FINITE ELEMENT METHOD

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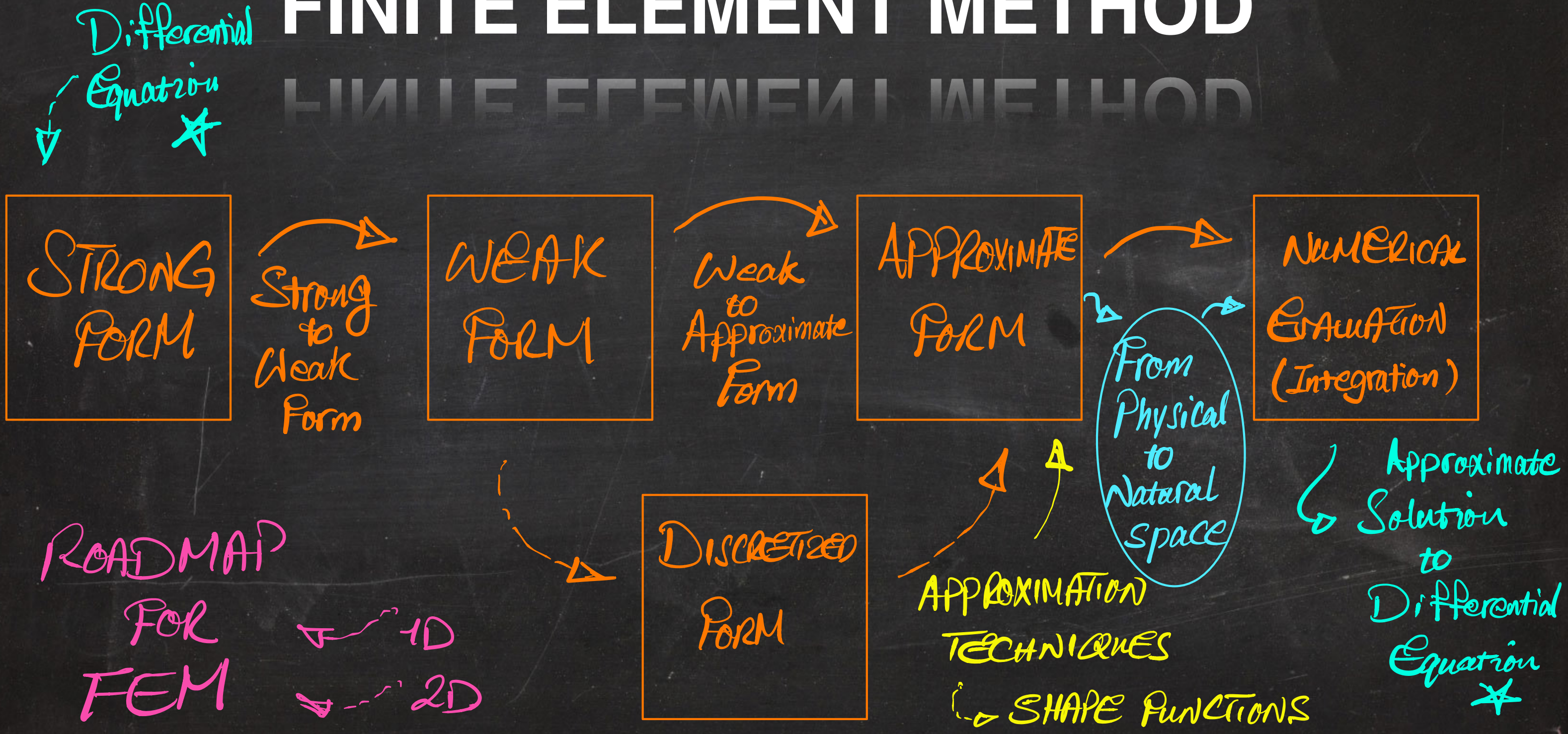

UNDERSTANDING FEM VIA AN ANALOGY (A BRUTAL SIMPLIFICATION)

Sin 45° → Calculator → 0.707 106 7812 ← Approximate Solution

→ Taylor Expansion → $x - \frac{x^3}{3!} + \frac{x^5}{5!}$ ← Approximate Equation

$$\sin x = \sum_{n}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!}$$

0.707 143 0458 ← Solution approximation

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Sin 45° → Calculator → 0.707 106 7812 ← Approximate Solution

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→ Input Approximation

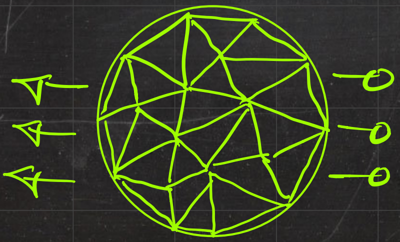
0.707 143 0458 ← Solution approximation

$45^\circ = \frac{\pi}{4} = 0.7853981634$

UNDERSTANDING FEM VIA AN ANALOGY (A BRUTAL SIMPLIFICATION)

