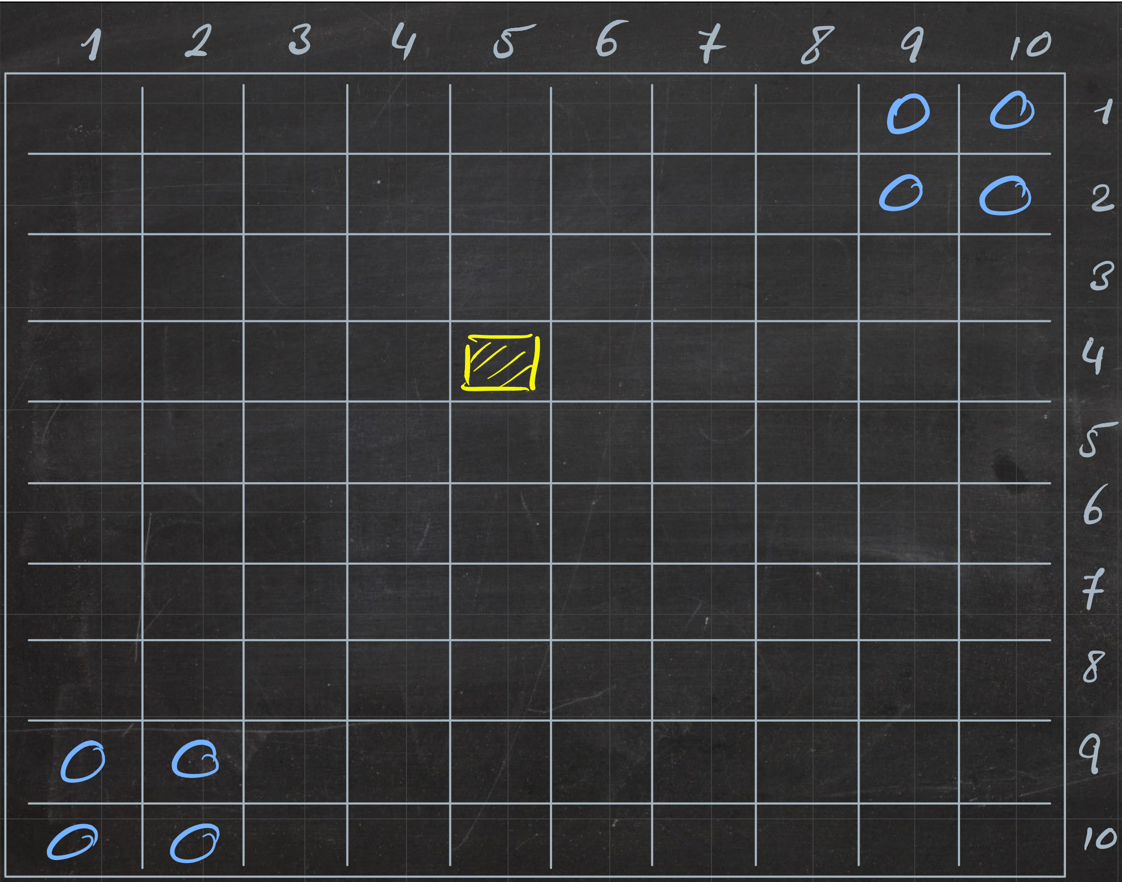
	1	2	3	4	5	6	7	8	9	10
1	K^2_{11}	K^2_{12}		K^2_{13}	K^2_{14}					
2	K^2_{21}	K^2_{22}		K^2_{23}	K^2_{24}					
3										
4	K^2_{31}	K^2_{32}		K^2_{33}	K^2_{34}					
5	K^2_{41}	K^2_{42}		K^2_{43}	K^2_{44}					
6										
7										
8										
9										
10										

$$\mathbb{K}^2 = \frac{EA}{1} \begin{array}{cccc|c} 1 & 2 & 4 & 5 & \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & -1 & 0 & 1 & 5 \end{array}$$

$$\Downarrow$$

El. No. 2

$$\mathbb{K}^2 = \begin{array}{cccc|c} 1 & 2 & 4 & 5 & \\ \hline K^2_{11} & K^2_{12} & K^2_{13} & K^2_{14} & 1 \\ K^2_{21} & K^2_{22} & K^2_{23} & K^2_{24} & 2 \\ K^2_{31} & K^2_{32} & K^2_{33} & K^2_{34} & 4 \\ K^2_{41} & K^2_{42} & K^2_{43} & K^2_{44} & 5 \end{array}$$

