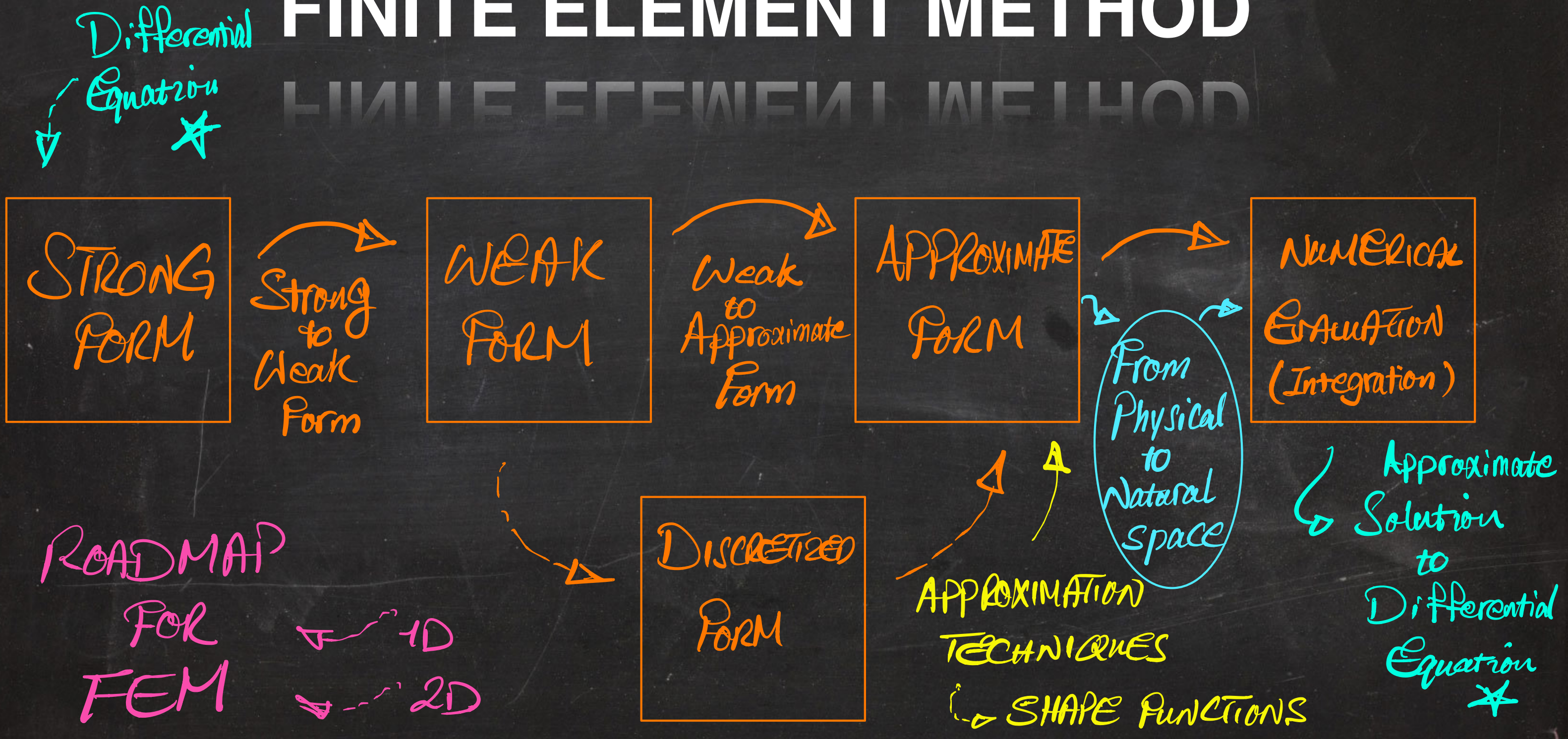
FINITE ELEMENT METHOD

22

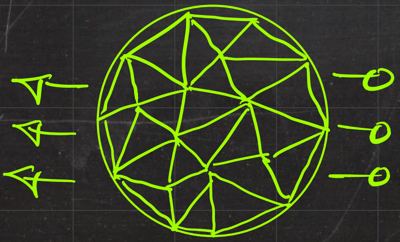
FINITE ELEMENT METHOD



UNDERSTANDING FEM VIA AN ANALOGY (A BRUTAL SIMPLIFICATION)

Approximations in FEM

- → Solution Approximation ⇝ inherent to numerical techniques using computers
- → Equation Approximation ⇝ diff equation is transformed to weak form + space approximation
- → Input Approximation ⇝ space discretization



Discretization (Approximation)

- → DOMAIN (X)
- → SOLUTION (u)
- → TEST (w)

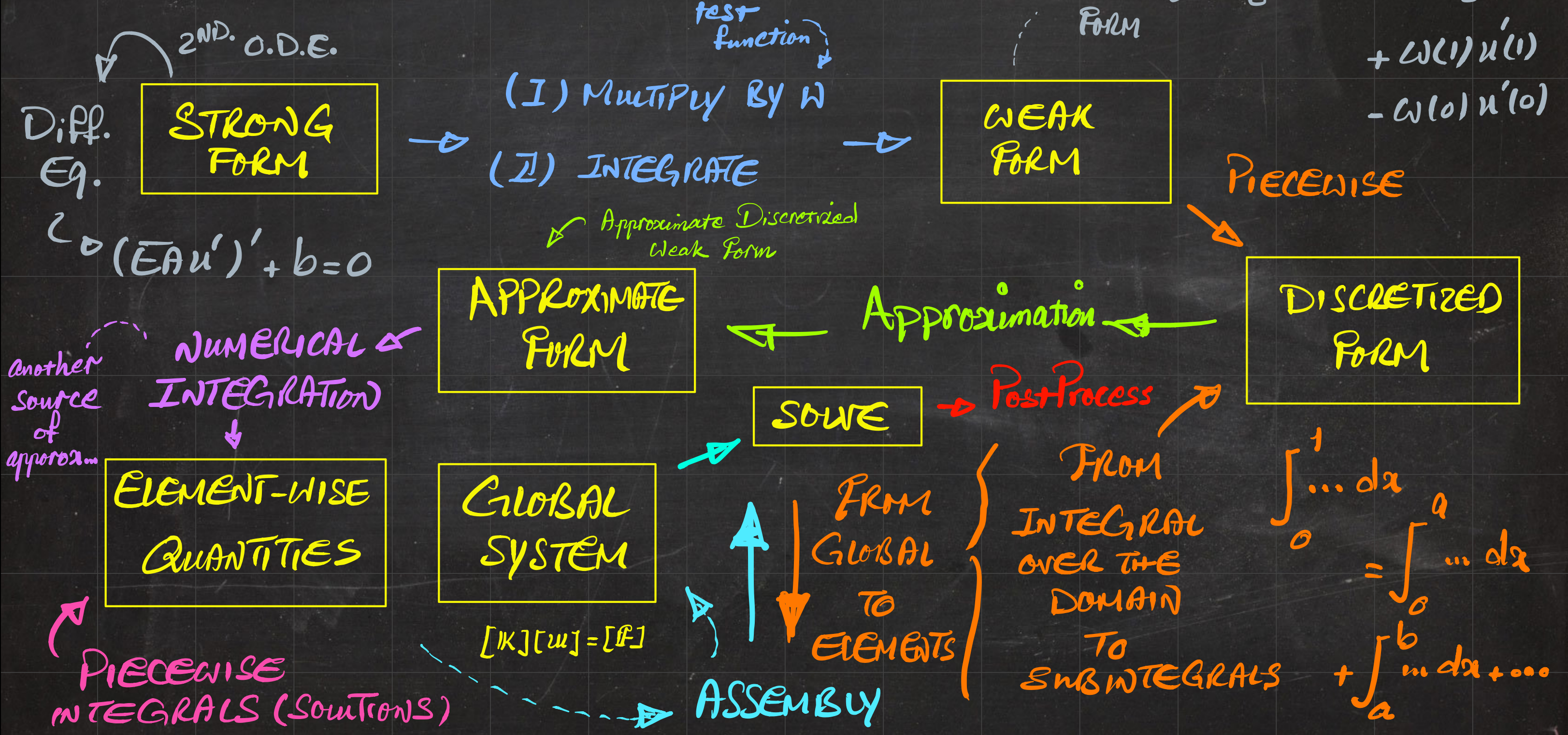
diff. Eq.

→ STRONG FORM

TO

integral → WEAK FORM

FINITE ELEMENT METHOD IN A NUTSHELL



$$(EAu')' + b = 0$$

$$\int_0^1 w'u'\,dx = \int_0^1 w\,dx + w(1)u'(1) - w(0)u'(0)$$

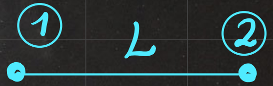
$$[\mathbb{K}][u] = [\mathbb{F}]$$

$$\int_0^1 \dots dx = \int_0^a \dots dx + \int_a^b \dots dx + \dots$$

1D FEM

Overview and Wrap-up

FROM STRONG FORM TO ELEMENT STIFFNESS ← IN PHYSICAL SPACE NOW



$EA\,u'' = 0$ SUBJECT TO BCs

MULTIPLY BY TEST FUNCTION w

DIRICHLET → u IS PRESCRIBED

NEUMANN → u' IS PRESCRIBED

$$EA\,w\,u'' = 0 \quad \leftarrow \quad w\,u'' = (w\,u')' - w'\,u'$$

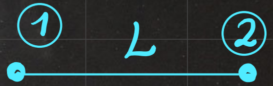
$$EA\left[(w\,u')' - w'\,u'\right] = 0 \Rightarrow EA\,w'\,u' = EA\,(w\,u')'$$

← INTEGRATE

$$\int_L EA\,w'\,u'\,dx = \int_L EA\,(w\,u')'\,dx = EA\,w\,u'\Big]_{①}^{②} = EA\,w^2\,u'^2 - EA\,w^1\,u'^1$$

FROM STRONG FORM TO ELEMENT STIFFNESS ← IN PHYSICAL SPACE NOW

$EAu'' = 0$ SUBJECT TO BCs



$$EA \begin{bmatrix} \int_L N^{1'} N^{1'}\,dx & \int_L N^{1'} N^{2'}\,dx \\ \int_L N^{2'} N^{1'}\,dx & \int_L N^{2'} N^{2'}\,dx \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} = \begin{bmatrix} F^1 \\ F^2 \end{bmatrix}$$

$$\begin{bmatrix} K^{11} & K^{12} \\ K^{21} & K^{22} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} = \begin{bmatrix} F^1 \\ F^2 \end{bmatrix} \leftarrow K^{ij} = EA \int_L N^{i'} N^{j'}\,dx$$

$$K^{ij} = EA \int_L N^{i\prime} N^{j\prime} \, dx$$

PHYSICAL

RECALL:

$$\int_{-1}^{1} g(\xi)\, d\xi = \sum_{gp=1}^{GPE} g(\xi_{gp})\, \alpha_{gp}$$

NATURAL

$$= EA \int_{-1}^{1} \frac{\partial N^i}{\partial \xi} \, \frac{\partial N^j}{\partial \xi} \, J^{-1} \, d\xi$$

$$= EA \sum_{gp=1}^{GPE} \left\{ \left[\frac{\partial N^i}{\partial \xi} \, \frac{\partial N^j}{\partial \xi} \, J^{-1} \right]_{gp} \times \alpha_{gp} \right\}$$

Loop over gp

⋮

END

eg. in MATLAB

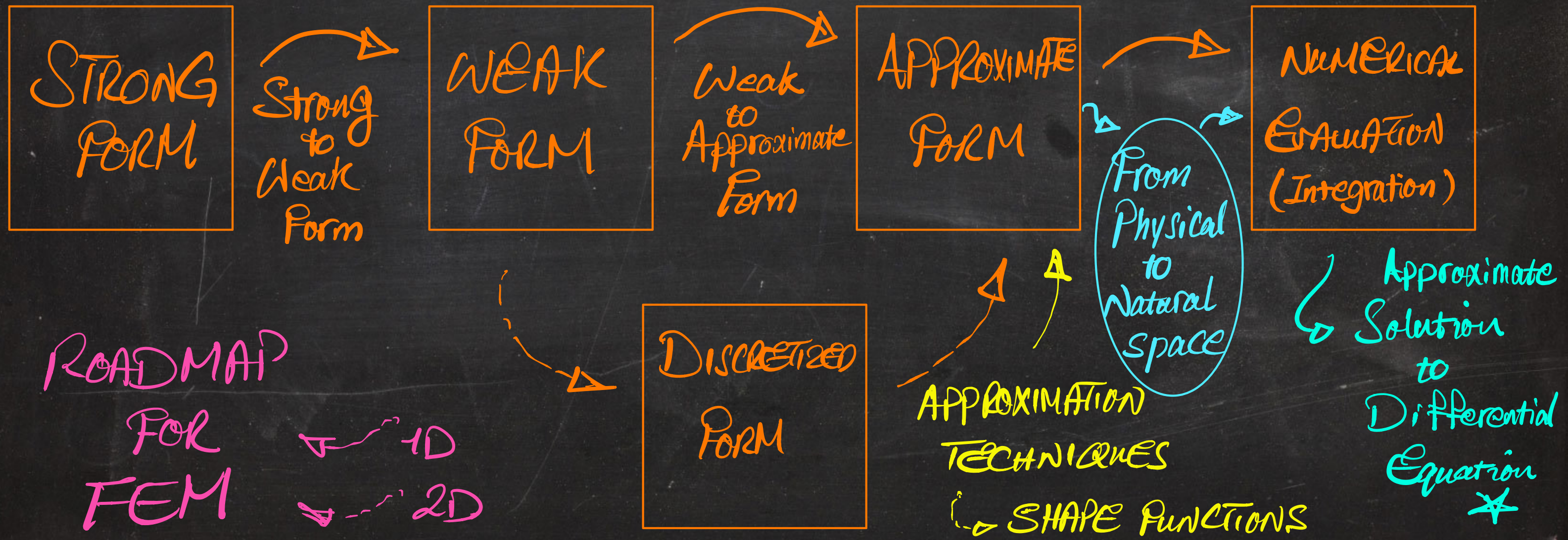
WHAT YOU SEE IN THE CODE !

```
For gp=1:GPE
  ⋮
End
```

2D FEM

FINITE ELEMENT METHOD

Differential Equation



MATHEMATICAL PRELIMINARIES

EINSTEIN SUMMATION CONVENTION

→ A little definition for notation convenience

→ A REPEATED INDEX TWICE MEANS SUMMATION OVER THAT INDEX

also, called "dummy index"

$$\sum_{i=1}^{3} u_i v_i = u_1 v_1 + u_2 v_2 + u_3 v_3 \equiv \underbrace{u_i v_i}_{}$$

← i is summation index

i: free index

$$\sum_{j=1}^{3} A_{ij} u_j \Rightarrow \begin{cases} i=1 \Rightarrow A_{11} u_1 + A_{12} u_2 + A_{13} u_3 \\ i=2 \Rightarrow A_{21} u_1 + A_{22} u_2 + A_{23} u_3 \\ i=3 \Rightarrow A_{31} u_1 + A_{32} u_2 + A_{33} u_3 \end{cases} \Rightarrow \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \equiv A_{ij} \underbrace{u_j}_{}$$

$$1 \leq i \leq 3$$

j: summation index

$$\mathbb{u} \circ \mathbb{v} = u_x v_x + u_y v_y + u_z v_z = u_1 v_1 + u_2 v_2 + u_3 v_3 = \sum_{i=1}^{3} u_i v_i = u_i v_i$$

↳ Dot Product ($\mathbb{u}, \mathbb{v}$) $\rightsquigarrow$ SCALAR $\leftarrow$ $u_i v_i$ $\leftarrow$ $\mathbb{u} \circ \mathbb{v}$

Double Dot Product ($\mathbb{A}, \mathbb{B}$) $\rightsquigarrow$ SCALAR $\leftarrow$ $A_{ij} B_{ij}$ $\leftarrow$ $\mathbb{A} : \mathbb{B}$

$\mathbb{u} \otimes \mathbb{v}$ Dyadic Product ($\mathbb{u}, \mathbb{v}$) $\rightsquigarrow$ MATRIX $\leftarrow$ $u_i v_j$ $\leftarrow$ $[\mathbb{u} \otimes \mathbb{v}]_{ij}$

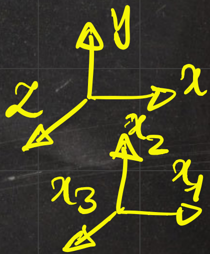
KRONECKER DELTA $\rightsquigarrow$ $\delta_{ij} = \Phi_i \cdot \Phi_j \Rightarrow \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

$\mathbb{u} \cdot \mathbb{v} = u_i v_i$ $\qquad [\mathbb{A} \cdot \mathbb{B}]_{ik} = A_{ij} B_{jk}$ $\qquad [\mathbb{u} \otimes \mathbb{v}]_{ij} = u_i v_j$

$\delta_{ij} = \mathbb{e}_i \cdot \mathbb{e}_j$ $\qquad [\mathbb{A} \cdot \mathbb{u}]_i = A_{ij} u_j$ $\qquad \mathbb{A} : \mathbb{B} = A_{ij} B_{ij}$

$\mathbb{E}$ ↝ FOURTH-ORDER TENSOR (ARRAY) ↝ $3 \times 3 \times 3 \times 3 = 81$ Components

$$\overset{\text{2nd.}}{\mathbb{B}} = \overset{\text{4th.}}{\mathbb{E}} : \overset{\text{2nd.}}{\mathbb{e}} \quad \rightsquigarrow \quad [\mathbb{B}]_{ij} = [\mathbb{E}]_{ijkl} [\mathbb{e}]_{kl}$$



