FINITE ELEMENT METHOD

5

Ali Javili

Bilkent University

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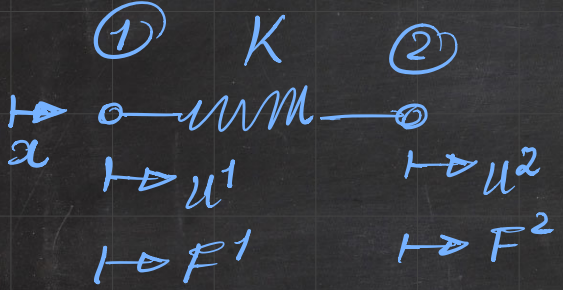
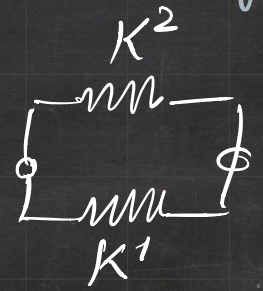
Understanding key ingredients of FEM using springs:



$$\begin{bmatrix} F^1 \\ F^2 \end{bmatrix} = \begin{bmatrix} K & -K \\ -K & K \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix}$$

$$F = K \cdot u$$

I)

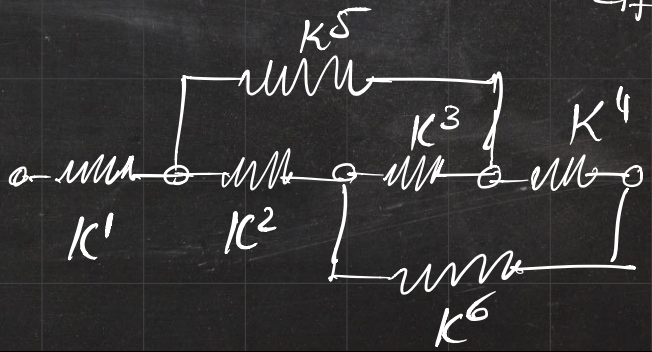


K_{eff} ?

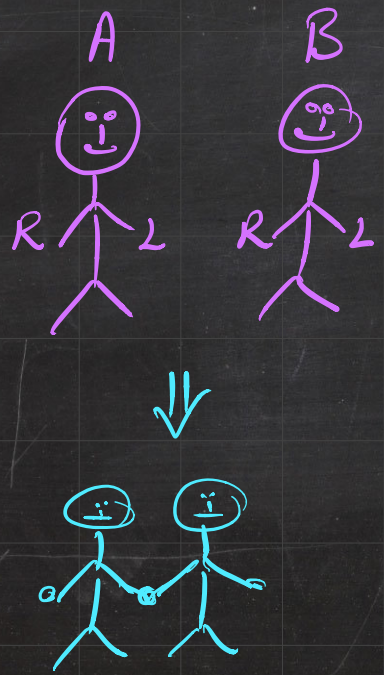
II)

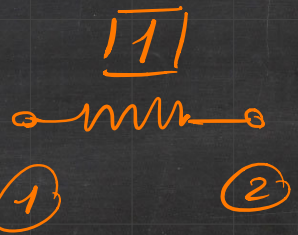
K_{eff} ?

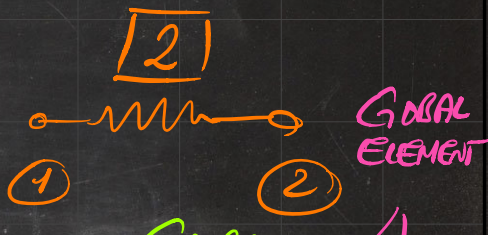
III)



TOWARDS AN ALGORITHMIC APPROACH TO ASSEMBLY:





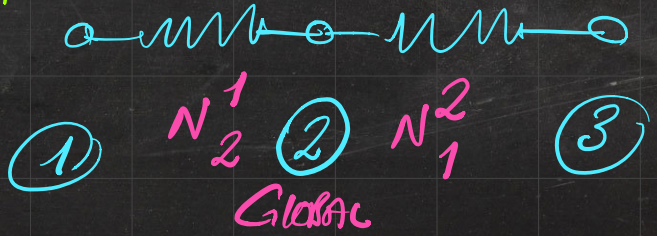


Superscript: GLOBAL

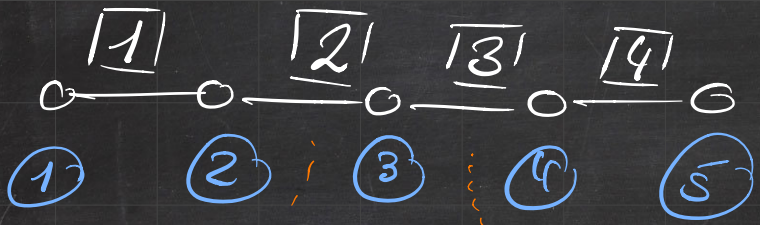
subscript: LOCAL

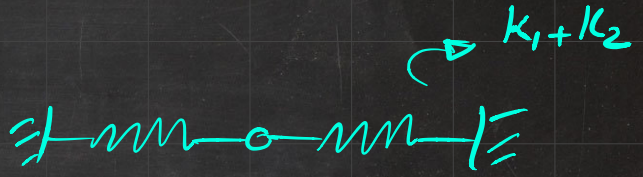
GLOBAL NODE

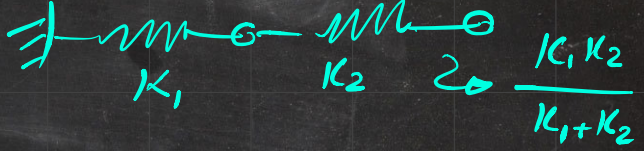
$N^2 = N_2^1 = N_1^2$



TOWARDS AN ALGORITHMIC APPROACH TO ASSEMBLY:

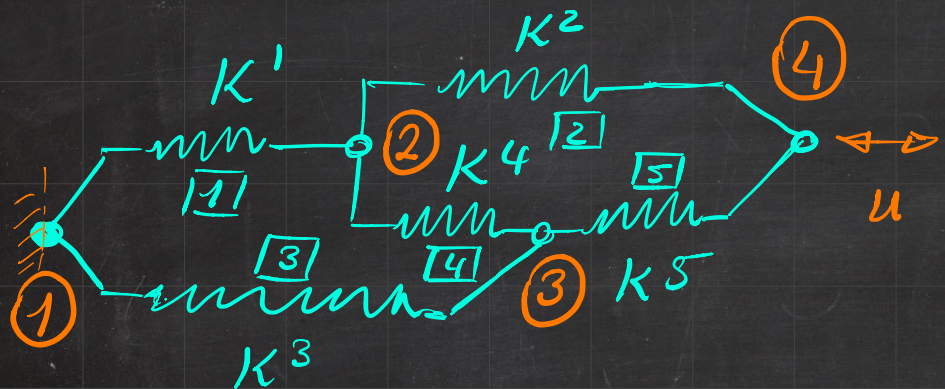






$$\Rightarrow K^3 = K_2^2 + K_1^3$$

TOWARDS AN ALGORITHMIC APPROACH TO ASSEMBLY:



ELEMENT 5

$$\|K^5 = \begin{bmatrix} K^5 & -K^5 \\ -K^5 & K^5 \end{bmatrix}$$

ELEMENT 1

$$\|K^1 = \begin{bmatrix} K^1 & -K^1 \\ -K^1 & K^1 \end{bmatrix}$$

ELEMENT 2

$$\|K^2 = \begin{bmatrix} K^2 & -K^2 \\ -K^2 & K^2 \end{bmatrix}$$

ELEMENT 3

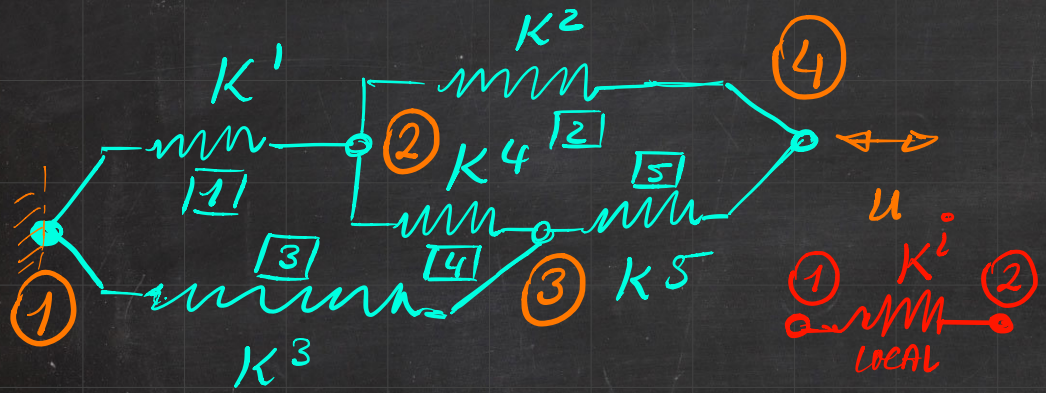
$$\|K^3 = \begin{bmatrix} K^3 & -K^3 \\ -K^3 & K^3 \end{bmatrix}$$

ELEMENT 4

$$\|K^4 = \begin{bmatrix} K^4 & -K^4 \\ -K^4 & K^4 \end{bmatrix}$$

BOUND

TOWARDS AN ALGORITHMIC APPROACH TO ASSEMBLY:



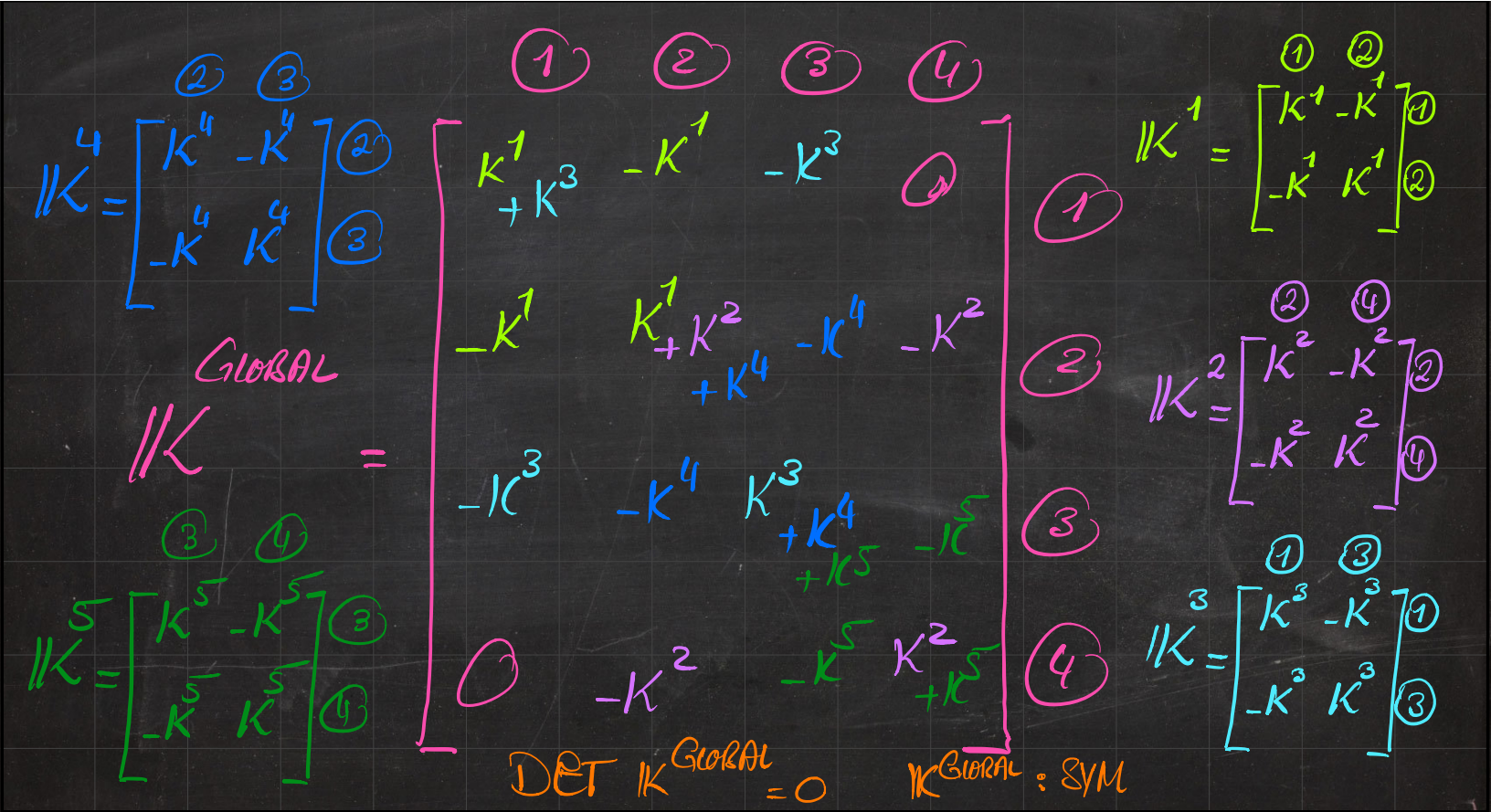
$$\|K^5 = \begin{bmatrix} K^5 & -K^5 \\ -K^5 & K^5 \end{bmatrix} \begin{matrix} 3 \\ 4 \end{matrix} \qquad \begin{matrix} 3 & 4 \end{matrix}$$

$$\|K^1 = \begin{bmatrix} K^1 & -K^1 \\ -K^1 & K^1 \end{bmatrix} \begin{matrix} 1 \\ 2 \end{matrix} \qquad \begin{matrix} 1 & 2 \end{matrix}$$

$$\|K^2 = \begin{bmatrix} K^2 & -K^2 \\ -K^2 & K^2 \end{bmatrix} \begin{matrix} 2 \\ 4 \end{matrix} \qquad \begin{matrix} 2 & 4 \end{matrix}$$

$$\|K^3 = \begin{bmatrix} K^3 & -K^3 \\ -K^3 & K^3 \end{bmatrix} \begin{matrix} 1 \\ 3 \end{matrix} \qquad \begin{matrix} 1 & 3 \end{matrix}$$

$$\|K^4 = \begin{bmatrix} K^4 & -K^4 \\ -K^4 & K^4 \end{bmatrix} \begin{matrix} 2 \\ 3 \end{matrix} \qquad \begin{matrix} 2 & 3 \end{matrix}$$


GLOBAL
DET K GLOBAL = 0
K GLOBAL : SYM

$$\begin{bmatrix} F^1 \\ F^2 \\ F^3 \\ F^4 \end{bmatrix} = \begin{bmatrix} K^{11} & K^{12} & K^{13} & K^{14} \\ K^{21} & K^{22} & K^{23} & K^{24} \\ K^{31} & K^{32} & K^{33} & K^{34} \\ K^{41} & K^{42} & K^{43} & K^{44} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \\ u^4 \end{bmatrix}$$

GLOBAL GLOBAL GLOBAL

$$[F] = [K] \cdot [u]$$

$[F]$: NoN x 1 $\rightarrow$ [PD x NoN] x 1

$[K]$: NoN x NoN

$[u]$: NoN x 1

NoN $\equiv$ NoN x PD

1D Problem $\rightarrow$ Dimension

$\Rightarrow$ 4 Eq. & 4 Unknowns $\leftarrow$ BCs ?

BCs ? $\rightarrow$ DISPLACEMENT BASED $\rightarrow$ DIRICHLET BCs.