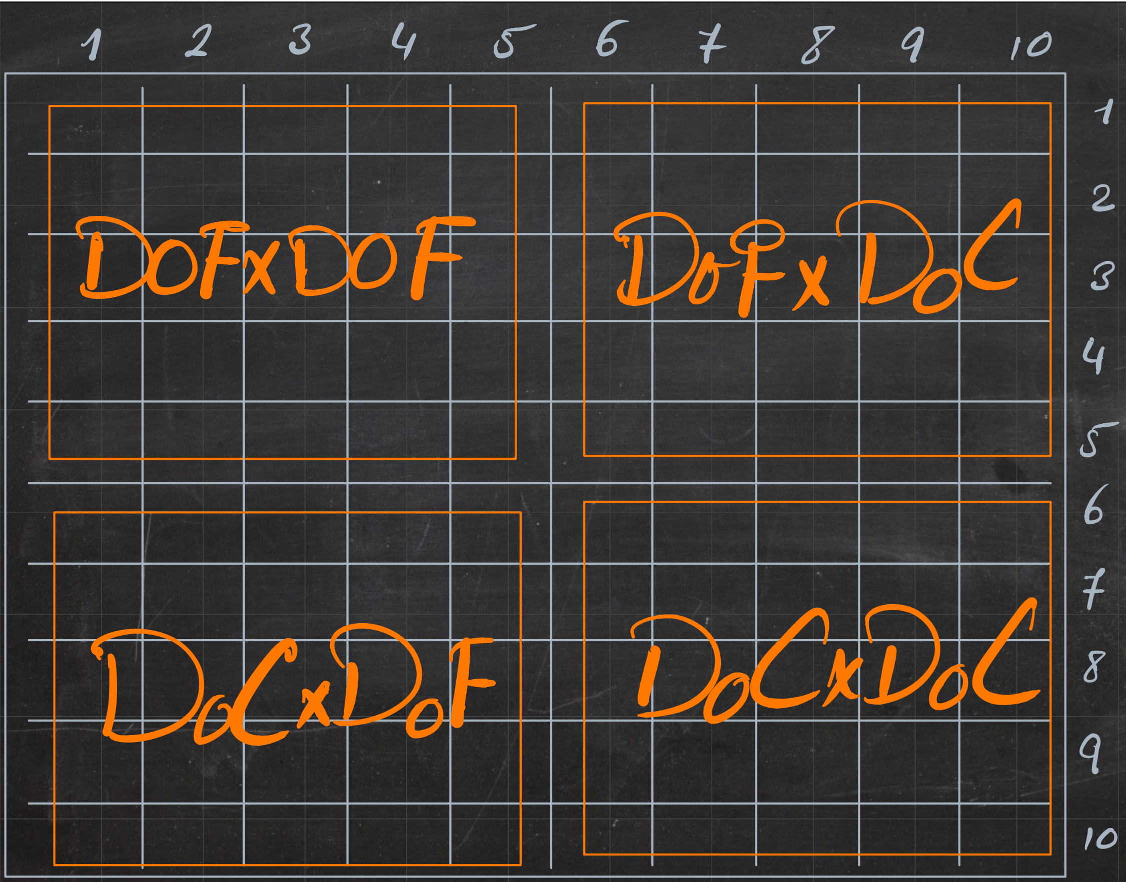


$$\begin{array}{ccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \\
 & & & & & & & & 0 & 0 & 1 \\
 & & & & & & & & 0 & 0 & 2 \\
 & & & & & & & & & & 3 \\
 & & & & ▨ & & & & & & 4 \\
 & & & & & & & & & & 5 \\
 & & & & & & & & & & 6 \\
 & & & & & & & & & & 7 \\
 & & & & & & & & & & 8 \\
0 & 0 & & & & & & & & & 9 \\
0 & 0 & & & & & & & & & 10
\end{array}$$

MIND THE ZEROS

▨

$$L_0\left[\frac{1}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}\right]EA$$