Approximations in FEM

- → Solution Approximation ⇝ inherent to numerical techniques using computers
- → Equation Approximation ⇝ diff equation is transformed to weak form + space approximation
- → Input Approximation ⇝ space discretization



Discretization (Approximation)

- → DOMAIN (X)
- → SOLUTION (u)
- → TEST (w)

diff. Eq. ⇓

→ STRONG FORM TO WEAK FORM

integral

STRONG FORM (Differential Equation in 1D)

$$F + dF - F + b\,dV = 0$$

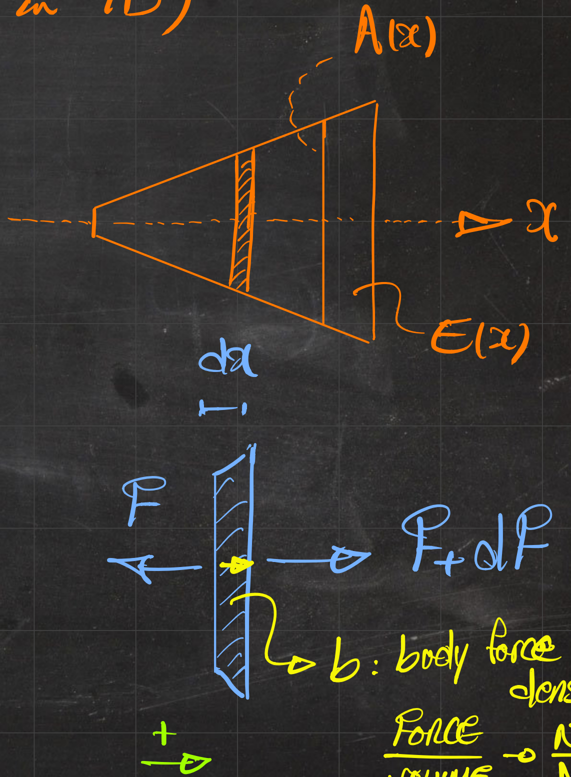
$dV \leftarrow A\,dx$

$$\frac{dF}{dx} + bA = 0$$

$bA \to w$ → force density per length

$[N/M]$

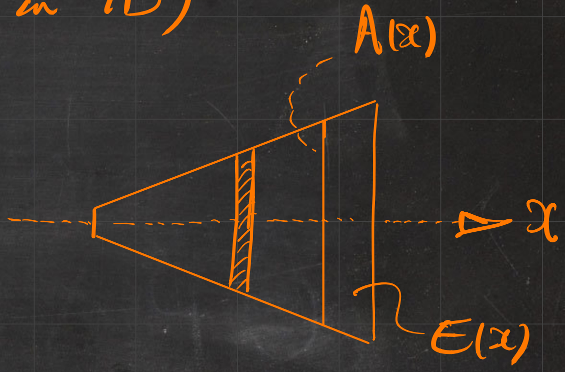
↑ 1D force density



→ b : body force density

$\frac{\text{FORCE}}{\text{VOLUME}} \to \frac{N}{M^3}$

STRONG FORM (Differential Equation in 1D)



$$\frac{dF}{dx} + bA = 0$$

$F = \sigma A$

$\sigma = E\varepsilon$

$\varepsilon = du/dx = u'$ ← 1D-Problem

$$\frac{d}{dx}(\sigma A) + bA = 0$$

$$\frac{d}{dx}(EA\varepsilon) + bA = 0 \Rightarrow \frac{d}{dx}\left(EA\frac{du}{dx}\right) + bA = 0$$

E, A : CONST.

$$Eu'' + b = 0$$

STRONG FORM (Differential Equation in 1D)

$$\frac{dF}{dx} + bA = 0$$

$F = \sigma A$

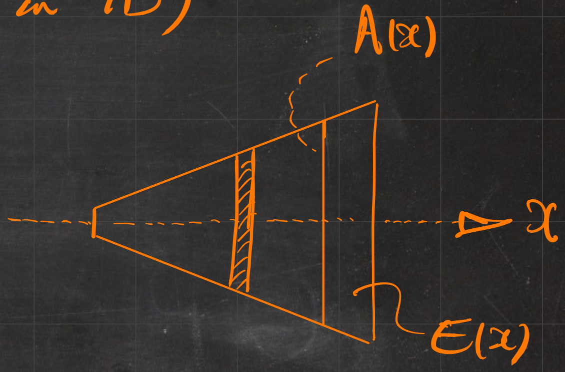
$\sigma = E\varepsilon$

$\varepsilon = u'$

ODE of 2nd order

$\Rightarrow \quad Eu'' + b = 0$ & 2BCs on boundary $\begin{cases} x = 0 \\ x = L \end{cases}$