BCs ? $\rightarrow$ FORCE BASED $\rightarrow$ NEUMANN BCs.

$$\begin{bmatrix} F^1 \\ F^2 \\ F^3 \\ F^4 \end{bmatrix} = \begin{bmatrix} K^{11} & K^{12} & K^{13} & K^{14} \\ K^{21} & K^{22} & K^{23} & K^{24} \\ K^{31} & K^{32} & K^{33} & K^{34} \\ K^{41} & K^{42} & K^{43} & K^{44} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \\ u^4 \end{bmatrix}$$

$u^1 = 0$

$u^4 = u$

Homogeneous — NON Homogeneous

DIRICHLET — Displacement — $= 0$ — $\neq 0$

NEUMANN — Force — $= 0$ — $\neq 0$

$$= \begin{bmatrix} K^{11} \\ K^{21} \\ K^{31} \\ K^{41} \end{bmatrix} u^1 + \begin{bmatrix} K^{12} \\ K^{22} \\ K^{32} \\ K^{42} \end{bmatrix} u^2 + \begin{bmatrix} K^{13} \\ K^{23} \\ K^{33} \\ K^{43} \end{bmatrix} u^3 + \begin{bmatrix} K^{14} \\ K^{24} \\ K^{34} \\ K^{44} \end{bmatrix} u^4$$

4 EQN. & 4 Unknowns

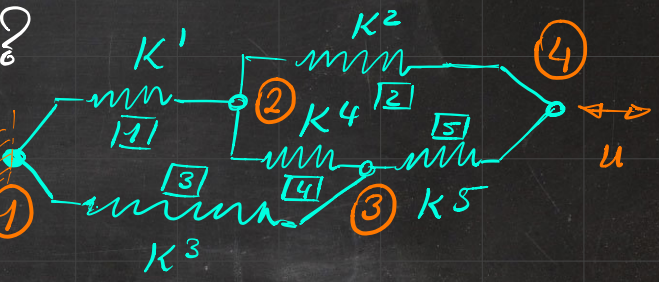
$$\begin{bmatrix} F^1 \\ F^2 \\ F^3 \\ F^4 \end{bmatrix} = \begin{bmatrix} K^{11} & K^{12} & K^{13} & K^{14} \\ K^{21} & K^{22} & K^{23} & K^{24} \\ K^{31} & K^{32} & K^{33} & K^{34} \\ K^{41} & K^{42} & K^{43} & K^{44} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \\ u^4 \end{bmatrix}$$

F^1: ?, F^2: 0, F^3: 0 (NEUMANN), F^4: ? (NEUMANN)

u^1: 0 → DIRICHLET (Prescribed Displacement)

u^2: ?, u^3: ?

u^4: u → DIRICHLET (Prescribed Displ.)



$$\begin{bmatrix} b \end{bmatrix} = \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} x \end{bmatrix} \quad (b: \checkmark,\ x: ?)$$

$$A \cdot x = b \Rightarrow x = A^{-1} \cdot b$$

4 EQN. & 4 Unknowns

$$\begin{bmatrix} F^1 \\ F^2 \\ F^3 \\ F^4 \end{bmatrix} = \begin{bmatrix} K^{11} & K^{12} & K^{13} & K^{14} \\ K^{21} & K^{22} & K^{23} & K^{24} \\ K^{31} & K^{32} & K^{33} & K^{34} \\ K^{41} & K^{42} & K^{43} & K^{44} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \\ u^4 \end{bmatrix}$$

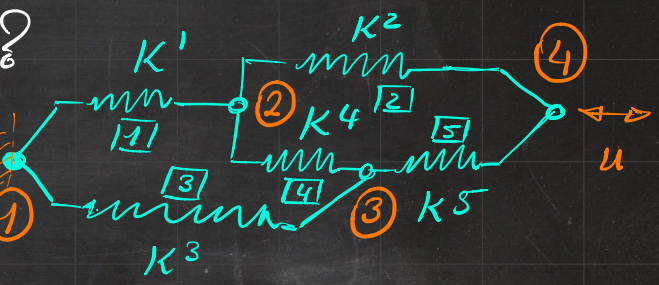
F^1 ?, F^2 = 0, F^3 = 0, F^4 ?

NEUMANN, NEUMANN

u^1 = 0, u^2 ?, u^3 ?, u^4 = u

DIRICHLET (Prescribed Displacement)

DIRICHLET (Prescribed Displ.)



$$\begin{bmatrix} F^p \\ F^u \end{bmatrix} = \begin{bmatrix} K^{pu} & K^{pp} \\ K^{uu} & K^{up} \end{bmatrix} \begin{bmatrix} u^u \\ u^p \end{bmatrix}$$

u^u unknown

u^p prescribed

4 EQN. & 4 Unknowns

$$\begin{bmatrix} F^1 \\ F^2 \\ F^3 \\ F^4 \end{bmatrix} = \begin{bmatrix} K^{11} & K^{12} & K^{13} & K^{14} \\ K^{21} & K^{22} & K^{23} & K^{24} \\ K^{31} & K^{32} & K^{33} & K^{34} \\ K^{41} & K^{42} & K^{43} & K^{44} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \\ u^4 \end{bmatrix}$$

F^1 ?, F^2 = 0, F^3 = 0, F^4 ?