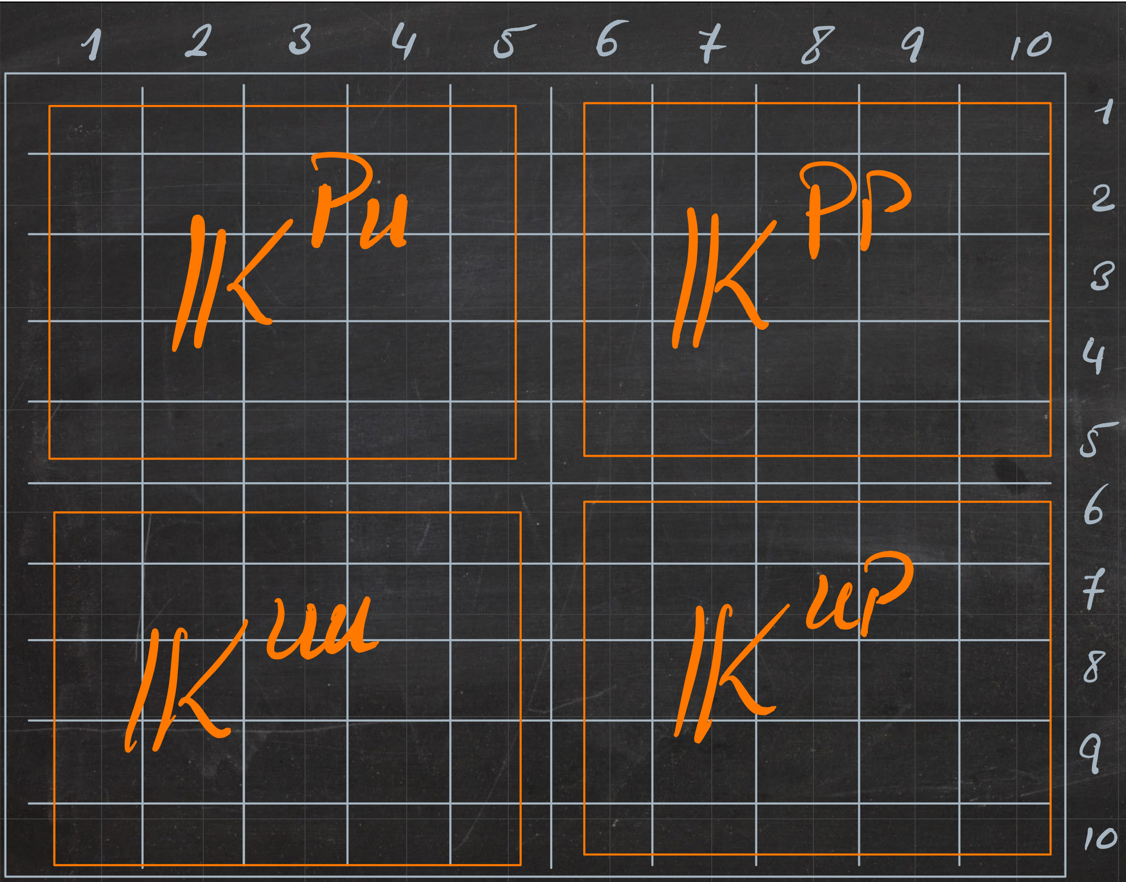


SEMI-Populated

Sparse

↓

lots of zeros



SEMI-
Populated

Sparse
↓
Lots of
zeros

$$\left[\begin{array}{c} \mathbb{F}^{p} \\ \hline \mathbb{F}^{u} \end{array}\right] = \left[\begin{array}{c|c} \mathbb{K}^{pu} & \mathbb{K}^{pp} \\ \hline \mathbb{K}^{up} & \mathbb{K}^{uu} \end{array}\right] \left[\begin{array}{c} u^{u} \\ \hline u^{p} \end{array}\right]$$

$$[\mathbb{K}^{pu}][u^{u}] + [\mathbb{K}^{pp}][u^{p}] = [\mathbb{F}^{p}]$$

$$\underbrace{[\mathbb{K}^{pu}][u^{u}]}_{\text{REDUCED}} = \underbrace{[\mathbb{F}^{p}] - [\mathbb{K}^{pp}][u^{p}]}_{\text{Prescribed}}$$

$\checkmark$

$$\mathbb{K}_{\text{RED}} \cdot u_{\text{RED}} = \mathbb{F}^{p}$$

STATIC CONDENSATION

$$\left[\begin{array}{c} \mathbb{F}^{p} \\ \hline \mathbb{F}^{u} \end{array}\right] = \left[\begin{array}{c:c} \mathbb{K}^{pu} & \mathbb{K}^{pp} \\ \hdashline \mathbb{K}^{up} & \mathbb{K}^{uu} \end{array}\right] \left[\begin{array}{c} \mathbb{U}^{u} \\ \hdashline \mathbb{U}^{p} \end{array}\right]$$

$$\mathbb{F}^{p}: \quad \begin{array}{c} 0 \\ -2 \\ 1 \\ -3 \\ 3\sqrt{3} \end{array}$$

$$\mathbb{U}^{u}: \quad \left.\begin{array}{c} -0.101 \\ 0.353 \\ -0.202 \\ -0.2176 \\ 0.5530 \end{array}\right\} \checkmark$$

$$\mathbb{U}^{p}: \quad \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}$$