Length of the domain



STRONG FORM

$$Eu'' + b = 0$$

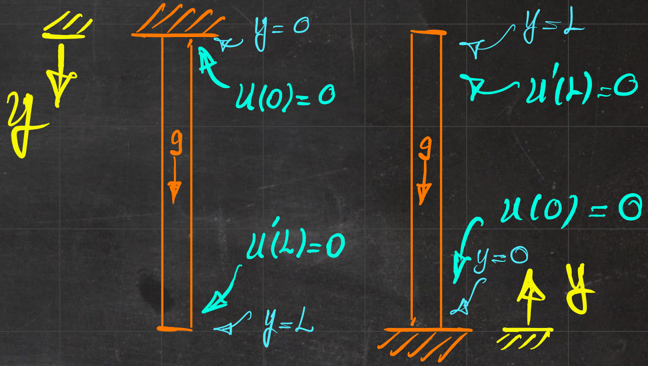
2 ends ← boundary condition

boundary

$$F = \sigma A = E\varepsilon A = EAu'$$

- Dirichlet ← Disp. ← u
- Neumann ← Force ← u'

Bar under its own weight



Hanging bar → Elongated

Standing bar → Compressed

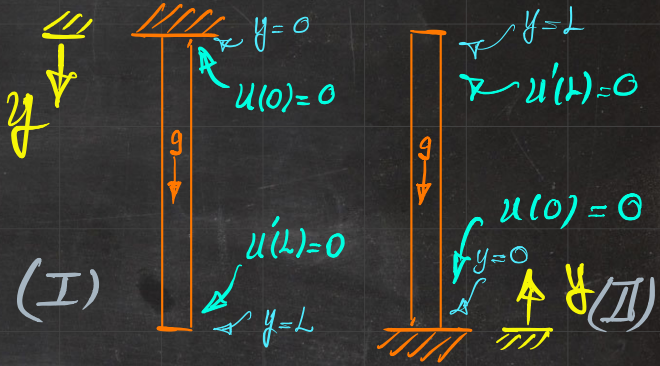
STRONG FORM

Bar under its own weight

$$Eu'' + b = 0$$

$+\rho g$

(I) Hanging bar ← $Eu'' + b = 0$



$$Eu'' + \rho g = 0 \quad \text{Subject to} \quad u(0)=0\,, \quad u'(L)=0$$

$$u = -\frac{1}{2}\frac{\rho g}{E}y^2 + C_1 y + C_2 \quad \overset{\text{BCs}}{\Longrightarrow} \quad u = -\frac{1}{2}\frac{\rho g}{E}y^2 + \frac{\rho g L}{E}y$$

$u(0)=0$

$u(L) = \frac{1}{2}\frac{\rho g}{E}L^2$

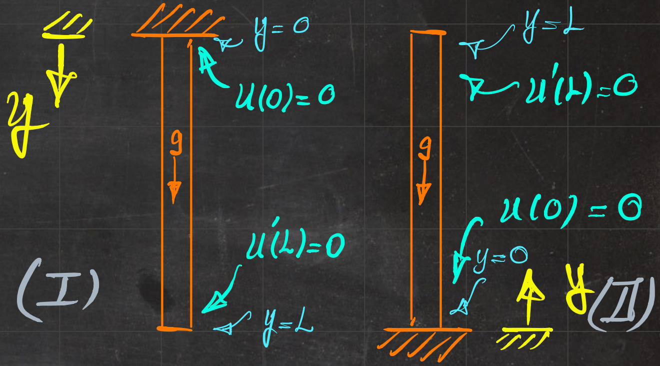
STRONG FORM

Bar under its own weight

$$E u'' + b = 0$$

$-\rho g$

(II) Standing bar ← $E u'' + b = 0$



$$E u'' - \rho g = 0 \quad \text{subject to} \quad u(0) = 0, \; u'(L) = 0$$

$$u = + \frac{1}{2} \frac{\rho g}{E} y^2 + C_1 y + C_2 \overset{\text{BCs}}{\Longrightarrow} u = \frac{1}{2} \frac{\rho g}{E} y^2 - \frac{\rho g L}{E} y$$