INSTEAD OF x, y, z ⟿ $1, 2, 3$ ↝ x_1, x_2, x_3 $\qquad \mathbb{e}_x \rightsquigarrow \mathbb{e}_1$

DERIVATIVES ← e.g. STRONG FORM ⟶ u''

$$y = f(x) \longrightarrow y' = f'(x) \quad \leftarrow \quad \{\circ\}' = \frac{\partial \{\circ\}}{\partial x}$$

$$\boldsymbol{u} = \boldsymbol{u}(x, y, z) \longrightarrow \text{GRAD } \boldsymbol{u} \quad \text{OR} \quad \text{DIV } \boldsymbol{u} \quad \text{OR} \quad \text{CURL } \boldsymbol{u}$$

$$\frac{\partial \boldsymbol{u}}{\partial x_i} \otimes \boldsymbol{e}_i \qquad \frac{\partial \boldsymbol{u}}{\partial x_i} \cdot \boldsymbol{e}_i \qquad \frac{\partial \boldsymbol{u}}{\partial x_i} \times \boldsymbol{e}_i$$

DERIVATIVES ⟵ e.g. STRONG FORM $\rightarrow u''$

$y = f(x) \rightarrow y' = f'(x)$ ⟵ $\{\circ\}' = \dfrac{\partial \{\circ\}}{\partial x}$

$\rightarrow \text{GRAD}\, \mathbf{u} = \dfrac{\partial \mathbf{u}}{\partial x_i} \otimes \mathbf{e}_i$

$\mathbf{u} = \mathbf{u}(x, y, z) \rightarrow$ GRAD $\mathbf{u}$ OR DIV $\mathbf{u}$

$\rightarrow \text{DIV}\, \mathbf{u} = \dfrac{\partial \mathbf{u}}{\partial x_i} \cdot \mathbf{e}_i$

$w = f(x, y, z) \rightarrow$ GRAD f ⟵ $\dfrac{\partial f}{\partial x_i} \mathbf{e}_i$

$\mathbb{A} = \mathbb{A}(x, y, z) \rightarrow$ GRAD $\mathbb{A}$ OR DIV $\mathbb{A}$

$\hookrightarrow \dfrac{\partial \mathbb{A}}{\partial x_i} \otimes \mathbf{e}_i$ $\hookrightarrow \dfrac{\partial \mathbb{A}}{\partial x_i} \cdot \mathbf{e}_i$

SCALAR → 0, VECTOR → 1, MATRIX → 2

GRADIENT INCREASES THE ORDER BY 1

DIVERGENCE REDUCES THE ORDER BY 1

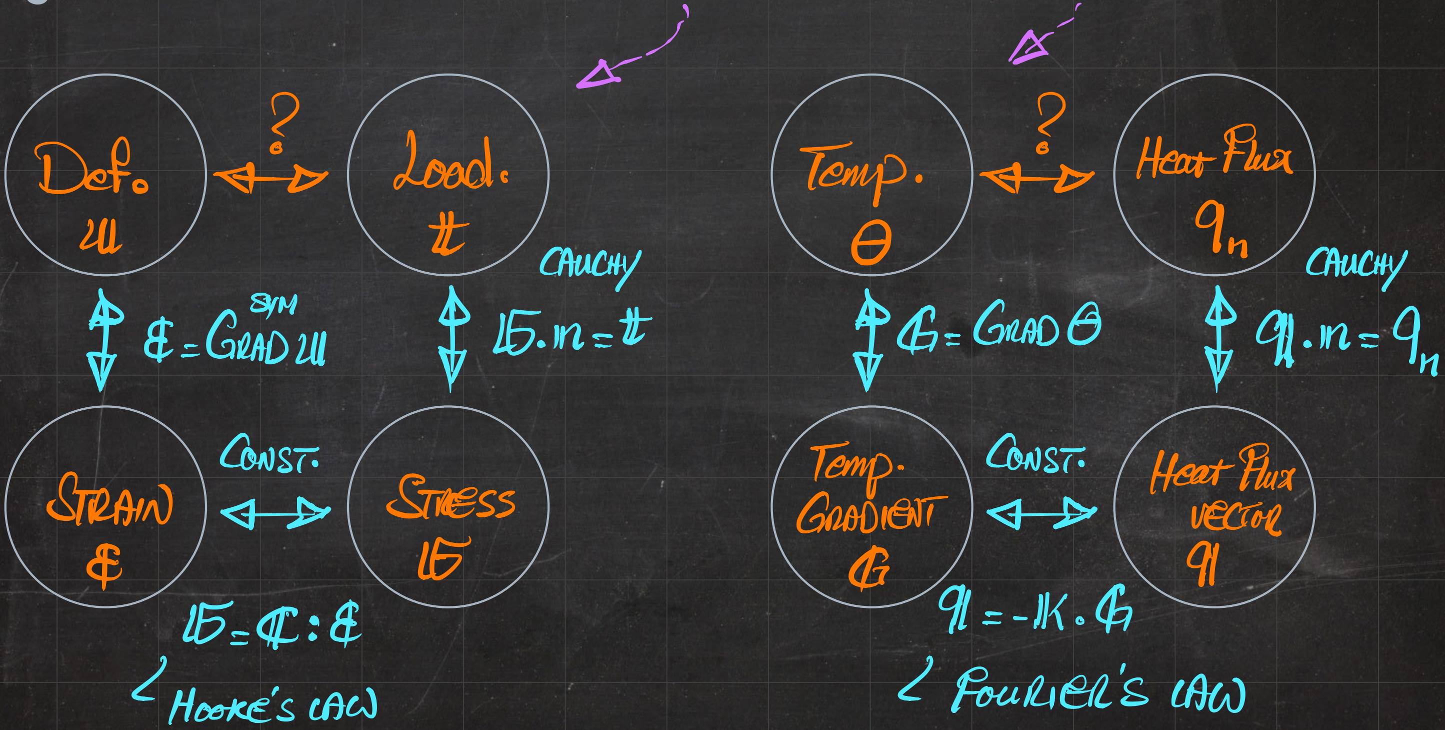
$$\text{GRAD}\,\phi = \begin{bmatrix} \partial\phi/\partial x_1 \\ \partial\phi/\partial x_2 \\ \partial\phi/\partial x_3 \end{bmatrix}$$

$$\text{GRAD}\,u = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

$$\text{DIV}\,u = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

$$\text{DIV}\,A = \begin{bmatrix} \frac{\partial A_{11}}{\partial x_1} + \frac{\partial A_{12}}{\partial x_2} + \frac{\partial A_{13}}{\partial x_3} \\ \frac{\partial A_{21}}{\partial x_1} + \frac{\partial A_{22}}{\partial x_2} + \frac{\partial A_{23}}{\partial x_3} \\ \frac{\partial A_{31}}{\partial x_1} + \frac{\partial A_{32}}{\partial x_2} + \frac{\partial A_{33}}{\partial x_3} \end{bmatrix}$$

Big Picture of Mechanics (Mechanical Problems & Thermal Problems)



STRONG FORM (GENERIC FORM) $\rightsquigarrow$ $\mathrm{Div}\,\boldsymbol{\sigma} + \mathbf{b} = \mathbf{0}$, $\mathrm{Div}\,\mathbf{q} + c = 0$

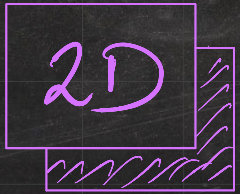
$$\frac{\partial \sigma_{jk}}{\partial x_k} + b_j = 0$$

$$\frac{\partial q_i}{\partial x_i} + c = 0$$

$$\begin{cases} \dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{xy}}{\partial y} + b_x = 0 \\[2ex] \dfrac{\partial \sigma_{yx}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} + b_y = 0 \end{cases}$$

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + c = 0$$

$$2\mathrm{D} \rightsquigarrow \left.\begin{array}{l} \text{Plane STRAIN} \\ \text{Plane STRESS} \end{array}\right\} \boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\varepsilon}$$



APPROXIMATION USING 2D FINITE ELEMENTS ← Shape Functions

EXAMPLE I:

$f(-1,-1) = 1$

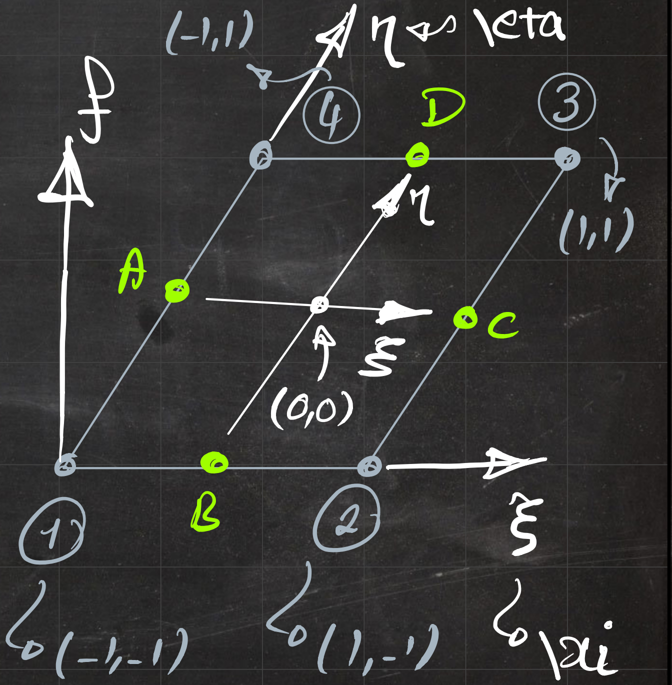
$f(1,-1) = 2$

$f(1,1) = -1$

$f(-1,1) = 4$

$$f(\xi,\eta) = N^1 f^1 + N^2 f^2 + N^3 f^3 + N^4 f^4$$

→ $f(0,0) = ?$, $f(\xi,\eta) = ?$



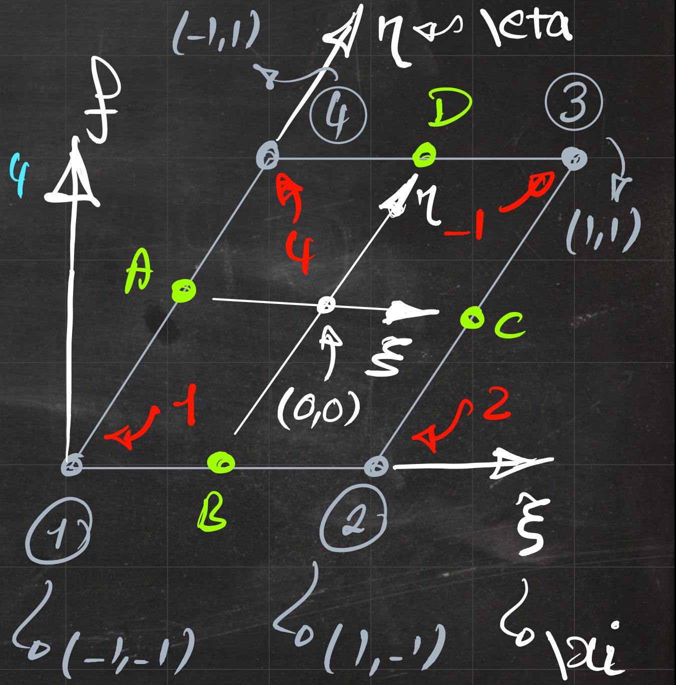
APPROXIMATION USING 2D FINITE ELEMENTS ← ✓ Shape Functions

EXAMPLE I:

$$N^1(\xi,\eta) = \frac{1}{4}[\xi-1][\eta-1]$$
$$N^2(\xi,\eta) = -\frac{1}{4}[\xi+1][\eta-1]$$
$$N^3(\xi,\eta) = \frac{1}{4}[\xi+1][\eta+1]$$
$$N^4(\xi,\eta) = -\frac{1}{4}[\xi-1][\eta+1]$$

$$f(\xi,\eta) = N^i f^i$$

$$= \frac{1}{4}[\xi-1][\eta-1] \times 1$$
$$- \frac{1}{4}[\xi+1][\eta-1] \times 2$$
$$+ \frac{1}{4}[\xi+1][\eta+1] \times -1$$
$$- \frac{1}{4}[\xi-1][\eta+1] \times 4$$
$$= \circ\circ\circ = f(\xi,\eta) \checkmark$$



2D Finite Element Library

two-dimensional finite elements library

- two-dimensional 4-noded quadrilateral element (D2QU4N)

  a.k.a. bilinear quadrilateral element

- two-dimensional 9-noded quadrilateral element (D2QU9N)

  a.k.a. Lagrange biquadratic quadrilateral element

- two-dimensional 8-noded quadrilateral element (D2QU8N)

  a.k.a. serendipity biquadratic quadrilateral element

- two-dimensional 3-noded triangular element (D2TR3N)

  a.k.a. constant strain triangle

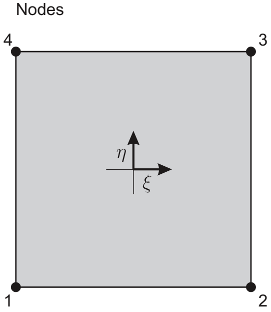
- two-dimensional 6-noded triangular element (D2TR6N)

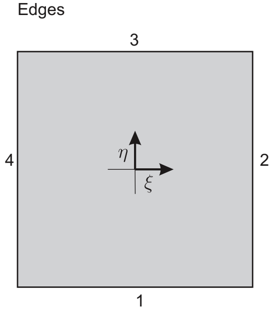
  a.k.a. quadratic triangle

- two-dimensional quadrature rule

D2QU4N

bilinear quadrilateral element





Node Number	Coordinates	
	ξ	η
1	-1	-1
2	1	-1
3	1	1
4	-1	1

$$N^1 = \frac{1}{4}\left(1-\xi\right)\left(1-\eta\right)$$

$$N^2 = \frac{1}{4}\left(1+\xi\right)\left(1-\eta\right)$$

$$N^3 = \frac{1}{4}\left(1+\xi\right)\left(1+\eta\right)$$

$$N^4 = \frac{1}{4}\left(1-\xi\right)\left(1+\eta\right)$$

$$N^1_{,\xi} = -\frac{1}{4}\left(1-\eta\right) \qquad N^1_{,\eta} = -\frac{1}{4}\left(1-\xi\right)$$

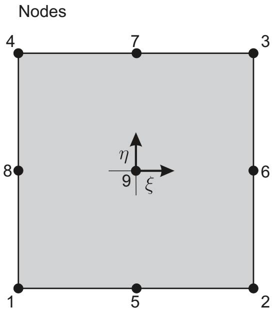
$$N^2_{,\xi} = +\frac{1}{4}\left(1-\eta\right) \qquad N^2_{,\eta} = -\frac{1}{4}\left(1+\xi\right)$$

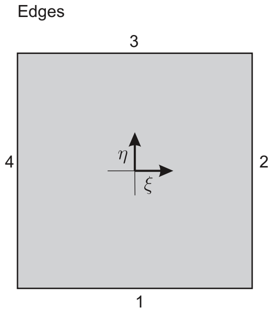
$$N^3_{,\xi} = +\frac{1}{4}\left(1+\eta\right) \qquad N^3_{,\eta} = +\frac{1}{4}\left(1+\xi\right)$$

$$N^4_{,\xi} = -\frac{1}{4}\left(1+\eta\right) \qquad N^4_{,\eta} = +\frac{1}{4}\left(1-\xi\right)$$

D2QU9N

Lagrange biquadratic quadrilateral element





Node Number	Coordinates	
	ξ	η
1	-1	-1
2	1	-1
3	1	1
4	-1	1
5	0	-1
6	1	0
7	0	1
8	-1	0
9	0	0

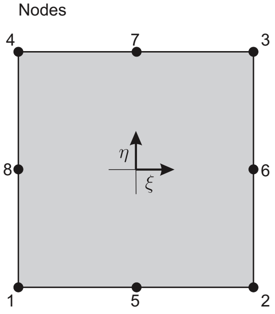
$$\begin{aligned}
N^1 &= +\frac{1}{4}\left(1-\xi\right)\xi\left(1-\eta\right)\eta \\
N^2 &= -\frac{1}{4}\left(1+\xi\right)\xi\left(1-\eta\right)\eta \\
N^3 &= +\frac{1}{4}\left(1+\xi\right)\xi\left(1+\eta\right)\eta \\
N^4 &= -\frac{1}{4}\left(1-\xi\right)\xi\left(1+\eta\right)\eta \\
N^5 &= -\frac{1}{2}\left(1-\xi\right)\left(1+\xi\right)\left(1-\eta\right)\eta \\
N^6 &= +\frac{1}{2}\left(1+\xi\right)\xi\left(1-\eta\right)\left(1+\eta\right) \\
N^7 &= +\frac{1}{2}\left(1-\xi\right)\left(1+\xi\right)\left(1+\eta\right)\eta \\
N^8 &= -\frac{1}{2}\left(1-\xi\right)\xi\left(1-\eta\right)\left(1+\eta\right) \\
N^9 &= \left(1-\xi\right)\left(1+\xi\right)\left(1-\eta\right)\left(1+\eta\right)
\end{aligned}$$