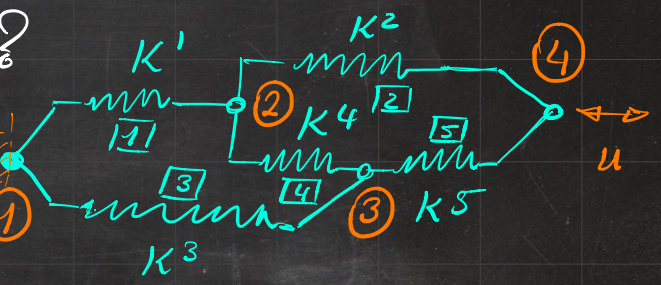
NEUMANN

NEUMANN

u^1 = 0 ← DIRICHLET (Prescribed Displacement)

u^2 ?, u^3 ?, u^4 = u

← DIRICHLET (Prescribed Displ.)



$$\begin{bmatrix} F^p \\ F^u \end{bmatrix} = \begin{bmatrix} K^{pu} & K^{pp} \\ K^{uu} & K^{up} \end{bmatrix} \begin{bmatrix} u^u \\ u^p \end{bmatrix}$$

NEUMANN

u^u } FREE NODES ⇒ DoF ← DEGREES OF FREEDOM

u^p } CONSTRAINED NODES ⇒ DoC ← DEGREES OF CONSTRAINT

DIRICHLET

4 EQN. & 4 Unknowns

$$\begin{bmatrix} F^1 \\ F^2 \\ F^3 \\ F^4 \end{bmatrix} = \begin{bmatrix} K^{11} & K^{12} & K^{13} & K^{14} \\ K^{21} & K^{22} & K^{23} & K^{24} \\ K^{31} & K^{32} & K^{33} & K^{34} \\ K^{41} & K^{42} & K^{43} & K^{44} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \\ u^4 \end{bmatrix}$$

NEUMANN NEUMANN

$u^1 = 0$, $F^2 = 0$, $F^3 = 0$, $u^4 = u$; F^1, u^2, u^3, F^4 = ?

$$[F^P] = [K^{Pu}][u^u] + [K^{PP}][u^P]$$

$$[K^{Pu}][u^u] = [F^P] - [K^{PP}][u^P]$$

A x b

$$[x] = [A]^{-1}[b] \Leftarrow A \cdot x = b$$

$$\begin{bmatrix} F^P \\ F^u \end{bmatrix} = \begin{bmatrix} K^{Pu} & K^{PP} \\ K^{uu} & K^{uP} \end{bmatrix} \begin{bmatrix} u^u \\ u^P \end{bmatrix}$$

$$\begin{bmatrix} DoF \\ DoC \end{bmatrix} = \begin{bmatrix} DoF \times DoF & DoF \times DoC \\ DoC \times DoF & DoC \times DoC \end{bmatrix} \begin{bmatrix} DoF \\ DoC \end{bmatrix}$$

FORCE STIFFNESS DISPLACEMENT

4 EQN. & 4 Unknowns

$$\begin{bmatrix} F^1 \\ F^2 \\ F^3 \\ F^4 \end{bmatrix} = \begin{bmatrix} K^{11} & K^{12} & K^{13} & K^{14} \\ K^{21} & K^{22} & K^{23} & K^{24} \\ K^{31} & K^{32} & K^{33} & K^{34} \\ K^{41} & K^{42} & K^{43} & K^{44} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \\ u^4 \end{bmatrix}$$

F^1 ?, F^2 ○, F^3 ○ NEUMANN, F^4 NEUMANN ?

u^1 ○, u^2 ?, u^3 ?, u^4 ○

$$[F^P] = [K^{Pu}][u^u] + [K^{PP}][u^P]$$

$$[K^{Pu}][u^u] = [F^P] - [K^{PP}][u^P]$$

REDUCED STIFFNESS

$$\Rightarrow [u^u] = [K^{Pu}]^{-1} \cdot \{[F^P] - [K^{PP}][u^P]\}$$

$$\begin{bmatrix} F^P \\ \hline F^u \end{bmatrix} = \begin{bmatrix} K^{Pu} & K^{PP} \\ \hline K^{uu} & K^{uP} \end{bmatrix} \begin{bmatrix} u^u \\ \hline u^P \end{bmatrix}$$

REDUCED SYSTEM

$$A \cdot x = b$$

DOF x DOF

4 EQN. & 4 Unknowns

$$\begin{bmatrix} F^1 \\ F^2 \\ F^3 \\ F^4 \end{bmatrix} = \begin{bmatrix} K^{11} & K^{12} & K^{13} & K^{14} \\ K^{21} & K^{22} & K^{23} & K^{24} \\ K^{31} & K^{32} & K^{33} & K^{34} \\ K^{41} & K^{42} & K^{43} & K^{44} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \\ u^4 \end{bmatrix}$$

F^1 ?, $F^2 = 0$, $F^3 = 0$ NEUMANN, F^4 ? NEUMANN

$u^1 = 0$, u^2 ?, u^3 ?, $u^4 = u$

$$[F^P] = [K^{Pu}][u^u] + [K^{PP}][u^P]$$

$$[K^{Pu}][u^u] = [F^P] - [K^{PP}][u^P]$$

REDUCED SYSTEM

$$\Rightarrow [u^u] = [K^{Pu}]^{-1} \cdot \{[F^P] - [K^{PP}][u^P]\}$$

$$\begin{bmatrix} F^P \\ \hline F^u \end{bmatrix} = \left[\begin{array}{c|c} K^{Pu} & K^{PP} \\ \hline K^{uu} & K^{uP} \end{array}\right] \begin{bmatrix} u^u \\ \hline u^P \end{bmatrix}$$