$u(0) = 0$

$u(L) = -\frac{1}{2} \frac{\rho g}{E} L^2$

STRONG FORM $Eu'' + b = 0$

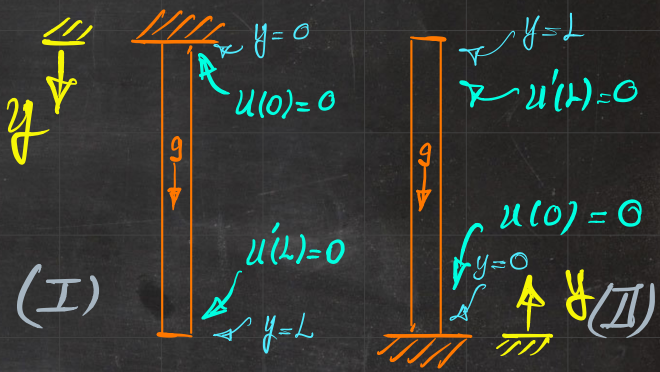
Bar under its own weight

(I) Hanging bar $\leftarrow Eu'' + \rho g = 0$

$$u = -\frac{1}{2}\frac{\rho g}{E}y^2 + \frac{\rho g L}{E}y$$

(II) Standing bar $\leftarrow Eu'' - \rho g = 0$

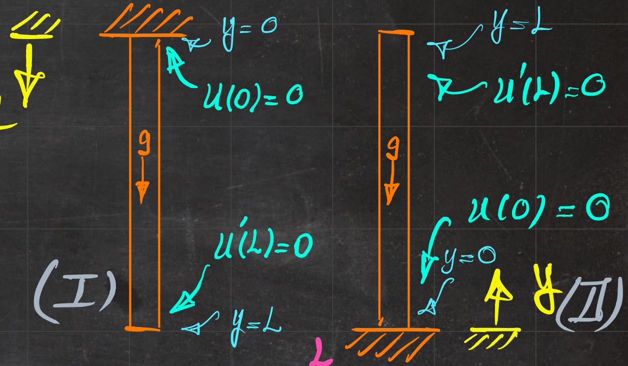
$$u = +\frac{1}{2}\frac{\rho g}{E}y^2 - \frac{\rho g L}{E}y$$

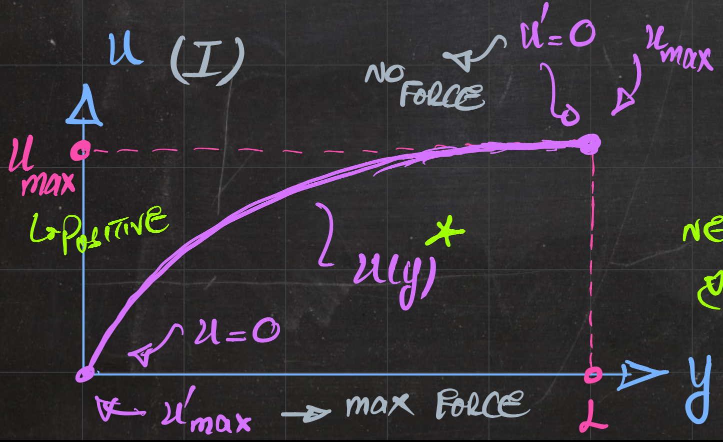


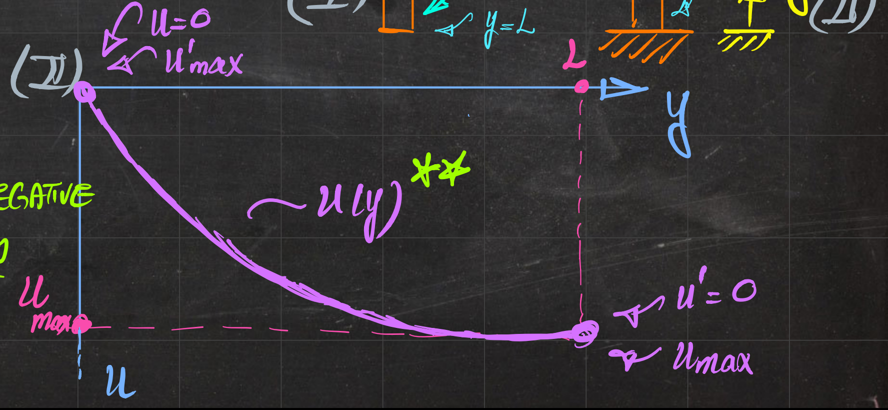
STRONG FORM $Eu'' + b = 0$ Bar under its own weight

(I) Hanging bar $u = -\frac{1}{2}\frac{\rho g}{E}y^2 + \frac{\rho g L}{E}y$ *

(II) Standing bar $u = +\frac{1}{2}\frac{\rho g}{E}y^2 - \frac{\rho g L}{E}y$ **







$$\frac{d}{dx}\left(EA\frac{du}{dx}\right) + bA = 0 \quad \text{Subject to BCs}$$

E, A: Const. $\rightsquigarrow EA\,u'' + bA = 0 \quad \leftarrow f := \frac{b}{E}$

STRONG FORM

$\rightsquigarrow$

$$u'' + f = 0 \qquad 0 \leqslant x \leqslant 1$$

D: $u(0) = u_0$ ← prescribed

N: $u'(1) = t$ ←