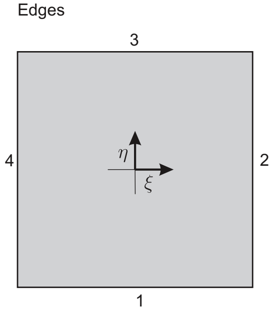
$$\begin{aligned}
N^1_{,\xi} &= +\frac{1}{4}\left(1-2\xi\right)\left(1-\eta\right)\eta \\
N^2_{,\xi} &= -\frac{1}{4}\left(1+2\xi\right)\left(1-\eta\right)\eta \\
N^3_{,\xi} &= +\frac{1}{4}\left(1+2\xi\right)\left(1+\eta\right)\eta \\
N^4_{,\xi} &= -\frac{1}{4}\left(1-2\xi\right)\left(1+\eta\right)\eta \\
N^5_{,\xi} &= \xi\eta\left(1-\eta\right) \\
N^6_{,\xi} &= \frac{1}{2}\left(1+2\xi\right)\left(1-\eta\right)\left(1+\eta\right) \\
N^7_{,\xi} &= -\xi\eta\left(1+\eta\right) \\
N^8_{,\xi} &= -\frac{1}{2}\left(1-2\xi\right)\left(1-\eta\right)\left(1+\eta\right) \\
N^9_{,\xi} &= -2\xi\left(1-\eta\right)\left(1+\eta\right)
\end{aligned}$$

$$\begin{aligned}
N^1_{,\eta} &= +\frac{1}{4}\left(1-\xi\right)\xi\left(1-2\eta\right) \\
N^2_{,\eta} &= -\frac{1}{4}\left(1+\xi\right)\xi\left(1-2\eta\right) \\
N^3_{,\eta} &= +\frac{1}{4}\left(1+\xi\right)\xi\left(1+2\eta\right) \\
N^4_{,\eta} &= -\frac{1}{4}\left(1-\xi\right)\xi\left(1+2\eta\right) \\
N^5_{,\eta} &= \frac{1}{2}\left(1-\xi\right)\left(1+\xi\right)\left(2\eta-1\right) \\
N^6_{,\eta} &= -\left(1+\xi\right)\xi\eta \\
N^7_{,\eta} &= \frac{1}{2}\left(1-\xi\right)\left(1+\xi\right)\left(1+2\eta\right) \\
N^8_{,\eta} &= \left(1-\xi\right)\xi\eta \\
N^9_{,\eta} &= -2\left(1-\xi\right)\left(1+\xi\right)\eta
\end{aligned}$$

D2QU8N

serendipity biquadratic quadrilateral element





Node Number	Coordinates	
	ξ	η
1	-1	-1
2	1	-1
3	1	1
4	-1	1
5	0	-1
6	1	0
7	0	1
8	-1	0

$$N^1 = -\frac{1}{4}\left(1-\xi\right)\left(1-\eta\right)\left(1+\xi+\eta\right) \qquad N^1_{,\xi} = +\frac{1}{4}\left(1-\eta\right)\left(2\xi+\eta\right) \qquad N^1_{,\eta} = +\frac{1}{4}\left(1-\xi\right)\left(\xi+2\eta\right)$$

$$N^2 = -\frac{1}{4}\left(1+\xi\right)\left(1-\eta\right)\left(1-\xi+\eta\right) \qquad N^2_{,\xi} = +\frac{1}{4}\left(1-\eta\right)\left(2\xi-\eta\right) \qquad N^2_{,\eta} = +\frac{1}{4}\left(1+\xi\right)\left(-\xi+2\eta\right)$$

$$N^3 = -\frac{1}{4}\left(1+\xi\right)\left(1+\eta\right)\left(1-\xi-\eta\right) \qquad N^3_{,\xi} = +\frac{1}{4}\left(1+\eta\right)\left(2\xi+\eta\right) \qquad N^3_{,\eta} = +\frac{1}{4}\left(1+\xi\right)\left(\xi+2\eta\right)$$

$$N^4 = -\frac{1}{4}\left(1-\xi\right)\left(1+\eta\right)\left(1+\xi-\eta\right) \qquad N^4_{,\xi} = +\frac{1}{4}\left(1+\eta\right)\left(2\xi-\eta\right) \qquad N^4_{,\eta} = +\frac{1}{4}\left(1-\xi\right)\left(-\xi+2\eta\right)$$

$$N^5 = \frac{1}{2}\left(1-\xi\right)\left(1+\xi\right)\left(1-\eta\right) \qquad N^5_{,\xi} = -\xi\left(1-\eta\right) \qquad N^5_{,\eta} = -\frac{1}{2}\left(1-\xi\right)\left(1+\xi\right)$$

$$N^6 = \frac{1}{2}\left(1+\xi\right)\left(1+\eta\right)\left(1-\eta\right) \qquad N^6_{,\xi} = +\frac{1}{2}\left(1-\eta\right)\left(1+\eta\right) \qquad N^6_{,\eta} = -\left(1+\xi\right)\eta$$

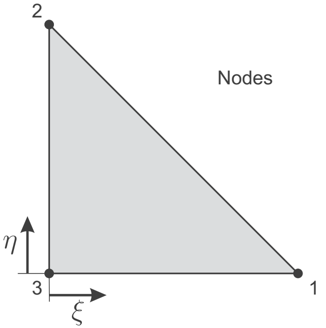
$$N^7 = \frac{1}{2}\left(1-\xi\right)\left(1+\xi\right)\left(1+\eta\right) \qquad N^7_{,\xi} = -\xi\left(1+\eta\right) \qquad N^7_{,\eta} = +\frac{1}{2}\left(1-\xi\right)\left(1+\xi\right)$$

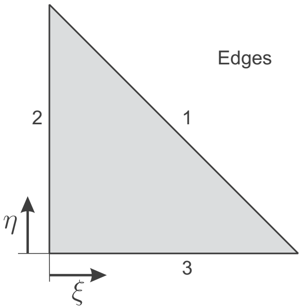
$$N^8 = \frac{1}{2}\left(1-\xi\right)\left(1+\eta\right)\left(1-\eta\right) \qquad N^8_{,\xi} = -\frac{1}{2}\left(1-\eta\right)\left(1+\eta\right) \qquad N^8_{,\eta} = -\left(1-\xi\right)\eta$$

2D Finite Element Library

D2TR3N

constant strain triangle (CST)





Node Number	Coordinates	
	ξ	η
1	1	0
2	0	1
3	0	0

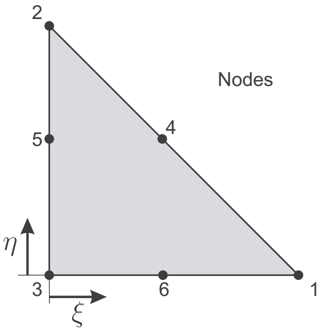
$$N^1 = \xi \qquad N^1_{,\xi} = 1 \qquad N^1_{,\eta} = 0$$

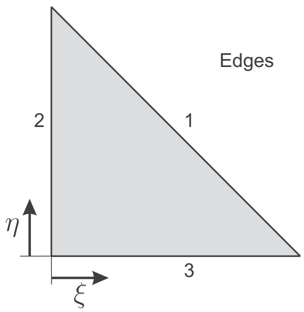
$$N^2 = \eta \qquad N^2_{,\xi} = 0 \qquad N^2_{,\eta} = 1$$

$$N^3 = (1 - \xi - \eta) \qquad N^3_{,\xi} = -1 \qquad N^3_{,\eta}(\xi, \eta) = -1$$

D2TR6N

quadratic triangle





Node Number	Coordinates	
	ξ	η
1	1	0
2	0	1
3	0	0
4	1/2	1/2
5	0	1/2
6	1/2	0

$$N^1 = \xi(2\xi - 1) \qquad N^1_{,\xi} = -1 + 4\xi \qquad N^1_{,\eta} = 0$$

$$N^2 = \eta(2\eta - 1) \qquad N^2_{,\xi} = 0 \qquad N^2_{,\eta} = -1 + 4\eta$$

$$N^3 = (1 - \xi - \eta)(1 - 2\xi - 2\eta) \qquad N^3_{,\xi} = -3 + 4\xi + 4\eta \qquad N^3_{,\eta} = -3 + 4\xi + 4\eta$$

$$N^4 = 4\xi\eta \qquad N^4_{,\xi} = 4\eta \qquad N^4_{,\eta} = 4\xi$$

$$N^5 = 4\eta(1 - \xi - \eta) \qquad N^5_{,\xi} = -4\eta \qquad N^5_{,\eta} = -4(-1 + 2\eta + \xi)$$

$$N^6 = 4\xi(1 - \xi - \eta) \qquad N^6_{,\xi} = -4(-1 + \eta + 2\xi) \qquad N^6_{,\eta} = -4\xi$$

two-dimensional quadrature rule i

Triangular Elements Gauss Point Rule

$$\int_0^1 \int_0^{1-\eta} \{\bullet\} \, \mathrm{d}\xi \, \mathrm{d}\eta \approx \frac{1}{2} \sum_{i=1}^{N_{\text{Gauss Points}}} \alpha_i \, \{\bullet\}|_{\text{Gauss Point}^i}$$

Gauss Point Number	Coordinates		Weight Factor
	ξ	η	α
1	1/3	1/3	1

Gauss Point Number	Coordinates		Weight Factor
	ξ	η	α
1	1/6	1/6	1/3
2	4/6	1/6	1/3
3	1/6	4/6	1/3

two-dimensional quadrature rule ii

Quadrilateral Elements Gauss Point Rule

$$\int_{-1}^{1}\int_{-1}^{1}\{\bullet\}\ \mathrm{d}\xi\ \mathrm{d}\eta \approx \sum_{i=1}^{N_{\text{Gauss Points}}} \alpha_i\ \{\bullet\}|_{\text{Gauss Point}^i}$$

Gauss Point Number	Coordinates		Weight Factor
	ξ	η	α
1	0	0	2×2

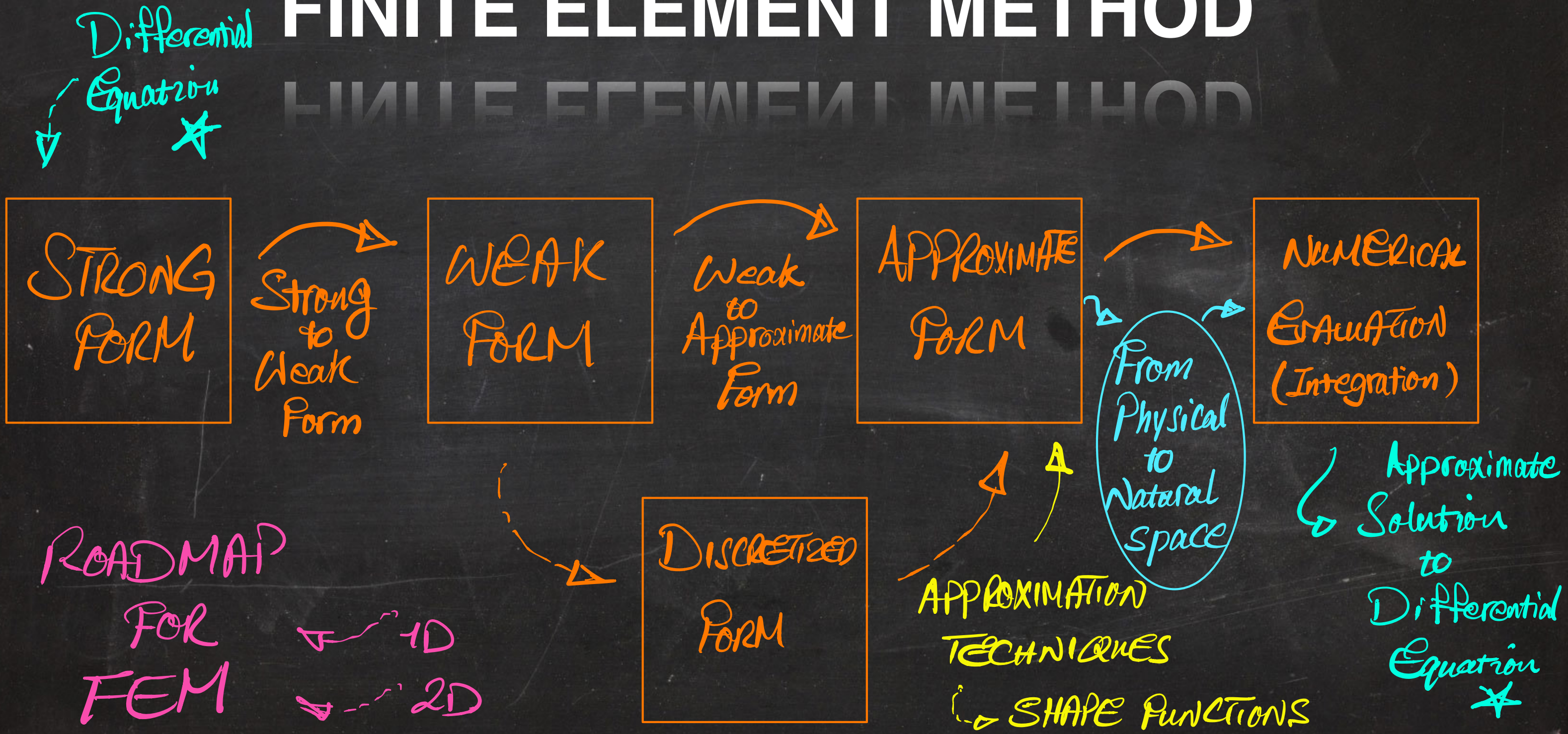
Gauss Point Number	Coordinates		Weight Factor
	ξ	η	α
1	$-1/\sqrt{3}$	$-1/\sqrt{3}$	1×1
2	$+1/\sqrt{3}$	$-1/\sqrt{3}$	1×1
3	$+1/\sqrt{3}$	$+1/\sqrt{3}$	1×1
4	$-1/\sqrt{3}$	$+1/\sqrt{3}$	1×1

2D Finite Element Library

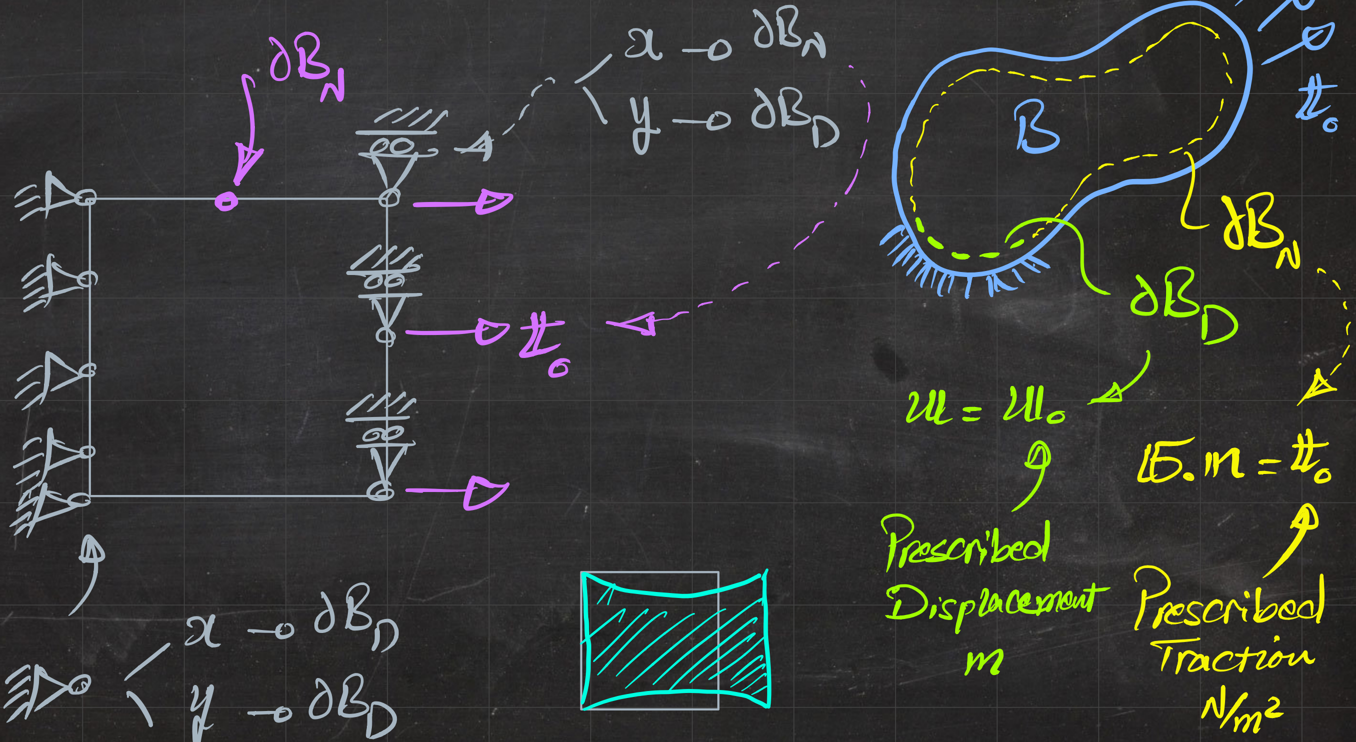
two-dimensional quadrature rule iii

Gauss Point Number	Coordinates		Weight Factor
	ξ	η	α
1	$-\sqrt{3/5}$	$-\sqrt{3/5}$	$5/9 \times 5/9$
2	$+\sqrt{3/5}$	$-\sqrt{3/5}$	$5/9 \times 5/9$
3	$\sqrt{3/5}$	$\sqrt{3/5}$	$5/9 \times 5/9$
4	$-\sqrt{3/5}$	$\sqrt{3/5}$	$5/9 \times 5/9$
5	0	$-\sqrt{3/5}$	$5/9 \times 8/9$
6	$+\sqrt{3/5}$	0	$5/9 \times 8/9$
7	0	$\sqrt{3/5}$	$5/9 \times 8/9$
8	$-\sqrt{3/5}$	0	$5/9 \times 8/9$
9	0	0	$8/9 \times 8/9$