$$\Rightarrow [F^u] = [K^{uu}][u^u] + [K^{uP}][u^P]$$

STATIC CONDENSATION ✓

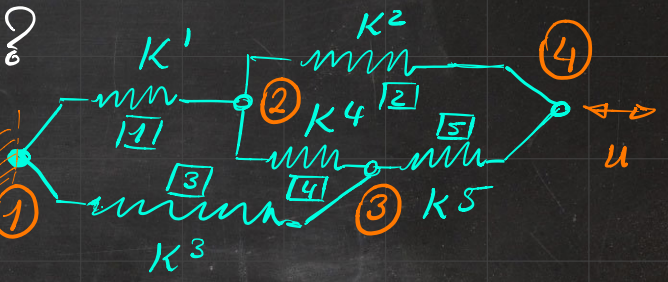
4 EQN. & 4 Unknowns

$$\begin{bmatrix} F^1 \\ F^2 \\ F^3 \\ F^4 \end{bmatrix} = \begin{bmatrix} K^{11} & K^{12} & K^{13} & K^{14} \\ K^{21} & K^{22} & K^{23} & K^{24} \\ K^{31} & K^{32} & K^{33} & K^{34} \\ K^{41} & K^{42} & K^{43} & K^{44} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \\ u^4 \end{bmatrix}$$

F^1 ?, F^2 0, F^3 0, F^4 ?

NEUMANN, NEUMANN

u^1 0, u^2 ?, u^3 ?, u^4 u

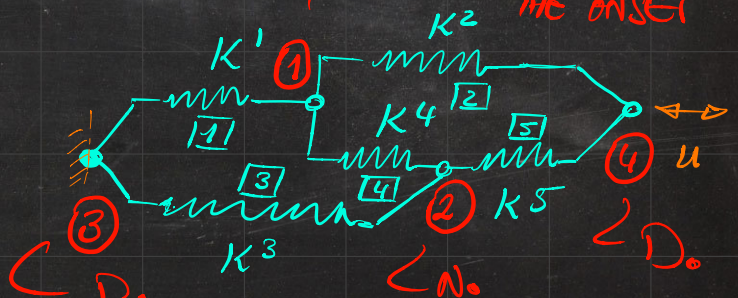


$$\begin{bmatrix} F^p \\ F^u \end{bmatrix} = \begin{bmatrix} K^{pu} & K^{pp} \\ K^{uu} & K^{up} \end{bmatrix} \begin{bmatrix} u^u \\ u^p \end{bmatrix}$$

u^u { N.

u^p { D.

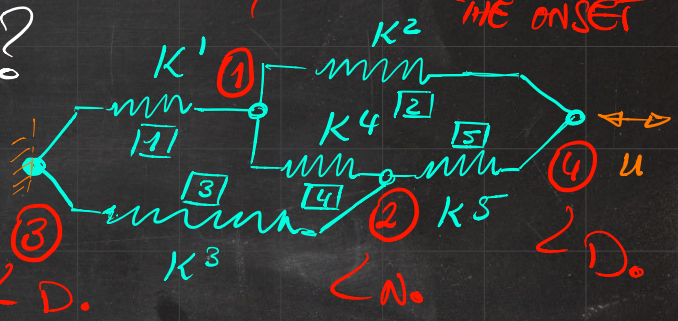
NUMBERING CAREFULLY FROM THE ONSET



4 EQN. & 4 Unknowns

$$\begin{bmatrix} F^1 \\ F^2 \\ F^3 \\ F^4 \end{bmatrix} = \begin{bmatrix} K^{11} & K^{12} & K^{13} & K^{14} \\ K^{21} & K^{22} & K^{23} & K^{24} \\ K^{31} & K^{32} & K^{33} & K^{34} \\ K^{41} & K^{42} & K^{43} & K^{44} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \\ u^4 \end{bmatrix}$$

NUMBERING CAREFULLY FROM THE ONSET



$$\begin{bmatrix} F^p \\ \hline F^u \end{bmatrix} = \begin{bmatrix} K^{pu} & K^{pp} \\ \hline K^{uu} & K^{up} \end{bmatrix} \begin{bmatrix} u^u \\ \hline u^p \end{bmatrix} \begin{matrix} \} \text{N.} \\ \} \text{D.} \end{matrix}$$

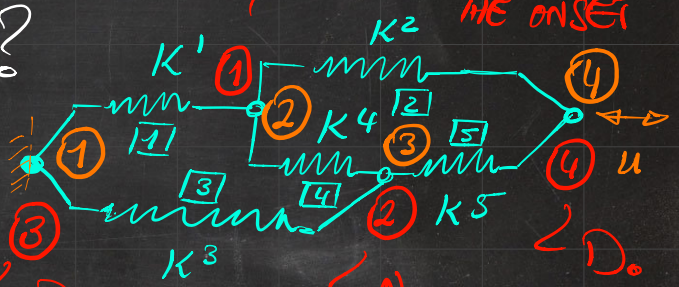
THE PROBLEM IS WE DO NOT KNOW A PRIORI HOW MANY DOFs WE HAVE! ✓ Efficiency Matters.

4 EQN. & 4 Unknowns

$$\begin{bmatrix} F^1 \\ F^2 \\ F^3 \\ F^4 \end{bmatrix} = \begin{bmatrix} K^{11} & K^{12} & K^{13} & K^{14} \\ K^{21} & K^{22} & K^{23} & K^{24} \\ K^{31} & K^{32} & K^{33} & K^{34} \\ K^{41} & K^{42} & K^{43} & K^{44} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \\ u^3 \\ u^4 \end{bmatrix}$$

$$\begin{bmatrix} F^p \\ \hline F^u \end{bmatrix} = \begin{bmatrix} K^{pu} & K^{pp} \\ \hline K^{uu} & K^{up} \end{bmatrix} \begin{bmatrix} u^u \\ \hline u^p \end{bmatrix}$$

NUMBERING CAREFULLY FROM THE ONSET



NO. DEGREE

NO.		DEGREE
(X,Y) 1	→	3
(X,Y) 2	→	1
(X,Y) 3	→	2
(X,Y) 4	→	4

ENL

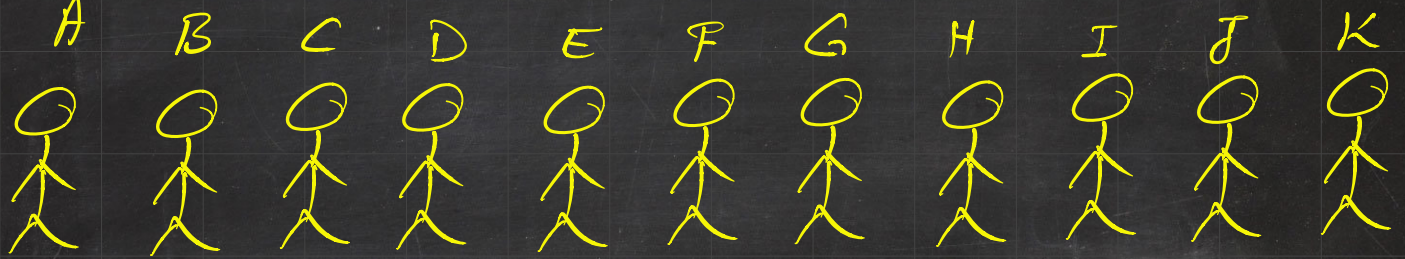
Loop over nodes

ASSIGN DEGREES TO NODES

end

EXTENDED NODE LIST

THE NAMING (NUMBERING) IS ARBITRARY!

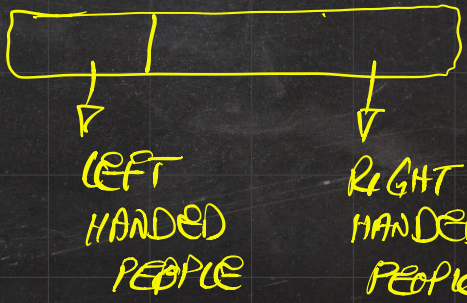


→ Every Person Can Say One Word ← Programming : One Loop !

How many people?

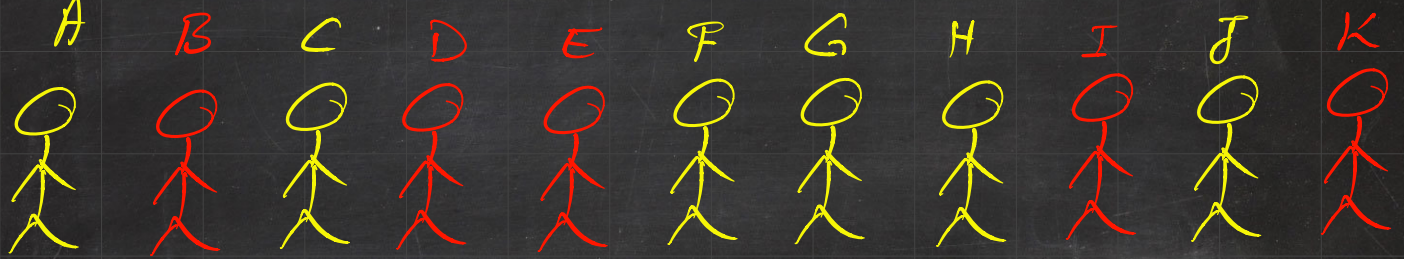
How many right-handed? → Assign Degrees

How many left-handed? → Assign Degrees



EXTENDED NODE LIST

THE NAMING (NUMBERING) IS ARBITRARY!



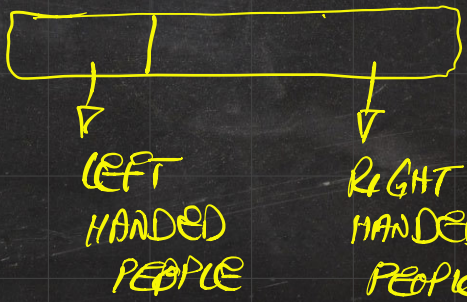
LEFT HANDED : NEUMANN-TYPE BC

RIGHT HANDED : DIRICHLET-TYPE BC

How many people?

How many right-handed? → Assign Degrees

How many left-handed? → Assign Degrees



EXTENDED NODE LIST

THE NAMING (NUMBERING) IS ARBITRARY!

↳ GLOBAL DEGREE

↳ 6 ↔ # DoFs

↳ 5 ↔ # DoCs

EXTENDED NODE LIST

NL	COOR.	BC INFO	TMP DEGREE	DEGREE
1	...	D	DoFs=0; DoCs=0;	
2	...	N	IF (D)	
3	...	N	DoCs = DoCs-1;	
4	...	D	IF (N)	
5	...	N	DoFs = DoFs+1;	

EXTENDED NODE LIST

NL	COOR.	BC INFO	TMP DEGREE	DEGREE
1	o o o	D	-1	
2	o o o	N	1	
3	o o o	N	2	
4	o o o	D	-2	
5	o o o	N	3	

```
DoFs=0; DoCs=0;
IF (D)
  DoCs = DoCs-1;
IF (N)
  DoFs = DoFs+1;
```