FINITE ELEMENT METHOD



FROM STRONG FORM TO WEAK FORM



$u = u_0$

Prescribed Displacement

m

$\sigma \cdot n = t_0$

Prescribed Traction

N/m²

FROM STRONG FORM TO WEAK FORM

$\mathrm{Div}\,\boldsymbol{\sigma} = 0$ in $\mathcal{B}$ subject to BCs

STRONG FORM IN THE ABSENCE OF BODY FORCES

$\boldsymbol{u} = \boldsymbol{u}_0$ at $\partial\mathcal{B}_D$

$\boldsymbol{\sigma}\cdot\boldsymbol{n} = \boldsymbol{t}_0$ at $\partial\mathcal{B}_N$

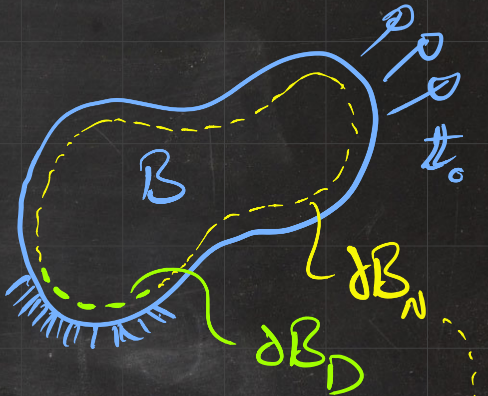
dot

$\boldsymbol{w}\cdot\mathrm{Div}\,\boldsymbol{\sigma} = 0$ SCALAR

TEST FUNCTION

$\forall\,\boldsymbol{w}$

$\boldsymbol{w}\big|_{\partial\mathcal{B}_D} = 0$



$\boldsymbol{u} = \boldsymbol{u}_0$

Prescribed Displacement

m

$\boldsymbol{\sigma}\cdot\boldsymbol{n} = \boldsymbol{t}_0$

Prescribed Traction

N/m^2

FROM STRONG FORM TO WEAK FORM

$\boldsymbol{u} = \boldsymbol{u}_0$ at ∂B_D

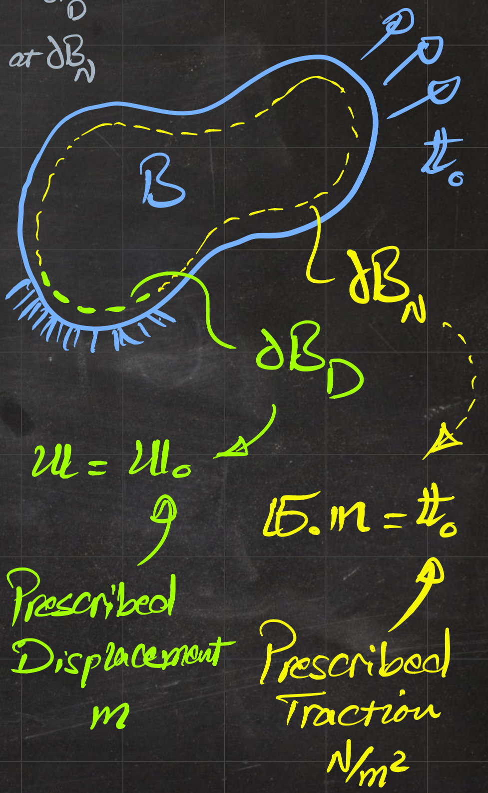
$\boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{t}_0$ at ∂B_N

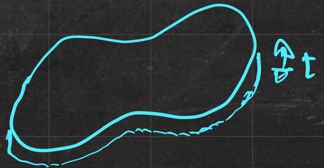
$\mathrm{Div}\, \boldsymbol{\sigma} = 0$ in B subject to BCs

$$\boldsymbol{w} \cdot \mathrm{Div}\, \boldsymbol{\sigma} = 0$$

$$\boldsymbol{w} \cdot \mathrm{Div}\, \boldsymbol{\sigma} = \mathrm{Div}(\boldsymbol{w} \cdot \boldsymbol{\sigma}) - [\mathrm{Grad}\, \boldsymbol{w}] : \boldsymbol{\sigma}$$

$$\int_B [\mathrm{Grad}\, \boldsymbol{w}] : \boldsymbol{\sigma} \, \underset{t\, dA}{dV} = \int_B \mathrm{Div}(\boldsymbol{w} \cdot \boldsymbol{\sigma})\, dV$$



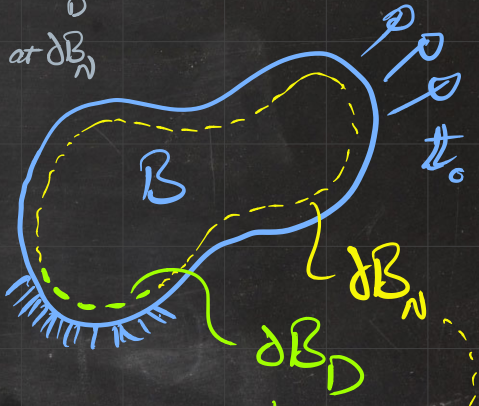


FROM STRONG FORM TO WEAK FORM

$\mathrm{Div}\,\boldsymbol{\sigma} = \boldsymbol{\phi}$ in B subject to BCs

$\boldsymbol{u} = \boldsymbol{u}_0$ at ∂B_D

$\boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{t}_0$ at ∂B_N



$\boldsymbol{w} \cdot \mathrm{Div}\,\boldsymbol{\sigma} = 0$

$$\int_B [\mathrm{Grad}\,\boldsymbol{w}] : \boldsymbol{\sigma}\, dA = \int_B \mathrm{Div}(\boldsymbol{w} \cdot \boldsymbol{\sigma})\, dA$$

$\boldsymbol{u} = \boldsymbol{u}_0$

Prescribed Displacement

m

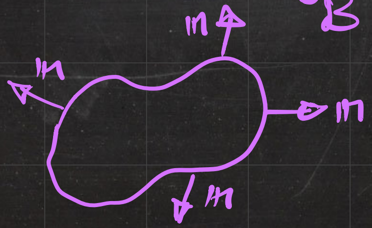
$\boldsymbol{\sigma} . \boldsymbol{n} = \boldsymbol{t}_0$

Prescribed Traction

N/m^2

Divergence Theorem $$\int_B \mathrm{Div}\{\circ\}\, dA = \int_{\partial B} \{\circ\} \cdot \boldsymbol{n}\, dL$$

unit outward normal vector to boundary



FROM STRONG FORM TO WEAK FORM

$\mathbb{u}=\mathbb{u}_0$ at ∂B_D

$\mathbb{P}\cdot\mathbb{n}=\mathbb{t}_0$ at ∂B_N

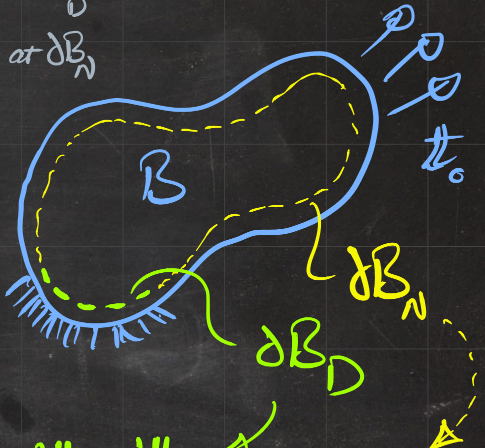
$\text{Div}\,\mathbb{P}=0$ in B subject to BCs ↗

$$\mathbb{w}\cdot\text{Div}\,\mathbb{P}=0$$

$$\int_B [\text{Grad}\,\mathbb{w}]:\mathbb{P}\,dA=\int_B \text{Div}(\mathbb{w}\cdot\mathbb{P})\,dA$$

$$\int_B [\text{Grad}\,\mathbb{w}]:\mathbb{P}\,dA=\int_{\partial B}\mathbb{w}\cdot\mathbb{P}\cdot\mathbb{n}\,dL$$

$$\partial B_D\cap\partial B_N=\emptyset \qquad \partial B=\partial B_D\cup\partial B_N$$



$\mathbb{u}=\mathbb{u}_0$

Prescribed Displacement

m

$\mathbb{P}\cdot\mathbb{n}=\mathbb{t}_0$

Prescribed Traction

N/m^2

FROM STRONG FORM TO WEAK FORM

$u = u_0$ at ∂B_D

$\mathbb{P} \cdot n = t_0$ at ∂B_N

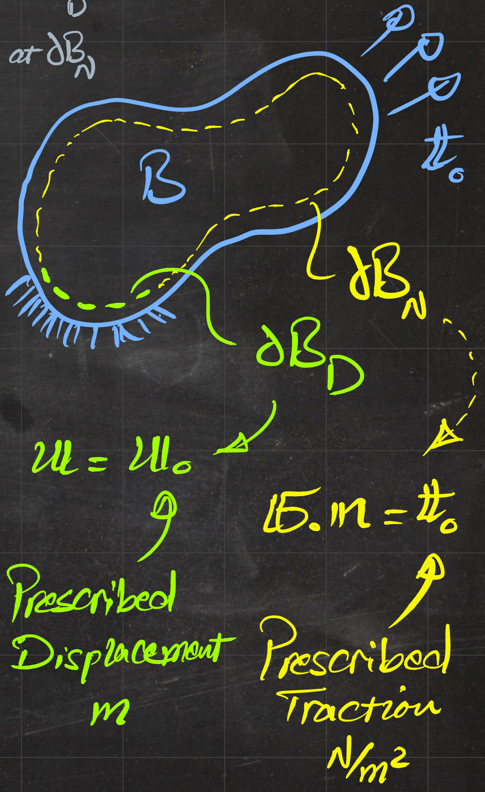
$\mathrm{Div}\, \mathbb{P} = 0$ in B subject to BCs

$$w \cdot \mathrm{Div}\, \mathbb{P} = 0 \quad \forall w, \; w|_{\partial B_D} = 0$$

$$\int_B [\mathrm{Grad}\, w] : \mathbb{P} \, dA = \int_B \mathrm{Div}(w \cdot \mathbb{P}) \, dA$$

$$\int_B [\mathrm{Grad}\, w] : \mathbb{P} \, dA = \int_{\partial B} w \cdot \mathbb{P} \cdot n \, dL$$

$$= \underbrace{\int_{\partial B_D} \overset{0}{w} \cdot \mathbb{P} \cdot n \, dL}_{0} + \int_{\partial B_N} w \cdot \underbrace{\mathbb{P} \cdot n}_{t_0} \, dL$$



FROM STRONG FORM TO WEAK FORM

$\mathbf{u} = \mathbf{u}_0$ at $\partial\mathcal{B}_D$

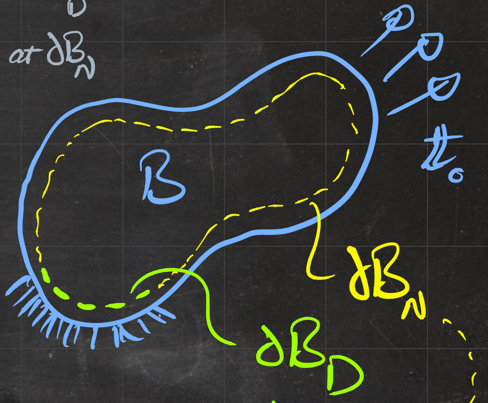
$\mathbb{P}\cdot\mathbf{n} = \mathbf{t}_0$ at $\partial\mathcal{B}_N$

$\text{Div}\,\mathbb{P} = 0$ in $\mathcal{B}$ subject to BCs

$$\mathbf{w}\cdot\text{Div}\,\mathbb{P} = 0 \quad \forall\mathbf{w}\,,\ \mathbf{w}|_{\partial\mathcal{B}_D} = 0$$

$$\int_{\mathcal{B}} [\text{Grad}\,\mathbf{w}] : \mathbb{P}\, dA = \int_{\mathcal{B}} \text{Div}(\mathbf{w}\cdot\mathbb{P})\, dA$$

$$\boxed{\int_{\mathcal{B}} [\text{Grad}\,\mathbf{w}] : \mathbb{P}\, dA = \int_{\partial\mathcal{B}_N} \mathbf{w}\cdot\mathbf{t}_0\, dL}$$



$\mathbf{u} = \mathbf{u}_0$

Prescribed Displacement

m

$\mathbb{P}\cdot\mathbf{n} = \mathbf{t}_0$

Prescribed Traction

N/m^2

FROM WEAK FORM TO APPROXIMATE FORM

$$\boxed{\int_{\mathcal{B}} [\mathrm{GRAD}\,\mathbf{w}] : \mathbf{P}\, dA = \int_{\partial\mathcal{B}_N} \mathbf{w}\cdot\mathbf{t}_0\, dL}$$

$$\mathbf{w}^j \otimes \mathrm{GRAD}\, N^j$$

$$\mathbf{P} = \mathbb{E} : \boldsymbol{\epsilon}$$

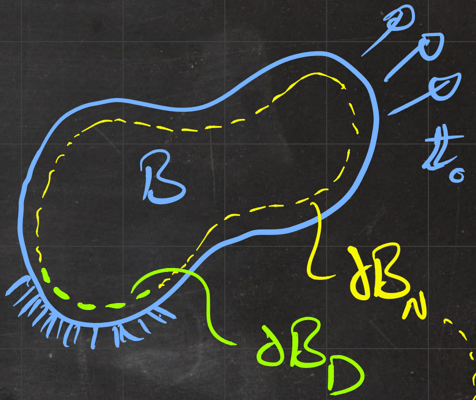
MAJOR
MINOR
SYMMETRY

$$\boldsymbol{\epsilon} = \overset{\mathrm{sym}}{\mathrm{GRAD}\,\mathbf{u}}$$

$$\mathbf{P} = \mathbb{E} : [\mathrm{GRAD}\,\mathbf{u}]$$

$$\mathbf{w} = N^j \mathbf{w}^j = N^1\mathbf{w}^1 + N^2\mathbf{w}^2 + \cdots$$

$$\mathrm{GRAD}\,\mathbf{w} = \mathbf{w}^j \otimes \mathrm{GRAD}\, N^j$$



$$\mathbf{u} = \mathbf{u}_0$$

Prescribed Displacement
m

$$\mathbf{P}\cdot\mathbf{n} = \mathbf{t}_0$$

Prescribed Traction
N/m^2

FROM WEAK FORM TO APPROXIMATE FORM

$$\boxed{\int_{\mathcal{B}} [\mathrm{Grad}\,\mathbf{w}] : \mathbf{P}\, dA = \int_{\partial\mathcal{B}_N} \mathbf{w}\cdot\mathbf{t}_0\, dL}$$

$$\mathbf{w}^j \otimes \mathrm{Grad}\, N^j$$

$$\mathbf{P} = \mathbb{E} : \boldsymbol{\varepsilon}$$

$$\mathbf{w} = N^j \mathbf{w}^j$$

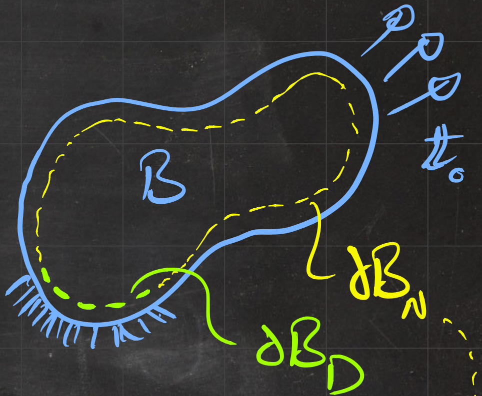
$$\mathbf{P} = \mathbb{E} : [\mathrm{Grad}\,\mathbf{u}]$$

$$\mathrm{Grad}\,\mathbf{w} = \mathbf{w}^j \otimes \mathrm{Grad}\, N^j$$

$$\mathrm{Grad}\,\mathbf{u} = \mathbf{u}^i \otimes \mathrm{Grad}\, N^i$$

$$\mathbf{u} = N^i \mathbf{u}^i$$

$$= N^1 \mathbf{u}^1 + N^2 \mathbf{u}^2 + \dots$$



$\mathbf{u} = \mathbf{u}_0$

Prescribed Displacement
m

$\mathbf{P}\cdot\mathbf{n} = \mathbf{t}_0$

Prescribed Traction
N/m²

FROM WEAK FORM TO APPROXIMATE FORM

$$\boxed{\int_{\mathcal{B}} [\mathrm{Grad}\,\mathbb{w}] : \mathbb{S}\, dA = \int_{\partial\mathcal{B}_N} \mathbb{w}\cdot \mathbb{t}_0 \, dL}$$

$$\mathbb{w}^{\dot{j}} \otimes \mathrm{Grad}\, N^{\dot{j}}$$

$$\mathbb{S} = \mathbb{E} : \boldsymbol{\varepsilon}$$

$$\Downarrow$$

$$\mathbb{S} = \mathbb{E} : [\mathrm{Grad}\,\mathbb{u}]$$

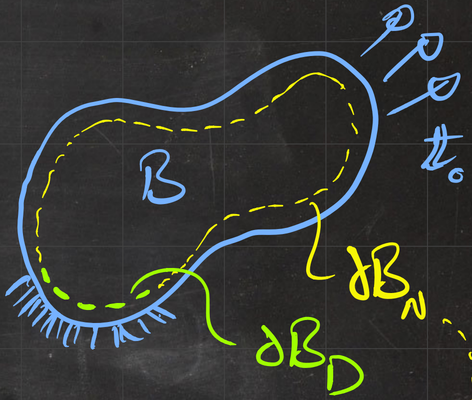
$$\Downarrow$$

$$\mathbb{S} = \mathbb{E} : [\mathbb{u}^{\dot{i}} \otimes \mathrm{Grad}\, N^{\dot{i}}]$$

$$\mathbb{w} = N^{\dot{j}}\, \mathbb{w}^{\dot{j}}$$

$$\downarrow$$

$$\mathrm{Grad}\,\mathbb{w} = \mathbb{w}^{\dot{j}} \otimes \mathrm{Grad}\, N^{\dot{j}}$$



$$\mathbb{u} = \mathbb{u}_0$$

Prescribed Displacement

m

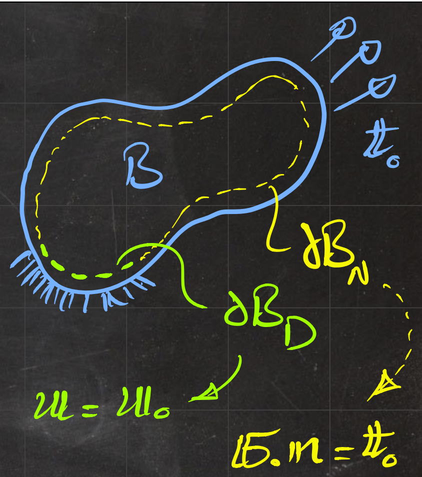
$$\mathbb{S}\cdot\mathbb{n} = \mathbb{t}_0$$

Prescribed Traction

N/m^2

FROM WEAK FORM TO APPROXIMATE FORM

$$\boxed{\int_{\mathcal{B}} [\text{GRAD}\,\mathbf{w}] : \mathbf{P}\, dA = \int_{\partial\mathcal{B}_N} \mathbf{w}\cdot\mathbf{t}_0\, dL}$$



$$\mathbf{w}^j \otimes \text{GRAD}\, N^j$$

$$\mathbf{P} = \mathbb{E} : \boldsymbol{\epsilon}$$

$$\mathbf{w} = N^j \mathbf{w}^j$$

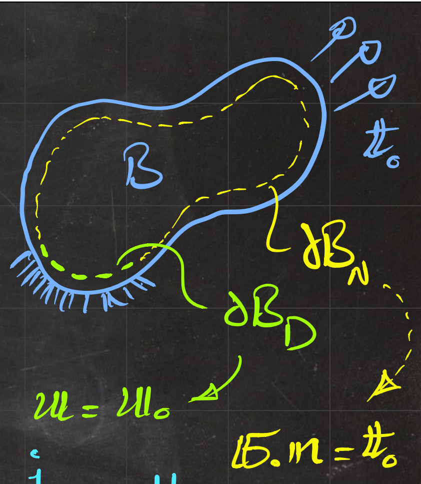
$$\mathbf{P} = \mathbb{E} : [\mathbf{u}^i \otimes \text{GRAD}\, N^i]$$

$$\text{GRAD}\,\mathbf{w} = \mathbf{w}^j \otimes \text{GRAD}\, N^j$$

$$\text{GRAD}\{\circ\} = \{\circ\}_{,\mathbf{x}}$$

$$\int_{\mathcal{B}} [\mathbf{w}^j \otimes N^j_{,\mathbf{x}}] : \mathbb{E} : [\mathbf{u}^i \otimes N^i_{,\mathbf{x}}]\, dA = \int_{\partial\mathcal{B}_N} N^j \mathbf{w}^j \cdot \mathbf{t}_0\, dL$$

FROM WEAK FORM TO APPROXIMATE FORM



$$\boxed{\int_{\mathcal{B}} [\mathrm{GRAD}\, w] : \mathbb{S}\, dA = \int_{\partial\mathcal{B}_N} w \cdot t_0\, dL}$$

$$E_{abcd}\, e_a \otimes e_b \otimes e_c \otimes e_d$$

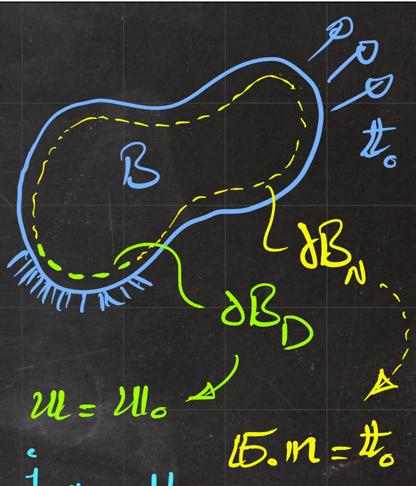
$$\underbrace{\int_{\mathcal{B}} [w^j \otimes N^j_{,x}] : \mathbb{E} : [u^i \otimes N^i_{,x}]}\, dA = \int_{\partial\mathcal{B}_N} N^j w^j \cdot t_0\, dL$$

$$\underset{2^{nd} O.}{\mathbb{A}} : \underset{4^{th} O.}{\mathbb{E}} : \underset{2^{nd} O.}{\mathbb{B}}$$

$$A_{ab}\, E_{abcd}\, B_{cd}$$

FROM WEAK FORM TO APPROXIMATE FORM

$$\boxed{\int_{\mathcal{B}} [\mathrm{Grad}\,\boldsymbol{w}] : \boldsymbol{\sigma}\, dA = \int_{\partial\mathcal{B}_N} \boldsymbol{w}\cdot\boldsymbol{t}_0\, dL}$$

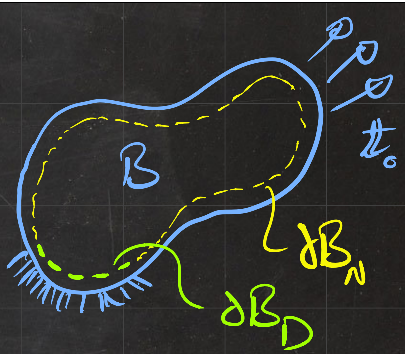


$$E_{abcd}\; \boldsymbol{e}_a \otimes \boldsymbol{e}_b \otimes \boldsymbol{e}_c \otimes \boldsymbol{e}_d$$

$$\int_{\mathcal{B}} [\boldsymbol{w}^j \otimes N^j_{,\boldsymbol{x}}] : \mathbb{E} : [\boldsymbol{u}^i \otimes N^i_{,\boldsymbol{x}}]\, dA = \int_{\partial\mathcal{B}_N} N^j \boldsymbol{w}^j \cdot \boldsymbol{t}_0\, dL$$

$$\int_{\mathcal{B}} [\boldsymbol{w}^j]_a\, [N^j_{,\boldsymbol{x}}]_b\, E_{abcd}\, [\boldsymbol{u}^i]_c\, [N^i_{,\boldsymbol{x}}]_d\, dA = \int_{\partial\mathcal{B}_N} N^j [\boldsymbol{w}^j]_q [\boldsymbol{t}_0]_q\, dL$$

FROM WEAK FORM TO APPROXIMATE FORM



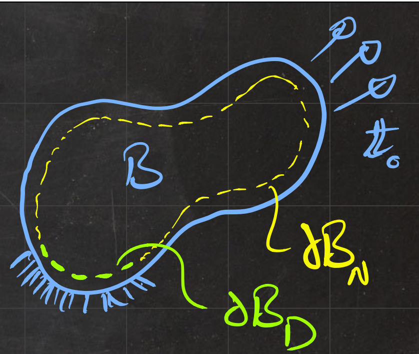
$$\boxed{\int_{\mathcal{B}} [\mathrm{GRAD}\,\mathbf{w}] : \mathbb{E} \, dA = \int_{\partial\mathcal{B}_N} \mathbf{w}\cdot\mathbf{t}_0 \, dL}$$

$$\int_{\mathcal{B}} [\mathbf{w}^{j}]_a \, [N^{j}_{,x}]_b \, E_{abcd} \, [\mathbf{u}^{i}]_c \, [N^{i}_{,x}]_d \, dA = \int_{\partial\mathcal{B}_N} N^{j} [\mathbf{w}^{j}]_q [\mathbf{t}_0]_q \, dL$$

$$[\mathbf{w}^{j}]_a \int_{\mathcal{B}} [N^{j}_{,x}]_b \, E_{abcd} \, [N^{i}_{,x}]_d \, dA \, [\mathbf{u}^{i}]_c = [\mathbf{w}^{j}]_q \int_{\partial\mathcal{B}_N} N^{j} [\mathbf{t}_0]_q \, dL$$

FROM WEAK FORM TO APPROXIMATE FORM

$$\boxed{\int_{\mathcal{B}} [\text{GRAD}\,\mathbf{w}] : \mathbb{S}\; dA = \int_{\partial\mathcal{B}_N} \mathbf{w}\cdot\mathbf{t}_0\; dL}$$



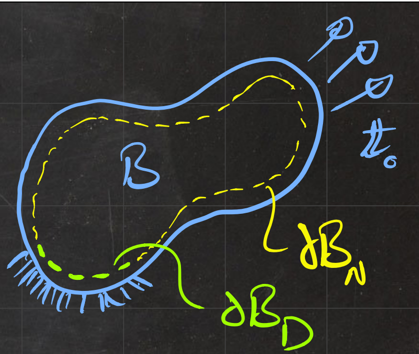
$$[w^j]_a \underbrace{\left[\int_{\mathcal{B}} \underbrace{[N^j_{,x}]_b\; E_{ab}}_{\text{dummy}}{}_{c}\underbrace{{}_{d}\;[N^i_{,x}]_d}_{\text{dummy}}\; dA\right]}_{[\mathbb{K}]^{ji}_{ac}} [u^i]_c = [w^j]_a \int_{\partial\mathcal{B}_N} N^j\, [t_0]_a\; dL$$

$$[\mathbb{K}]^{ji}_{ac}$$

$ji \leftarrow i, j \in \{1, 2, \dots\}$

$ac \leftarrow a, c \in \{1, 2\}$

FROM WEAK FORM TO APPROXIMATE FORM



$$\boxed{\int_B [\mathrm{GRAD}\, w] : \mathbb{P} \, dA = \int_{\partial B_N} w \cdot t_0 \, dL}$$

$$[w^j]_a \left[\int_B [N^j_{,\alpha}]_b \, E_{abcd} \, [N^i_{,\alpha}]_d \, dA \right] [u^i]_c = [w^j]_q \int_{\partial B_N} N^j [t_a]_q \, dL$$

$$[w^j]_a \, [\mathbb{K}^{ji}]_{ac} \, [u^i]_c = [w^j]_q \int_{\partial B_N} N^j [t_a]_q \, dL$$

FROM WEAK FORM TO APPROXIMATE FORM

$$\boxed{\int_{\mathcal{B}} [\mathrm{GRAD}\, w] : \mathbb{S}\, dA = \int_{\partial\mathcal{B}_N} w \cdot t_0 \, dL}$$

$$[\mathbb{K}^{ji}]_{ac} = \int_{\mathcal{B}} [N^j_{,x}]_b \, E_{abcd} \, [N^i_{,x}]_d \, dA$$

$$[w^j]_a \, [\mathbb{K}^{ji}]_{ac} \, [u^i]_c = [w^j]_a \int_{\partial\mathcal{B}_N} N^j \, [t_0]_a \, dL$$

$$i, j \in \{1, \dots, NPE\}$$

$$[w^j]_a$$

at node j

in direction a

↳ x, y

$[w^1]_1 = 1$ AND ALL THE REST ARE ZERO

$$w^1 = \begin{bmatrix}1\\0\end{bmatrix} \quad w^2 = \begin{bmatrix}0\\0\end{bmatrix} \quad w^3 = \begin{bmatrix}0\\0\end{bmatrix} \quad \dots$$

FROM WEAK FORM TO APPROXIMATE FORM

$$\boxed{\int_{\mathcal{B}} [\text{GRAD}\,\boldsymbol{w}] : \mathbb{S} \, dA = \int_{\partial\mathcal{B}_N} \boldsymbol{w} \cdot \boldsymbol{t}_0 \, dL}$$

$$[\mathbb{K}^{ji}]_{ac} = \int_{\mathcal{B}} [N^j_{,x}]_b \, E_{abcd} \, [N^i_{,x}]_d \, dA$$

$$[\boldsymbol{w}^j]_a \, [\mathbb{K}^{ji}]_{ac} \, [\boldsymbol{u}^i]_c = [\boldsymbol{w}^j]_q \int_{\partial\mathcal{B}_N} N^j \, [\boldsymbol{t}_0]_q \, dL$$

$$[\mathbb{K}^{1i}]_{1c} \, [\boldsymbol{u}^i]_c = \int_{\partial\mathcal{B}_N} N^1 \, [\boldsymbol{t}_0]_1 \, dL$$

$$F^1_x$$

$$[F^1]_1$$

$$\boldsymbol{w}^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \boldsymbol{w}^2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \boldsymbol{w}^3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \dots$$

FROM WEAK FORM TO APPROXIMATE FORM

$$\boxed{\int_B [\mathrm{GRAD}\,\mathbb{w}] : \mathbb{S} \, dA = \int_{\partial B_N} \mathbb{w} \cdot \mathbb{t}_o \, dL}$$

$$[\mathbb{K}^{ji}]_{ac} = \int_B [N^j_{,x}]_b \, E_{abcd} \, [N^i_{,x}]_d \, dA$$

$$[\mathbb{w}^j]_a \, [\mathbb{K}^{ji}]_{ac} \, [\mathbb{u}^i]_c = [\mathbb{w}^j]_q \int_{\partial B_N} N^j \, [\mathbb{t}_o]_q \, dL$$

$$[\mathbb{K}^{1i}]_{1c} \, [\mathbb{u}^i]_c = [F^1]_1 \quad \rightsquigarrow \quad \sum_i \sum_c \dots$$

$$\mathbb{w}^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbb{w}^2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \mathbb{w}^3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \dots$$

$$[K^{11}]_{11} \, [u^1]_1 + [K^{11}]_{12} \, [u^1]_2 + [K^{12}]_{11} \, [u^2]_1 + \dots$$

FROM WEAK FORM TO APPROXIMATE FORM

$$\boxed{\int_B [\mathrm{Grad}\, w] : \mathbb{P}\, dA = \int_{\partial B_N} w \cdot \mathbb{t}_0\, dL}$$

$$[\mathbb{K}^{ji}]_{ac} = \int_B [N^j_{,x}]_b\, E_{abcd}\, [N^i_{,x}]_d\, dA$$

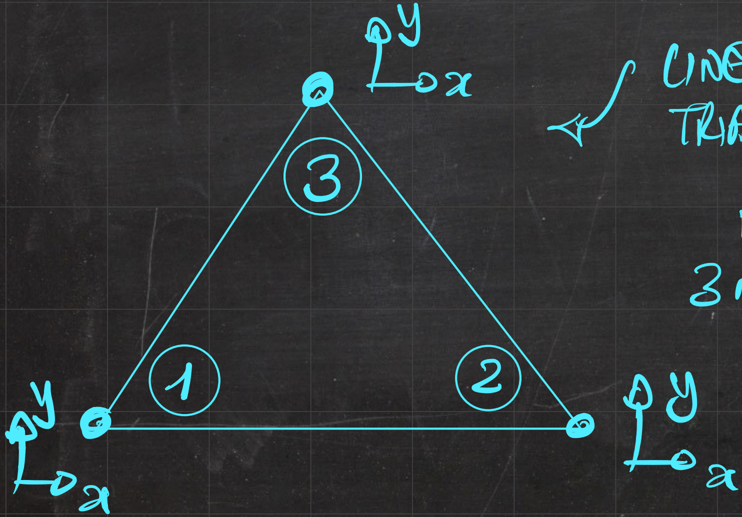
$$[w^j]_a\, [\mathbb{K}^{ji}]_{ac}\, [u^i]_c = [w^j]_q \int_{\partial B_N} N^j\, [\mathbb{t}_0]_q\, dL$$

$$[\mathbb{K}^{1i}]_{1c}\, [u^i]_c = [F^1]_1 \rightsquigarrow \sum_i \sum_c \dots$$

$$w^1 = \begin{bmatrix}1\\0\end{bmatrix} \qquad w^2 = \begin{bmatrix}0\\0\end{bmatrix} \qquad w^3 = \begin{bmatrix}0\\0\end{bmatrix} \quad \dots$$

$$[\;.\;.\;.\;.\;.\;.\;.\;]\begin{bmatrix}\vdots\\ \vdots\end{bmatrix} = [F^1]_1$$

$$\begin{bmatrix} K^{11}_{11} & K^{11}_{12} & K^{12}_{11} & K^{12}_{12} & K^{13}_{11} & K^{13}_{12} \end{bmatrix} \begin{bmatrix} u^1_1 \\ u^1_2 \\ u^2_1 \\ u^2_2 \\ u^3_1 \\ u^3_2 \end{bmatrix} = F^1_1$$



LINEAR TRIANGLE

⇓

3 NODES

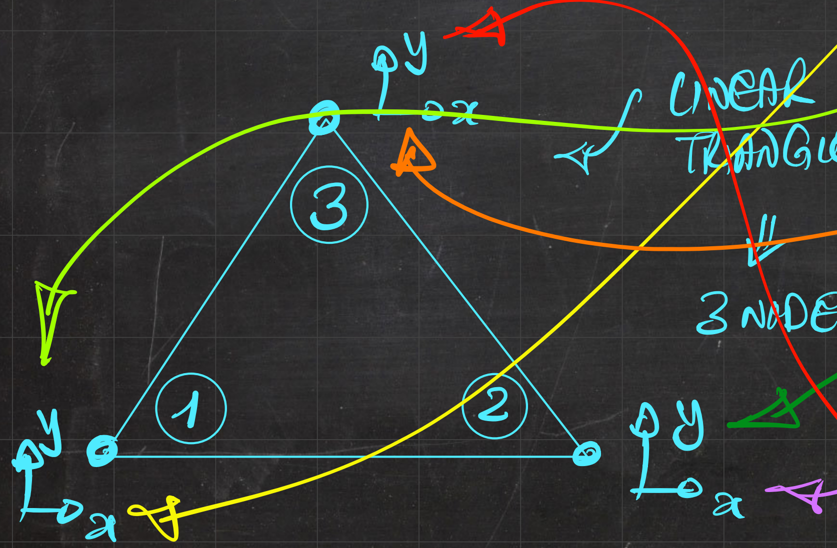
$$\begin{bmatrix} K^{11}_{11} & K^{11}_{12} & K^{12}_{11} & K^{12}_{12} & K^{13}_{11} & K^{13}_{12} \end{bmatrix} \begin{bmatrix} u^1_1 \\ u^1_2 \\ u^2_1 \\ u^2_2 \\ u^3_1 \\ u^3_2 \end{bmatrix} = F^1_1$$