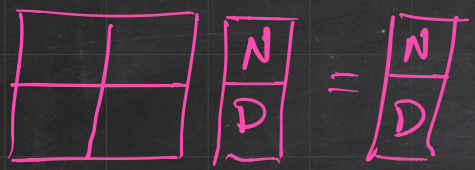
⇒ { 3 DoFs, 2 DoCs }

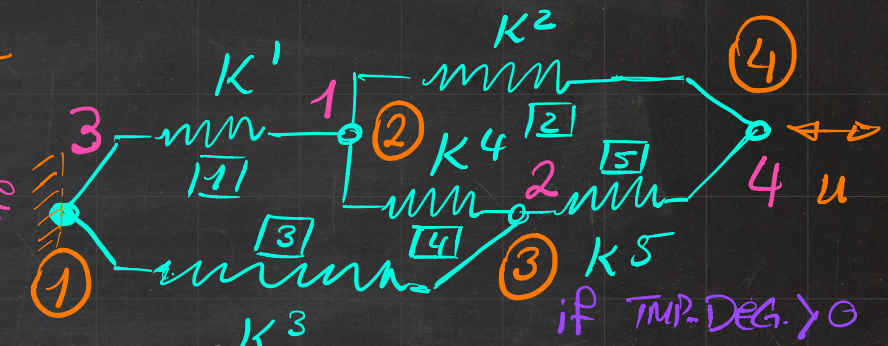
EXTENDED NODE LIST

if TMP. DEG. > 0
DEG = TMP. DEG
else
DEG = DOFs + |TMP. D|

NL	COOR.	BC INFO	TMP DEGREE	DEGREE
1	o o o	D	-1	4
2	o o o	N	1	1
3	o o o	N	2	2
4	o o o	D	-2	5
5	o o o	N	3	3

EXTENDED NODE LIST

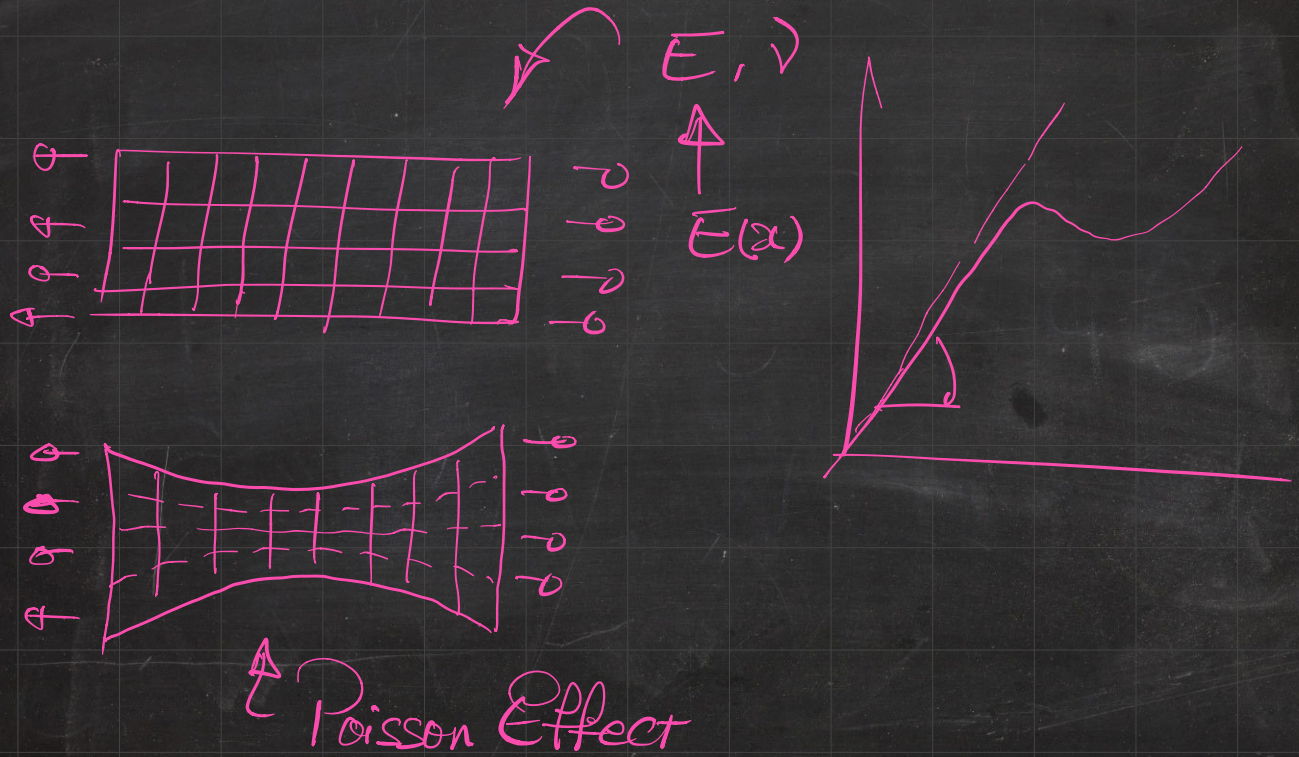




NL	COOR.	BC INFO	TMP DEGREE	DEGREE
1	...	D	-1	3
2	...	N	1	1
3	...	N	2	2
4	...	D	-2	4

if TMP.DEG. > 0
DEG = TMP.DEG
else
DEG = DOFS + |TMP.D|

```
DOFS=0; DOCS=0;
IF (D)
  DOCS = DOCS-1;
IF (N)
  DOFS = DOFS+1;
```


E, ν
E(x)
Poisson Effect