1st NODE

x

FORCE OF NODE 1 IN DIRECTION 1

x



LINEAR TRIANGLE

⇓

3 NODES

FROM WEAK FORM TO APPROXIMATE FORM

$$\boxed{\int_B [\mathrm{GRAD}\, w] : \mathbb{P} \, dA = \int_{\partial B_N} w \cdot t_0 \, dL}$$

$$[\mathbb{K}^{ji}]_{ac} = \int_B [N^j_{,x}]_b \, E_{abcd} \, [N^i_{,x}]_d \, dA$$

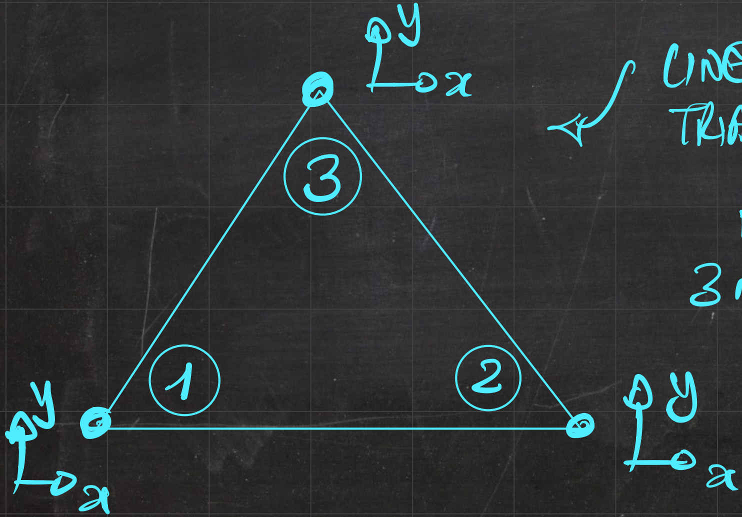
$$[w^j]_a \, [\mathbb{K}^{ji}]_{ac} \, [u^i]_c = [w^j]_q \int_{\partial B_N} N^j \, [t_0]_q \, dL$$

$$[\mathbb{K}^{1i}]_{2c} \, [u^i]_c = [F^1]_2 \rightsquigarrow \sum_i \sum_c \ldots$$

$$w^1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad w^2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad w^3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \ldots$$

$$\begin{bmatrix} \cdots\cdots\cdots \end{bmatrix} \begin{bmatrix} \vdots \end{bmatrix} = [F^1]_2$$

$$\begin{bmatrix} K^{11}_{21} & K^{11}_{22} & K^{12}_{21} & K^{12}_{22} & K^{13}_{21} & K^{13}_{22} \end{bmatrix} \begin{bmatrix} u^1_1 \\ u^1_2 \\ u^2_1 \\ u^2_2 \\ u^3_1 \\ u^3_2 \end{bmatrix} = F^1_2$$



LINEAR TRIANGLE

$\Downarrow$

3 NODES

$$\begin{bmatrix} \boxed{\begin{matrix} K^{11}_{11} & K^{11}_{12} \\ K^{11}_{21} & K^{11}_{22} \end{matrix}} & \begin{matrix} K^{12}_{11} & K^{12}_{12} & K^{13}_{11} & K^{13}_{12} \\ K^{12}_{21} & K^{12}_{22} & K^{13}_{21} & K^{13}_{22} \end{matrix} \end{bmatrix} \begin{bmatrix} u^1_1 \\ u^1_2 \\ u^2_1 \\ u^2_2 \\ u^3_1 \\ u^3_2 \end{bmatrix} = \mathcal{F}^1_2$$

$$[\mathbb{K}]_{6\times6} \checkmark$$

$$[\mathbb{K}^{11}]_{2\times2} \longleftarrow$$

$$\mathbb{K}^{11} = \begin{bmatrix} K^{11}_{xx} & K^{11}_{xy} \\ K^{11}_{yx} & K^{11}_{yy} \end{bmatrix}$$

FROM WEAK FORM TO APPROXIMATE FORM

$$\boxed{\int_B [\text{GRAD}\,\mathbb{w}] : \mathbb{S}\, dA = \int_{\partial B_N} \mathbb{w} \cdot \mathbb{t}_0 \, dL}$$

$$[\mathbb{K}^{ji}]_{ac} = \int_B [N^j_{,\mathbb{x}}]_b \, E_{abcd} \, [N^i_{,\mathbb{x}}]_d \, dA$$

$$[\mathbb{w}^j]_a \, [\mathbb{K}^{ji}]_{ac} \, [\mathbb{u}^i]_c = [\mathbb{w}^j]_q \int_{\partial B_N} N^j \, [\mathbb{t}_a]_q \, dL$$

$$[\mathbb{K}^{ji}]_{ac} \, [\mathbb{u}^i]_c = [\mathbb{F}^j]_a$$

$\mathbb{K}^{ji}$ ← STIFFNESS BETWEEN NODES i & j

↳ 2x2 matrix

FROM WEAK FORM TO APPROXIMATE FORM

$$\boxed{\int_B [\mathrm{GRAD}\,\boldsymbol{w}] : \mathbb{S}\, dA = \int_{\partial B_N} \boldsymbol{w} \cdot \boldsymbol{t}_0 \, dL}$$

$$[\mathbb{K}^{ji}]_{ac} = \int_B [N^j_{,x}]_b \, E_{abcd} \, [N^i_{,x}]_d \, dA$$

$$\boxed{[\mathbb{K}^{ji}]_{ac} \, [u^i]_c = [F^j]_a}$$

$\mathbb{K}^{ji}$ ↙ STIFFNESS BETWEEN NODES i & j

↳ 2x2 matrix

$$[\mathbb{K}^{ji}] = \begin{bmatrix} K^{ji}_{11} & K^{ji}_{12} \\ K^{ji}_{21} & K^{ji}_{22} \end{bmatrix}$$

FROM WEAK FORM TO APPROXIMATE FORM

$$\boxed{\int_{\mathcal{B}} [\text{GRAD}\, w] : \mathbb{S} \, dA = \int_{\partial \mathcal{B}_N} w \cdot t_0 \, dL}$$

$$[\mathbb{K}^{ij}]_{ac} = \int_{\mathcal{B}} [N^i_{,x}]_b \, E_{abcd} \, [N^j_{,x}]_d \, dA$$

$$\boxed{[\mathbb{K}^{ij}]_{ac} \, [u^j]_c = [F^i]_a}$$

$\mathbb{K}^{ij}$ ↙ STIFFNESS BETWEEN NODES i & j

↳ 2x2 matrix

$$\{i, j\} \in \{1, 2, \dots, NPE\}$$

$$[\mathbb{K}^{ij}] = \begin{bmatrix} \mathbb{K}^{ij}_{11} & \mathbb{K}^{ij}_{12} \\ \mathbb{K}^{ij}_{21} & \mathbb{K}^{ij}_{22} \end{bmatrix}$$

APPROXIMATE FORM

$$\boxed{[\mathbb{K}^{ij}]_{ac}\,[u^{j}]_{c} = [\mathbb{F}^{i}]_{a}}$$

↳ $[\mathbb{K}][u] = [\mathbb{F}]$

$\{i,j\} \in \{1,2,\dots,NPE\}$

$$[\mathbb{K}^{ij}]_{ac} = \int_{B} [N^{i}_{,x}]_{b}\, E_{abcd}\, [N^{j}_{,x}]_{d}\, dA$$

$\mathbb{K}^{ij}$ ↙ STIFFNESS BETWEEN NODES i & j

↳ 2x2 matrix

$$[\mathbb{K}^{ij}] = \begin{bmatrix} K^{ij}_{11} & K^{ij}_{12} \\ K^{ij}_{21} & K^{ij}_{22} \end{bmatrix}$$

UNDERSTANDING $[\mathbb{K}^{ij}]_{ac}$

$\leftarrow \mathbb{K}^{ij}_{ac}$

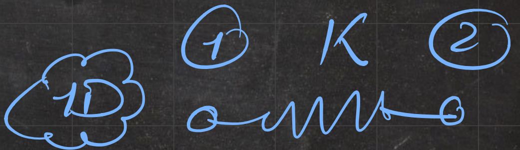
$\leftarrow$ SUPERSCRIPTS → NODES

$\leftarrow$ SUBSCRIPTS → DIRECTIONS

$[\mathbb{K}^{ij}]_{ac}$

↳ STIFFNESS BETWEEN NODES i & j AND FOR DIRECTIONS a & c RESPECTIVELY

$$[\mathbb{K}^{ij}]_{ac} = \int_{\mathcal{B}} [N^i_{,x}]_b \, E_{abcd} \, [N^j_{,x}]_d \, dA$$

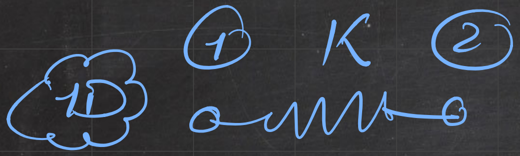


$$\begin{bmatrix} K^{11} & K^{12} \\ K^{21} & K^{22} \end{bmatrix} \leftarrow \mathbb{K} = \begin{bmatrix} K & -K \\ -K & K \end{bmatrix}$$

UNDERSTANDING $[\mathbb{K}^{ij}]_{ac} \leftarrow \mathbb{K}^{ij}_{ac}$

$\leftarrow$ SUPERSCRIPTS $\rightarrow$ NODES

$\leftarrow$ SUBSCRIPTS $\rightarrow$ DIRECTIONS



$K^{11} \rightarrow$ STIFFNESS BETWEEN NODE 1&1

$$\begin{bmatrix} K^{11} & K^{12} \\ K^{21} & K^{22} \end{bmatrix} \leftarrow \mathbb{K} = \begin{bmatrix} K & -K \\ -K & K \end{bmatrix}$$

$$\begin{bmatrix} K^{11} & K^{12} \\ K^{21} & K^{22} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} = \begin{bmatrix} F^1 \\ F^2 \end{bmatrix}$$

$K^{11} = ?$ $K^{12} = ?$

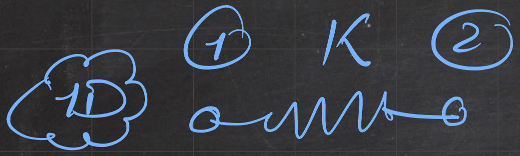
└ STIFFNESS BETWEEN NODES 1&2

UNDERSTANDING $[\mathbb{K}^{ij}]_{ac}$

$[\mathbb{K}^{ij}]_{ac} \leftarrow \mathbb{K}^{ij}_{ac}$

ij ← SUPERSCRIPTS ← NODES

ac ← SUBSCRIPTS ← DIRECTIONS



K^{11} ⟶ STIFFNESS BETWEEN NODE 1&1

$$K^{ij} = \frac{\text{Change of Force @ NODE } i}{\text{Change of Disp. @ NODE } j} \cong \frac{\delta F^i}{\delta u^j}$$

$$\begin{bmatrix} K^{11} & K^{12} \\ K^{21} & K^{22} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} = \begin{bmatrix} F^1 \\ F^2 \end{bmatrix}$$

$K^{12} = ?$

↳ STIFFNESS BETWEEN NODES 1&2

UNDERSTANDING $[\mathbb{K}^{ij}]_{ac} \leftarrow \mathbb{K}^{ij}_{ac}$

$^{ij} \leftarrow$ SUPERSCRIPTS $\leftarrow$ NODES

$_{ac} \leftarrow$ SUBSCRIPTS $\leftarrow$ DIRECTIONS

$$[K^{ij}]_{ac}$$

↳ STIFFNESS BETWEEN NODES i & j AND FOR DIRECTIONS a & c RESPECTIVELY

$$[\mathbb{K}^{ij}]_{ac} = \int_{B} [N^{i}_{,x}]_b \, E_{abcd} \, [N^{j}_{,x}]_d \, dA$$

↳ STIFFNESS BETWEEN DIRECTION a of NODE i & DIRECTION c of NODE j

UNDERSTANDING $[\mathbb{K}^{ij}]_{ac} \leftarrow \mathbb{K}^{ij}_{ac}$ ← SUPERSCRIPTS ← NODES

← SUBSCRIPTS ← DIRECTIONS

$$[\mathbb{K}^{ij}]_{ac}$$

$$[\mathbb{K}^{ij}]_{ac} = \int_{\mathcal{B}} [N^{i}_{,x}]_b \, E_{abcd} \, [N^{j}_{,x}]_d \, dA$$

$$K^{ij}_{ac} = \frac{\delta F^{i}_{a}}{\delta u^{j}_{c}}$$

→ STIFFNESS BETWEEN

DIRECTION a of NODE i

&

DIRECTION c of NODE j

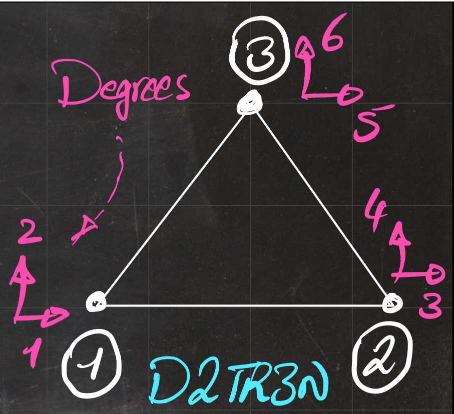
$$\mathbb{K}_{\Delta} = \begin{bmatrix} \mathbb{K}^{11} & \mathbb{K}^{12} & \mathbb{K}^{13} \\ \mathbb{K}^{21} & \mathbb{K}^{22} & \mathbb{K}^{23} \\ \mathbb{K}^{31} & \mathbb{K}^{32} & \mathbb{K}^{33} \end{bmatrix}$$

D2TR3N

6x6

↳ NoN x PD
↑3 ↑2

$$\mathbb{K}^{11} = \frac{\delta \mathbb{F}^1}{\delta \mathbb{u}^1}$$

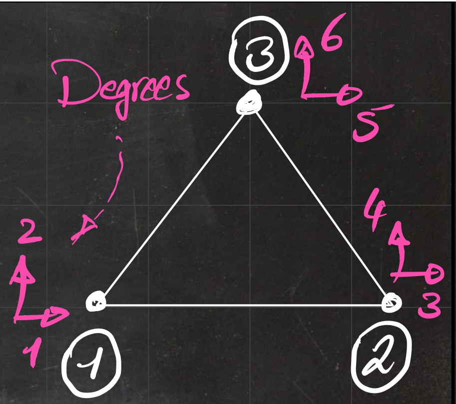


$$\mathbb{K}^{ij} = \begin{bmatrix} K^{ij}_{11} & K^{ij}_{12} \\ K^{ij}_{21} & K^{ij}_{22} \end{bmatrix} = \begin{bmatrix} K^{ij}_{xx} & K^{ij}_{xy} \\ K^{ij}_{yx} & K^{ij}_{yy} \end{bmatrix}$$

1 ⟶ NODE^1_x
2 ⟶ NODE^1_y
3 ⟶ NODE^2_x
4 ⟶ NODE^2_y
5 ⟶ NODE^3_x
6 ⟶ NODE^3_y

$$\mathbb{K}_{\Delta} = \begin{bmatrix} \mathbb{K}^{11} & \mathbb{K}^{12} & \mathbb{K}^{13} \\ \mathbb{K}^{21} & \mathbb{K}^{22} & \mathbb{K}^{23} \\ \mathbb{K}^{31} & \mathbb{K}^{32} & \mathbb{K}^{33} \end{bmatrix} \longleftarrow 6 \times 6$$

↳ $\underset{\underset{3}{\uparrow}}{\mathrm{NoN}} \times \underset{\underset{2}{\uparrow}}{\mathrm{PD}}$



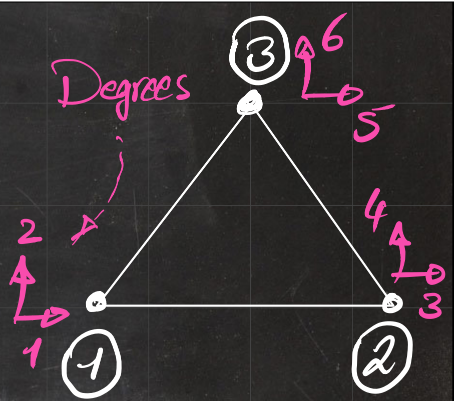
$$\mathbb{K}_{\Delta} = \begin{bmatrix} K^{11}_{11} & K^{11}_{12} & K^{12}_{11} & K^{12}_{12} & K^{13}_{11} & K^{13}_{12} \\ K^{11}_{21} & K^{11}_{22} & K^{12}_{21} & K^{12}_{22} & K^{13}_{21} & K^{13}_{22} \\ K^{21}_{11} & K^{21}_{12} & K^{22}_{11} & K^{22}_{12} & K^{23}_{11} & K^{23}_{12} \\ K^{21}_{21} & K^{21}_{22} & K^{22}_{21} & K^{22}_{22} & K^{23}_{21} & K^{23}_{22} \\ K^{31}_{11} & K^{31}_{12} & K^{32}_{11} & K^{32}_{12} & K^{33}_{11} & K^{33}_{12} \\ K^{31}_{21} & K^{31}_{22} & K^{32}_{21} & K^{32}_{22} & K^{33}_{21} & K^{33}_{22} \end{bmatrix}$$

1 → NODE^1 x
2 → NODE^1 y

3 → NODE^2 x
4 → NODE^2 y

5 → NODE^3 x
6 → NODE^3 y

$$\mathbb{K}^{\Delta} = \begin{bmatrix} \mathbb{K}^{11} & \mathbb{K}^{12} & \mathbb{K}^{13} \\ \mathbb{K}^{21} & \mathbb{K}^{22} & \mathbb{K}^{23} \\ \mathbb{K}^{31} & \mathbb{K}^{32} & \mathbb{K}^{33} \end{bmatrix} \leftarrow 6\text{x}6 \to \underset{3}{\uparrow}\!\!\!\!\text{NoN} \text{x} \underset{2}{\uparrow}\!\!\text{PD}$$



$$\mathbb{K}^{\Delta} = \left[\begin{array}{cccccc} 1_x & 1_y & 2_x & 2_y & 3_x & 3_y \\ K^{11}_{11} & K^{11}_{12} & K^{12}_{11} & K^{12}_{12} & K^{13}_{11} & K^{13}_{12} \\ K^{11}_{21} & K^{11}_{22} & K^{12}_{21} & K^{12}_{22} & K^{13}_{21} & K^{13}_{22} \\ K^{21}_{11} & K^{21}_{12} & K^{22}_{11} & K^{22}_{12} & K^{23}_{11} & K^{23}_{12} \\ K^{21}_{21} & K^{21}_{22} & K^{22}_{21} & K^{22}_{22} & K^{23}_{21} & K^{23}_{22} \\ K^{31}_{11} & K^{31}_{12} & K^{32}_{11} & K^{32}_{12} & K^{33}_{11} & K^{33}_{12} \\ K^{31}_{21} & K^{31}_{22} & K^{32}_{21} & K^{32}_{22} & K^{33}_{21} & K^{33}_{22} \end{array}\right] \begin{array}{c} \\ 1_x \\ 1_y \\ 2_x \\ 2_y \\ 3_x \\ 3_y \end{array}$$

1 → NODE^1_x

2 → NODE^1_y

3 → NODE^2_x

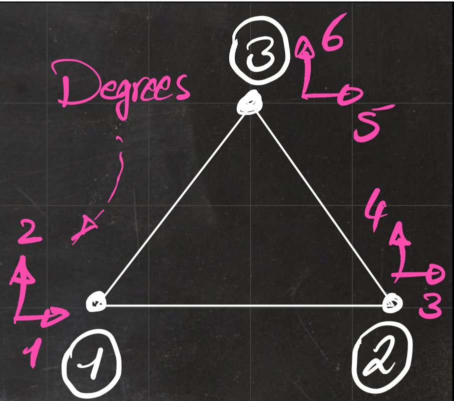
4 → NODE^2_y

5 → NODE^3_x

6 → NODE^3_y

$$\underset{\Delta}{\mathbb{K}} = \begin{bmatrix} \mathbb{K}^{11} & \mathbb{K}^{12} & \mathbb{K}^{13} \\ \mathbb{K}^{21} & \mathbb{K}^{22} & \mathbb{K}^{23} \\ \mathbb{K}^{31} & \mathbb{K}^{32} & \mathbb{K}^{33} \end{bmatrix} \longleftarrow 6\times 6$$

$\hookrightarrow \underset{\uparrow 3}{\text{NoN}} \times \underset{\uparrow 2}{\text{PD}}$



$$\underset{\Delta}{\mathbb{K}} = \begin{array}{c|cccccc|c} & 1_x & 1_y & 2_x & 2_y & 3_x & 3_y & \\ \hline & K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} & 1_x \\ & K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} & 1_y \\ & K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} & 2_x \\ & K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} & 2_y \\ & K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} & 3_x \\ & K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} & 3_y \end{array}$$

$1 \to \text{NODE}^{1}\ x$

$2 \to \text{NODE}^{1}\ y$

$3 \to \text{NODE}^{2}\ x$

$4 \to \text{NODE}^{2}\ y$

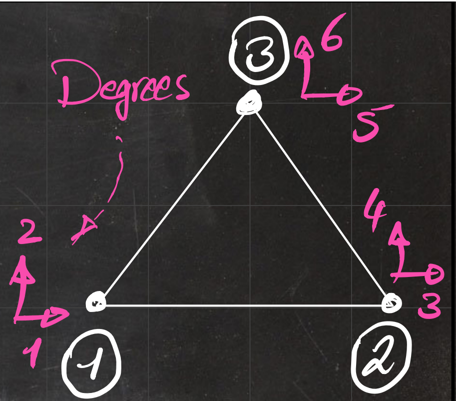
$5 \to \text{NODE}^{3}\ x$

$6 \to \text{NODE}^{3}\ y$

$$\mathbb{K}_{\Delta} = \begin{bmatrix} \mathbb{K}^{11} & \mathbb{K}^{12} & \mathbb{K}^{13} \\ \mathbb{K}^{21} & \mathbb{K}^{22} & \mathbb{K}^{23} \\ \mathbb{K}^{31} & \mathbb{K}^{32} & \mathbb{K}^{33} \end{bmatrix}$$

6x6

↳ $\underset{\uparrow 3}{\text{NoN}} \times \underset{\uparrow 2}{\text{PD}}$



$$\mathbb{K}_{\Delta} = \begin{array}{c} \begin{array}{cccccc} 1_x & 1_y & 2_x & 2_y & 3_x & 3_y \end{array} \\ \left[\begin{array}{cccccc} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} \end{array} \right] \end{array} \begin{array}{c} \\ 1_x \\ 1_y \\ 2_x \\ 2_y \\ 3_x \\ 3_y \end{array}$$

1 → NODE^1 x

2 → NODE^1 Y

3 → NODE^2 x

4 → NODE^2 y

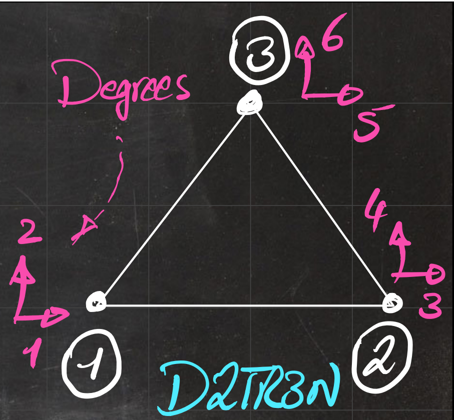
5 → NODE^3_x

6 → NODE^3_y

$$\underset{\Delta}{\mathbb{K}} = \begin{bmatrix} \mathbb{K}^{11} & \mathbb{K}^{12} & \mathbb{K}^{13} \\ \mathbb{K}^{21} & \mathbb{K}^{22} & \mathbb{K}^{23} \\ \mathbb{K}^{31} & \mathbb{K}^{32} & \mathbb{K}^{33} \end{bmatrix} \leftarrow 6\times6$$

$\hookrightarrow \text{NoN} \times \text{PD}$

$\uparrow 3 \quad \uparrow 2$



$$\underset{\Delta}{\mathbb{K}} = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \leftarrow \text{DEGREES} \\ \left[\begin{array}{cccccc} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

$1 \rightarrow \text{NODE}^1\, x$

$2 \rightarrow \text{NODE}^1\, y$

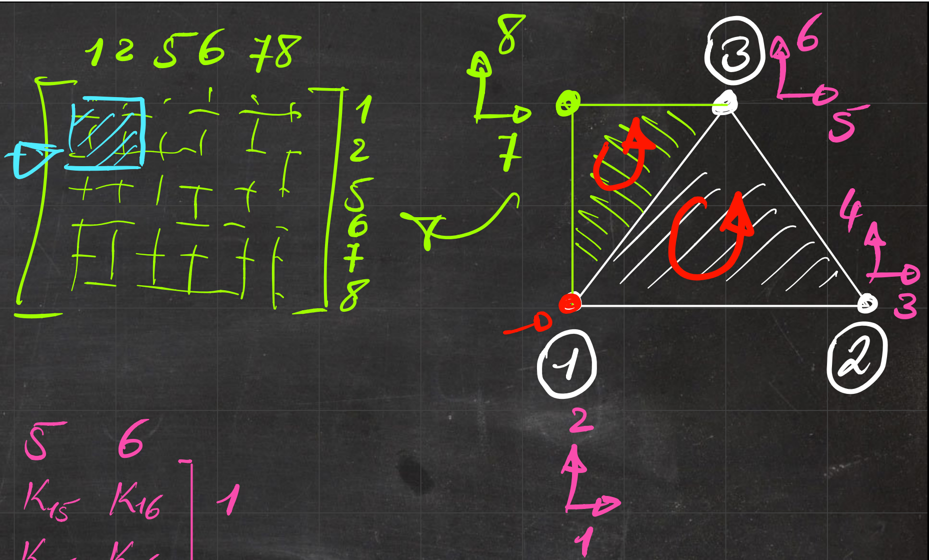
$3 \rightarrow \text{NODE}^2\, x$

$4 \rightarrow \text{NODE}^2\, y$

$5 \rightarrow \text{NODE}^3\, x$

$6 \rightarrow \text{NODE}^3\, y$

ASSEMBLY ✓



$$\mathbb{K}_{\Delta} = \begin{array}{c|cccccc|c} & 1 & 2 & 3 & 4 & 5 & 6 & \\ \hline & K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} & 1 \\ & K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} & 2 \\ & K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} & 3 \\ & K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} & 4 \\ & K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} & 5 \\ & K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} & 6 \end{array}$$

→ Question

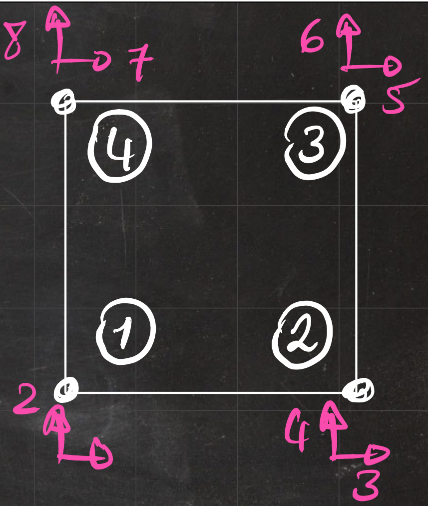
& ELEMENT STIFFNESS

$$\underset{\square}{\mathbb{K}} = \begin{bmatrix} \mathbb{K}^{11} & \mathbb{K}^{12} & \mathbb{K}^{13} & \mathbb{K}^{14} \\ \mathbb{K}^{21} & \mathbb{K}^{22} & \mathbb{K}^{23} & \mathbb{K}^{24} \\ \mathbb{K}^{31} & \mathbb{K}^{32} & \mathbb{K}^{33} & \mathbb{K}^{34} \\ \mathbb{K}^{41} & \mathbb{K}^{42} & \mathbb{K}^{43} & \mathbb{K}^{44} \end{bmatrix} \rightarrow 8\times 8$$

D2QU4N

$\hookrightarrow$ NoN x PD

$\hookrightarrow 4$ $\hookrightarrow 2$



$$\mathbb{K}^{ij} = \begin{bmatrix} K^{ij}_{11} & K^{ij}_{12} \\ K^{ij}_{21} & K^{ij}_{22} \end{bmatrix} = \begin{bmatrix} K^{ij}_{xx} & K^{ij}_{xy} \\ K^{ij}_{yx} & K^{ij}_{yy} \end{bmatrix}$$

1, 2 → NODE^{1} x,y

3, 4 → NODE^{2} x,y

5, 6 → NODE^{3} x,y

7, 8 → NODE^{4} x,y

D2TR3N ← $[\mathbb{K}]_{6x6}$

D2TR6N ← $[\mathbb{K}]_{12x12}$

D2QU4N ← $[\mathbb{K}]_{8x8}$

D2QU8N ← $[\mathbb{K}]_{16x16}$

D2QU9N ← $[\mathbb{K}]_{18x18}$

PROBLEMS TO ADDRESS

- ↳ INTEGRAL ↳ GAUSS QUADRATURE RULE
- ↳ $f(x)$ ↳ $x \to \$$
- ↳ E_{abcd} ↳ ?

$$[\mathbb{K}^{ij}]_{ac} = \int_{B} [N^{i}_{,x}]_{b} \, E_{abcd} \, [N^{j}_{,x}]_{d} \, dA$$

D2TR3N ← $[\mathbb{K}]_{6\times6}$

D2TR6N ← $[\mathbb{K}]_{12\times12}$

D2QU4N ← $[\mathbb{K}]_{8\times8}$

D2QU8N ← $[\mathbb{K}]_{16\times16}$

D2QU9N ← $[\mathbb{K}]_{18\times18}$

$$E_{abcd} = \frac{E}{2[1+\nu]}\left[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}\right] + \frac{E\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

↑ CONSTITUTIVE TENSOR

↑ 4th O.

↑ 2x2x2x2 = 16 COMPONENTS

E → Young's Modulus

ν: Poisson's Ratio \nu

δ: Kronecker Delta

$$[\mathbb{K}^{ij}]_{ac} = \int_{\mathcal{B}} [N^{i}_{,x}]_b \, E_{abcd} \, [N^{j}_{,x}]_d \, dA$$

D2TR3N $\leftarrow [\mathbb{K}]_{6\times 6}$

D2TR6N $\leftarrow [\mathbb{K}]_{12\times 12}$

D2QU4N $\leftarrow [\mathbb{K}]_{8\times 8}$

D2QU8N $\leftarrow [\mathbb{K}]_{16\times 16}$

D2QU9N $\leftarrow [\mathbb{K}]_{18\times 18}$

$$E_{abcd} = \frac{E}{2[1+\nu]}\left[\delta_{ad}\,\delta_{bc} + \delta_{ac}\,\delta_{bd}\right] + \frac{E\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

$$E_{\underset{a}{1}\underset{b}{1}\underset{c}{1}\underset{d}{1}} = \frac{E}{2[1+\nu]}\left[1\times1 + 1\times1\right] + \frac{E\nu}{1-\nu^2}\,1\times1$$

$$[\mathbb{K}^{ij}]_{ac} = \int_{\mathcal{B}} [N^{i}_{,a}]_b \, E_{abcd} \, [N^{j}_{,a}]_d \, dA$$

D2TR3N $\leftarrow [\mathbb{K}]_{6\times6}$

D2TR6N $\leftarrow [\mathbb{K}]_{12\times12}$

D2QU4N $\leftarrow [\mathbb{K}]_{8\times8}$

D2QU8N $\leftarrow [\mathbb{K}]_{16\times16}$

D2QU9N $\leftarrow [\mathbb{K}]_{18\times18}$

$$\mathbb{E}_{abcd} = \frac{E}{2[1+\nu]}\left[\delta_{ad}\,\delta_{bc}+\delta_{ac}\,\delta_{bd}\right] + \frac{E\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

$$\nu = \frac{\nu_{3D}}{1-\nu_{3D}}$$

PLANE STRAIN

$$\mathbb{E}_{\underset{a\,b\,c\,d}{1111}} = \frac{E}{[1+\nu]} + \frac{E\nu}{1-\nu^2} = \frac{E[1-\nu]+E\nu}{1-\nu^2} = \frac{E}{1-\nu^2}$$

$$[\mathbb{K}^{ij}]_{ac} = \int_{\mathcal{B}} [N^{i}_{,x}]_b \, \mathbb{E}_{abcd} \, [N^{j}_{,x}]_d \, dA$$

D2TR3N ← $[\mathbb{K}]_{6\times6}$

D2TR6N ← $[\mathbb{K}]_{12\times12}$

D2QU4N ← $[\mathbb{K}]_{8\times8}$

D2QU8N ← $[\mathbb{K}]_{16\times16}$

D2QU9N ← $[\mathbb{K}]_{18\times18}$

$$\mathbb{E}_{abcd} = \frac{E}{2[1+\nu]}\left[\delta_{ad}\,\delta_{bc}+\delta_{ac}\,\delta_{bd}\right] + \frac{E\nu}{1-\nu^2}\,\delta_{ab}\,\delta_{cd}$$

$$\mathbb{E}_{1112} = \frac{E}{2[1+\nu]}\left[0\times1 + 1\times0\right] + \frac{E\nu}{1-\nu^2}\,1\times0 = 0$$

a b c d

$$[\mathbb{K}^{ij}]_{ac} = \int_{\mathcal{B}} [N^{i}_{,\alpha}]_b\, \mathbb{E}_{abcd}\, [N^{j}_{,\alpha}]_d \, dA$$

$$[\mathbb{K}^{ij}]_{ac} = \int_{\mathcal{B}} [N^{i}_{,x}]_b \, E_{abcd} \, [N^{j}_{,x}]_d \, dA$$

$$E_{abcd} = \frac{E}{2[1+\nu]} \left[\delta_{ad}\delta_{bc} + \delta_{ac}\delta_{bd} \right] + \frac{E\nu}{1-\nu^2} \delta_{ab}\delta_{cd}$$

$$E_{1111} = \frac{E}{[1-\nu^2]} \qquad E_{1112} = 0 \qquad E_{1121} = 0 \qquad E_{1122} = \frac{E}{[1-\nu^2]}$$

$$E_{1211} = 0 \qquad E_{1212} = \frac{E}{2[1+\nu]} \qquad E_{1221} = \frac{E}{2[1+\nu]} \qquad E_{1222} = 0$$

$$E_{2111} = 0 \qquad E_{2112} = \frac{E}{2[1+\nu]} \qquad E_{2121} = \frac{E}{2[1+\nu]} \qquad E_{2122} = 0$$

$$E_{2211} = \frac{E}{[1-\nu^2]} \qquad E_{2212} = 0 \qquad E_{2221} = 0 \qquad E_{2222} = \frac{E}{[1-\nu^2]}$$

FROM STRONG FORM TO ELEMENT STIFFNESS ← IN PHYSICAL SPACE now

$EA\,u'' = 0$ SUBJECT TO BCs

$$EA \begin{bmatrix} \int_L N^{1\prime} N^{1\prime}\, dx & \int_L N^{1\prime} N^{2\prime}\, dx \\ \int_L N^{2\prime} N^{1\prime}\, dx & \int_L N^{2\prime} N^{2\prime}\, dx \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} = \begin{bmatrix} F^1 \\ F^2 \end{bmatrix}$$

$$\begin{bmatrix} K^{11} & K^{12} \\ K^{21} & K^{22} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} = \begin{bmatrix} F^1 \\ F^2 \end{bmatrix} \leftarrow K^{ij} = EA \int_L N^{i\prime} N^{j\prime}\, dx$$

$$K^{ij} = EA \int_L N^{i\prime} N^{j\prime} \, dx$$ ← PHYSICAL

RECALL:

$$\int_{-1}^{1} g(\xi)\, d\xi = \sum_{gp=1}^{GPE} g(\xi)_{gp} \, \alpha_{gp}$$

$$= EA \int_{-1}^{1} \frac{\partial N^i}{\partial \xi} \frac{\partial N^j}{\partial \xi} J^{-1} \, d\xi$$ ← NATURAL

$$= EA \sum_{gp=1}^{GPE} \left\{ \left[\frac{\partial N^i}{\partial \xi} \frac{\partial N^j}{\partial \xi} J^{-1} \right]_{gp} \times \alpha_{gp} \right\}$$

← Loop over gp

⋮

END

WHAT YOU SEE IN THE CODE ? →

For gp=1:GPE

⋮

End

← e.g. in MATLAB

$$[\mathbb{K}^{ij}]_{ac} = \int_{\mathcal{B}} [N^{i}_{,\boldsymbol{x}}]_{b} \, E_{abcd} \, [N^{j}_{,\boldsymbol{x}}]_{d} \, dA$$

$$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{\xi})$$

$$x = x(\xi,\eta)$$

$$y = y(\xi,\eta)$$

$$N^{i}_{,\boldsymbol{x}} \rightarrow \begin{bmatrix} \dfrac{\partial N^{i}}{\partial x} \\[2ex] \dfrac{\partial N^{i}}{\partial y} \end{bmatrix}$$

$$\frac{\partial N^{i}}{\partial x} = \frac{\partial N^{i}}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial N^{i}}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$\frac{\partial N^{i}}{\partial y} = \frac{\partial N^{i}}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial N^{i}}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$\xi = \xi(x,y)$$

$$\eta = \eta(x,y)$$

$$\boldsymbol{\xi} = \boldsymbol{\xi}(\boldsymbol{x})$$

$N^{i}_{,\boldsymbol{x}}$

$$[\mathbb{K}^{ij}]_{ac} = \int_{\mathcal{B}} [N^{i}_{,\boldsymbol{x}}]_{b} \, E_{abcd} \, [N^{j}_{,\boldsymbol{x}}]_{d} \, dA$$

$x = x(\xi,\eta)$

$y = y(\xi,\eta)$

$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{\xi})$

$$\begin{bmatrix} \dfrac{\partial N^{i}}{\partial x} \\[2ex] \dfrac{\partial N^{i}}{\partial y} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \eta}{\partial x} \\[2ex] \dfrac{\partial \xi}{\partial y} & \dfrac{\partial \eta}{\partial y} \end{bmatrix} \begin{bmatrix} \dfrac{\partial N^{i}}{\partial \xi} \\[2ex] \dfrac{\partial N^{i}}{\partial \eta} \end{bmatrix}$$

$$\frac{\partial N^{i}}{\partial x} = \frac{\partial N^{i}}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial N^{i}}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$\frac{\partial N^{i}}{\partial y} = \frac{\partial N^{i}}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial N^{i}}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$\xi = \xi(x,y)$

$\eta = \eta(x,y)$

$\boldsymbol{\xi} = \boldsymbol{\xi}(\boldsymbol{x})$

$N^i_{,\mathbf{x}}$

$$[\mathbb{K}^{ij}]_{ac} = \int_B [N^i_{,\mathbf{x}}]_b \, E_{abcd} \, [N^j_{,\mathbf{x}}]_d \, dA$$

$$x = x(\xi, \eta)$$

$$y = y(\xi, \eta)$$

$$\mathbf{x} = \mathbf{x}(\boldsymbol{\xi}) \leftarrow$$

$$\updownarrow$$

$$\underbrace{\begin{bmatrix} \dfrac{\partial N^i}{\partial x} \\[2ex] \dfrac{\partial N^i}{\partial y} \end{bmatrix}}_{N^i_{,\mathbf{x}}} = \begin{bmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \eta}{\partial x} \\[2ex] \dfrac{\partial \xi}{\partial y} & \dfrac{\partial \eta}{\partial y} \end{bmatrix} \underbrace{\begin{bmatrix} \dfrac{\partial N^i}{\partial \xi} \\[2ex] \dfrac{\partial N^i}{\partial \eta} \end{bmatrix}}_{N^i_{,\boldsymbol{\xi}}}$$

$$\frac{\partial N^i}{\partial \mathbf{x}} \leftarrow N^i_{,\mathbf{x}}$$

$$\frac{\partial N^i}{\partial \boldsymbol{\xi}} \leftarrow N^i_{,\boldsymbol{\xi}}$$

$$\xi = \xi(x, y)$$

$$\eta = \eta(x, y)$$

$$\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{x})$$

$$N^i_{,\boldsymbol{x}}$$

$$[\mathbb{K}^{ij}]_{ac} = \int_B [N^i_{,\boldsymbol{x}}]_b \, E_{abcd} \, [N^j_{,\boldsymbol{x}}]_d \, dA$$

$$x = x(\xi,\eta)$$

$$y = y(\xi,\eta)$$

$$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{\xi})$$

$$\underbrace{\begin{bmatrix} \dfrac{\partial N^i}{\partial x} \\[2ex] \dfrac{\partial N^i}{\partial y} \end{bmatrix}}_{N^i_{,\boldsymbol{x}}} = \begin{bmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \eta}{\partial x} \\[2ex] \dfrac{\partial \xi}{\partial y} & \dfrac{\partial \eta}{\partial y} \end{bmatrix} \underbrace{\begin{bmatrix} \dfrac{\partial N^i}{\partial \xi} \\[2ex] \dfrac{\partial N^i}{\partial \eta} \end{bmatrix}}_{N^i_{,\boldsymbol{\xi}}}$$

$$N^i_{,\boldsymbol{x}} = \begin{bmatrix} \quad \end{bmatrix} N^i_{,\boldsymbol{\xi}}$$

$$\xi = \xi(x,y)$$

$$\eta = \eta(x,y)$$

$$\boldsymbol{\xi} = \boldsymbol{\xi}(\boldsymbol{x})$$

$N^{i}_{,\mathbf{x}}$

$$[\mathbb{K}^{ij}]_{ac} = \int_{\mathcal{B}} [N^{i}_{,\mathbf{x}}]_b \, E_{abcd} \, [N^{j}_{,\mathbf{x}}]_d \, dA$$

$x = x(\xi,\eta)$

$y = y(\xi,\eta)$

$\mathbf{x} = \mathbf{x}(\boldsymbol{\xi})$

$$\left[\frac{\partial \mathbf{u}}{\partial \mathbf{v}}\right]_{\alpha\beta} = \frac{\partial u_\alpha}{\partial v_\beta}$$

$$\mathbb{J}^{-1} = \frac{\partial \boldsymbol{\xi}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{bmatrix}$$

$$N^{i}_{,\mathbf{x}} = \left[\cdots\right] N^{i}_{,\boldsymbol{\xi}}$$

$$\begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{bmatrix}$$

$$\mathbb{J} = \frac{\partial \mathbf{x}}{\partial \boldsymbol{\xi}} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

$\xi = \xi(x,y)$

$\eta = \eta(x,y)$

$\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{x})$

$$N^{i}_{,\mathbf{x}}$$

$$[\mathbb{K}^{ij}]_{ac} = \int_{B} [N^{i}_{,\mathbf{x}}]_{b} \, E_{abcd} \, [N^{j}_{,\mathbf{x}}]_{d} \, dA$$

$$x = x(\xi,\eta)$$

$$y = y(\xi,\eta)$$

$$\mathbf{x} = \mathbf{x}(\boldsymbol{\xi})$$

$$\begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{bmatrix} \leftarrow \mathbb{J}^{-1}$$

$$\mathbb{J} = \frac{\partial \mathbf{x}}{\partial \boldsymbol{\xi}}$$

$$\mathbb{J}^{-1} = \frac{\partial \boldsymbol{\xi}}{\partial \mathbf{x}}$$

$$N^{i}_{,\mathbf{x}} = \left[\cdots \right] N^{i}_{,\boldsymbol{\xi}}$$

$$\begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{bmatrix} = \mathbb{J}^{-t}$$

$$\xi = \xi(x,y)$$

$$\eta = \eta(x,y)$$

$$\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{x})$$

$$[\mathbb{K}^{ij}]_{ac} = \int_{\mathcal{B}} [N^{i}_{,\boldsymbol{x}}]_b \, E_{abcd} \, [N^{j}_{,\boldsymbol{x}}]_d \, dA$$

$$[\mathbb{K}^{ij}]_{ac} = \int_{\mathcal{B}} [\vec{\mathbb{J}}^{-t} \cdot N^{i}_{,\boldsymbol{\xi}}]_b \, E_{abcd} \, [\vec{\mathbb{J}}^{-t} \cdot N^{j}_{,\boldsymbol{\xi}}]_d \, dA$$

$$\mathbb{J} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}} \quad \hookrightarrow \quad \boldsymbol{x} = \boldsymbol{x}(\boldsymbol{\xi}) \quad \checkmark \quad \boldsymbol{x} = N^{s} \boldsymbol{x}^{s}$$

$$\hookrightarrow N^{s}(\xi, \eta)$$

$$[\mathbb{K}^{ij}]_{ac} = \int_{B} [\mathbb{J}^{-t} \cdot N^{i}_{,\xi}]_{b} \, E_{abcd} \, [\mathbb{J}^{-t} \cdot N^{j}_{,\xi}]_{d} \, dA$$

$$J_{11} = \left[\frac{\partial x}{\partial \xi}\right]_{11} = \frac{\partial x}{\partial \xi} = \frac{\partial}{\partial \xi}\left(N^{s} x^{s}\right) = x^{s} \frac{\partial N^{s}}{\partial \xi}$$

$$\mathbb{J} = \frac{\partial x}{\partial \xi} \hookrightarrow x = x(\xi) \nearrow x = N^{s} x^{s}$$

$$\hookrightarrow N^{s}(\xi, \eta)$$

$$[\mathbb{K}^{ij}]_{ac} = \int_B [\mathbb{J}^{-t} \cdot N^i_{,\xi}]_b \, \mathbb{E}_{abcd} \, [\mathbb{J}^{-t} \cdot N^j_{,\xi}]_d \, dA$$

$$J_{11} = \left[\frac{\partial x}{\partial \xi}\right]_{11} = \frac{\partial x}{\partial \xi} = \frac{\partial}{\partial \xi}\left(N^s x^s\right) = x^s \frac{\partial N^s}{\partial \xi}$$

$$= x^1 \frac{\partial N^1}{\partial \xi} + x^2 \frac{\partial N^2}{\partial \xi} + \circ\circ\circ + x^{NPE} \frac{\partial N^{NPE}}{\partial \xi}$$

$$[\mathbb{K}^{ij}]_{ac} = \int_{\mathcal{B}} [\mathbf{J}^{-t} \cdot N^{i}_{,\xi}]_{b} \, \mathbb{E}_{abcd} \, [\mathbf{J}^{-t} \cdot N^{j}_{,\xi}]_{d} \, dA$$

$$J_{11} = \frac{\partial x}{\partial \xi} = x^{1} \frac{\partial N^{1}}{\partial \xi} + x^{2} \frac{\partial N^{2}}{\partial \xi} + \cdots + x^{NPE} \frac{\partial N^{NPE}}{\partial \xi}$$

$$J_{12} = \frac{\partial x}{\partial \eta} = x^{1} \frac{\partial N^{1}}{\partial \eta} + x^{2} \frac{\partial N^{2}}{\partial \eta} + \cdots + x^{NPE} \frac{\partial N^{NPE}}{\partial \eta}$$

$$J_{21} = \frac{\partial y}{\partial x} = y^{1} \frac{\partial N^{1}}{\partial \xi} + y^{2} \frac{\partial N^{2}}{\partial \xi} + \cdots + y^{NPE} \frac{\partial N^{NPE}}{\partial \xi}$$

$$J_{22} = \frac{\partial y}{\partial \eta} = y^{1} \frac{\partial N^{1}}{\partial \eta} + y^{2} \frac{\partial N^{2}}{\partial \eta} + \cdots + y^{NPE} \frac{\partial N^{NPE}}{\partial \eta}$$

$$[\mathbb{K}^{ij}]_{ac} = \int_{\mathcal{B}} [\mathbb{J}^{-t} \cdot N^{i}_{,\xi}]_{b} \, \mathbb{E}_{abcd} \, [\mathbb{J}^{-t} \cdot N^{j}_{,\xi}]_{d} \, dA$$

$$\underbrace{\begin{bmatrix} J_{11} & J_{21} \\ J_{12} & J_{22} \end{bmatrix}}_{\mathbb{J}^{t}} = \underbrace{\begin{bmatrix} \dfrac{\partial N^{1}}{\partial \xi} & \dfrac{\partial N^{2}}{\partial \xi} & \cdots & \dfrac{\partial N^{NPE}}{\partial \xi} \\[2ex] \dfrac{\partial N^{1}}{\partial \eta} & \dfrac{\partial N^{2}}{\partial \eta} & \cdots & \dfrac{\partial N^{NPE}}{\partial \eta} \end{bmatrix}}_{2 \times NPE} \underbrace{\begin{bmatrix} x^{1} & y^{1} \\ x^{2} & y^{2} \\ \vdots & \vdots \\ x^{NPE} & y^{NPE} \end{bmatrix}}_{NPE \times 2}$$

$$K^{ij}_{ac} = \sum_{gp=1}^{GPE} [\bar{J}^{-t} \cdot N^{i}_{,\xi}]_b \, E_{abcd} \, [\bar{J}^{-t} \cdot N^{i}_{,\xi}]_b \, \mathrm{Det}\,\bar{J} \times \alpha_{gp} \times \frac{1}{2}$$

IF TRIANGLE

JACOBIAN $\frac{\partial x}{\partial \xi} \longrightarrow$

$$\bar{J} = \begin{bmatrix} x^1 \cdots x^{NPE} \\ y^1 \cdots y^{NPE} \end{bmatrix} \begin{bmatrix} N^1_{,\xi} & N^1_{,\eta} \\ \vdots & \vdots \\ N^{NPE}_{,\xi} & N^{NPE}_{,\eta} \end{bmatrix}$$

$$E_{abcd} = \frac{E}{2[1+\nu]} [\delta_{ad}\delta_{bc} + \delta_{ac}\delta_{bd}] + \frac{E\nu}{1-\nu^2}\delta_{ab}\delta_{cd}$$

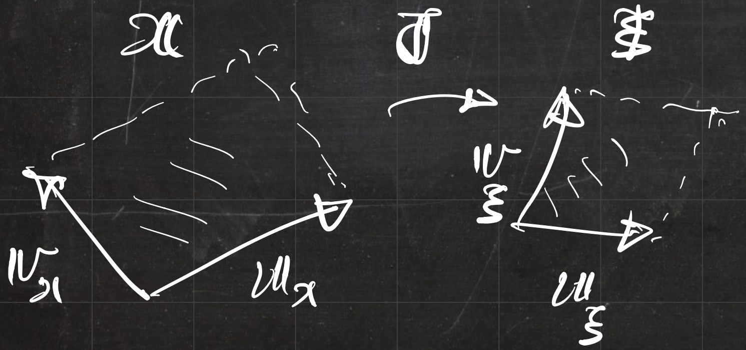
$$\begin{cases} E_{1111} = \frac{E}{[1-\nu^2]} & E_{1112} = 0 & E_{1121} = 0 & E_{1122} = \frac{E}{[1-\nu^2]} \\ E_{1211} = 0 & E_{1212} = \frac{E}{2[1+\nu]} & E_{1221} = \frac{E}{2[1+\nu]} & E_{1222} = 0 \\ E_{2111} = 0 & E_{2112} = \frac{E}{2[1+\nu]} & E_{2121} = \frac{E}{2[1+\nu]} & E_{2122} = 0 \\ E_{2211} = \frac{E}{[1-\nu^2]} & E_{2212} = 0 & E_{2221} = 0 & E_{2222} = \frac{E}{[1-\nu^2]} \end{cases}$$

$$dA_x = \text{Det}\, J \; dA_\xi$$

$$\begin{vmatrix} i & j & k \\ x & x & x \\ x & x & x \end{vmatrix}$$

↳ EXERCISE ↳ $J_{11}J_{22} - J_{12}J_{21}$

$$dA_x = | u_x \times v_x |$$



$$dA_\xi = | u_\xi \times v_\xi |$$

$$dA_x = \underbrace{\text{Det } J}_{25/4} \; dA_\xi$$